Optimal Disturbances in Compressible Boundary Layers – Complete Energy Norm Analysis

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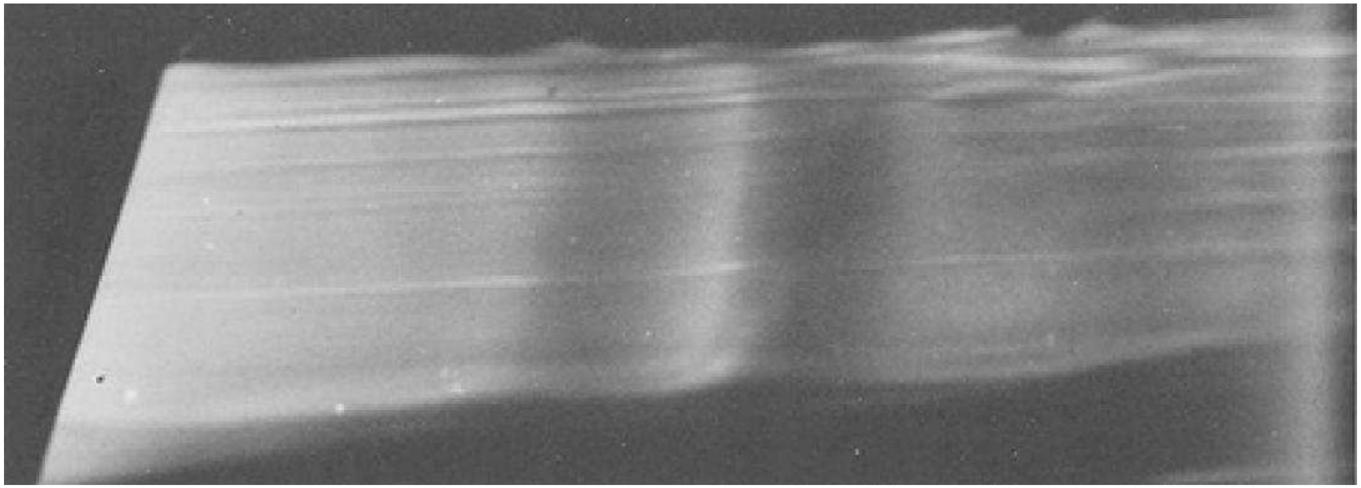
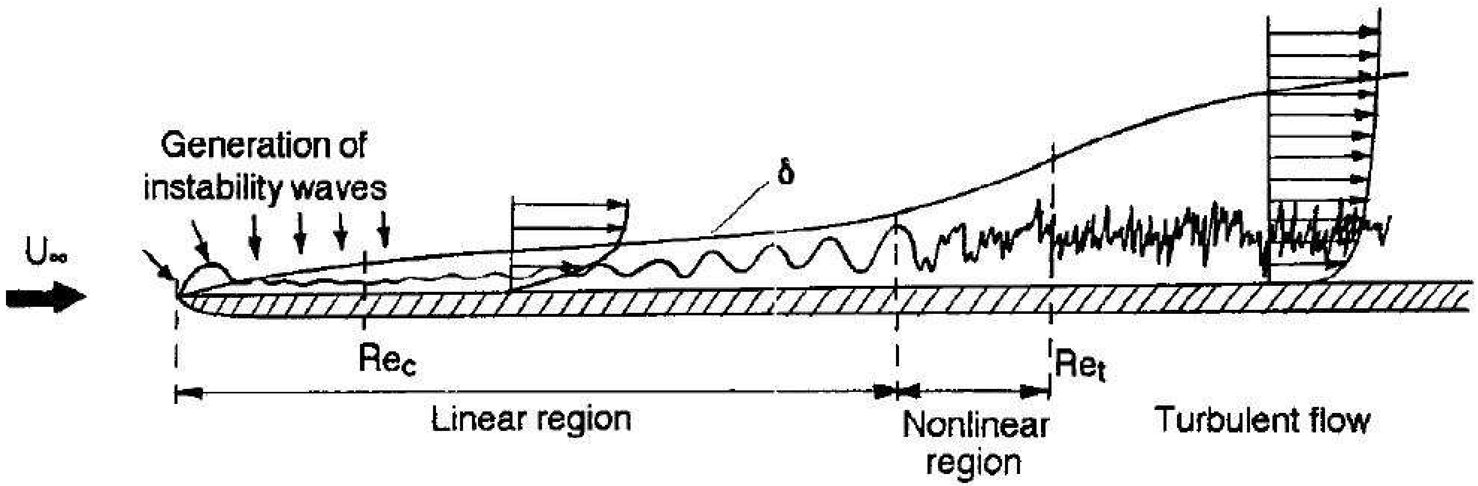
A classical transition mechanism

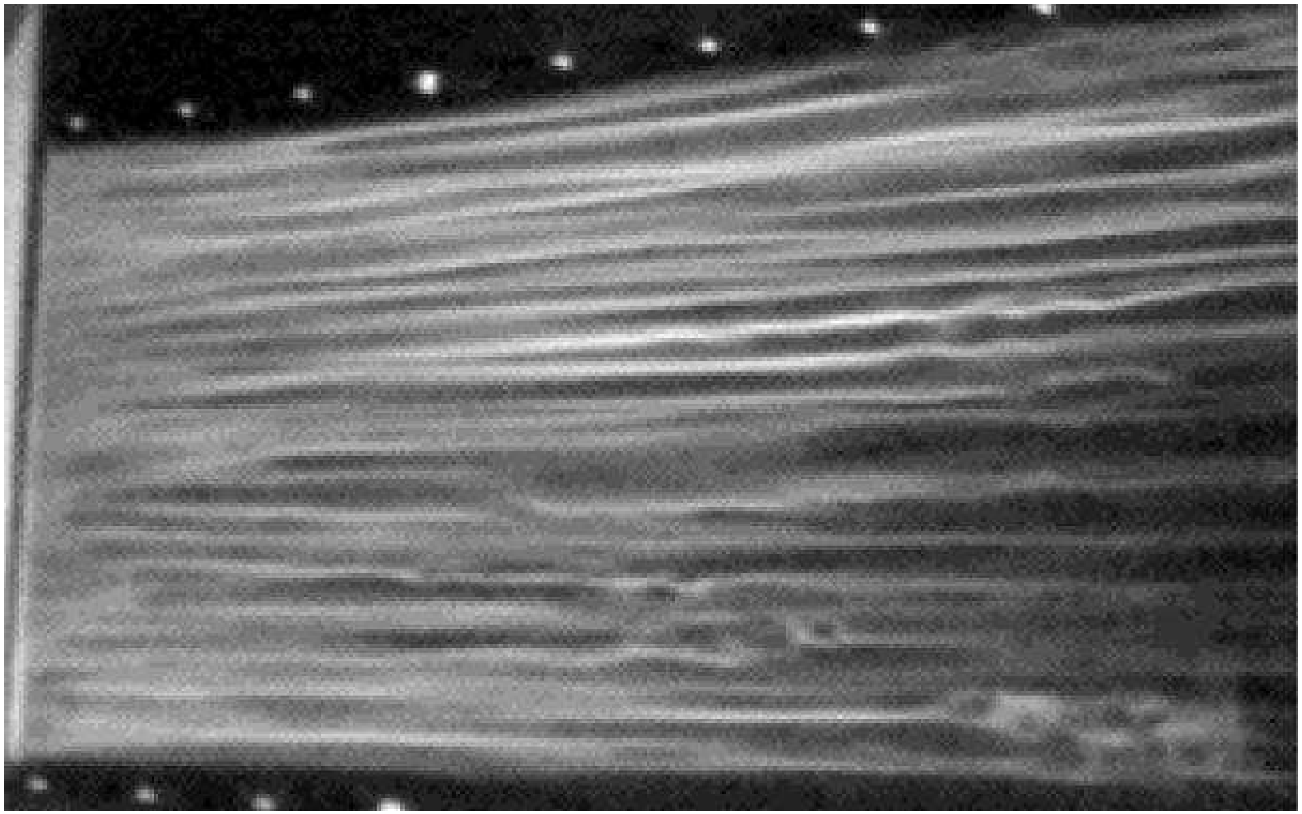




Tollmien-Schlicting (TS) waves first experimentally detected by Schubauer and Skramstad (1947), "Laminar boundary-layer oscillations and transition on a flat plate", *J. Res. Nat. Bur. Stand* 38:251–92, originally issued as NACA-ACR, 1943.

Are TS waves the only mechanism?

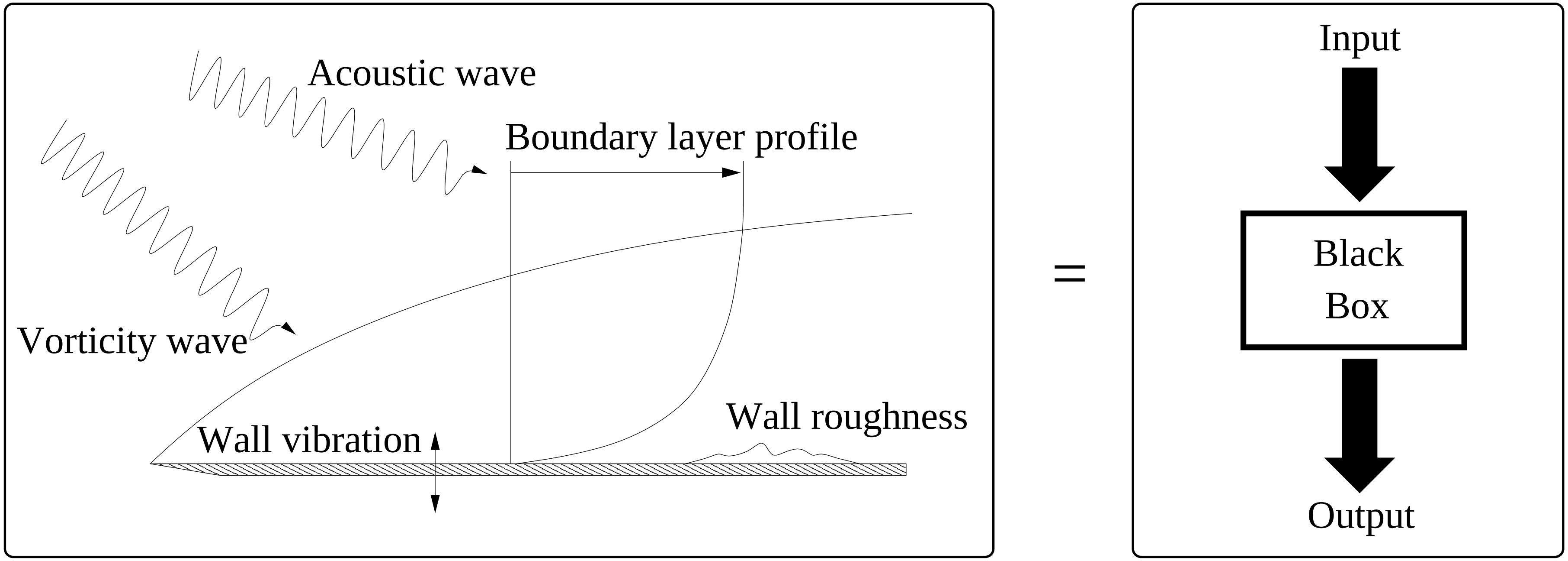
If the disturbances are not really infinitesimal (real world!)...



...streaks (instead of waves) can develop where the flow is stable according to the classical neutral stability curve.

Alternative mechanism to TS waves: Transient growth.

Modelling



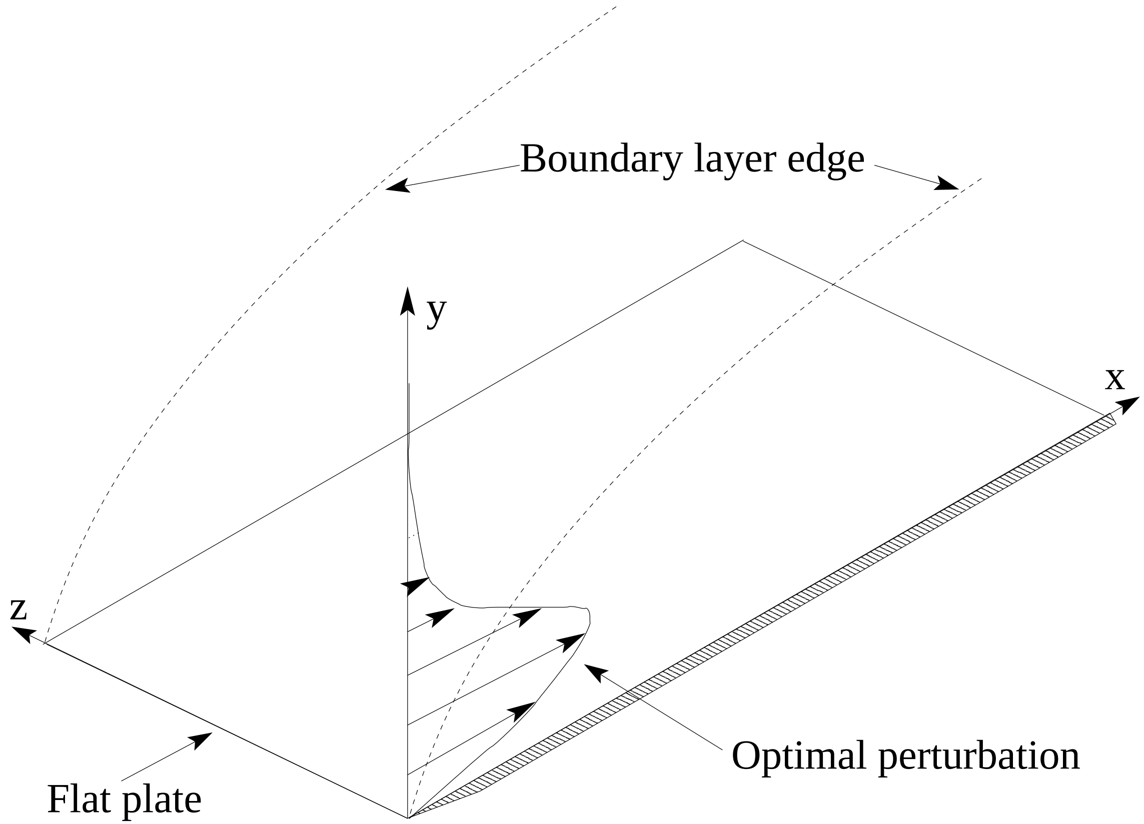
A boundary layer, and its governing equations, can be thought in an **input/output** fashion.

- *Inputs*. Initial conditions and boundary conditions.
- *Outputs*. Flow field, which can be measured by a norm.

Optimal perturbations

Question.

What is the most disrupting, steady initial condition, which maximizes the energy growth for a *given* initial energy of the perturbation?



In this sense the perturbations are optimal.

Goals/Tools

Goals

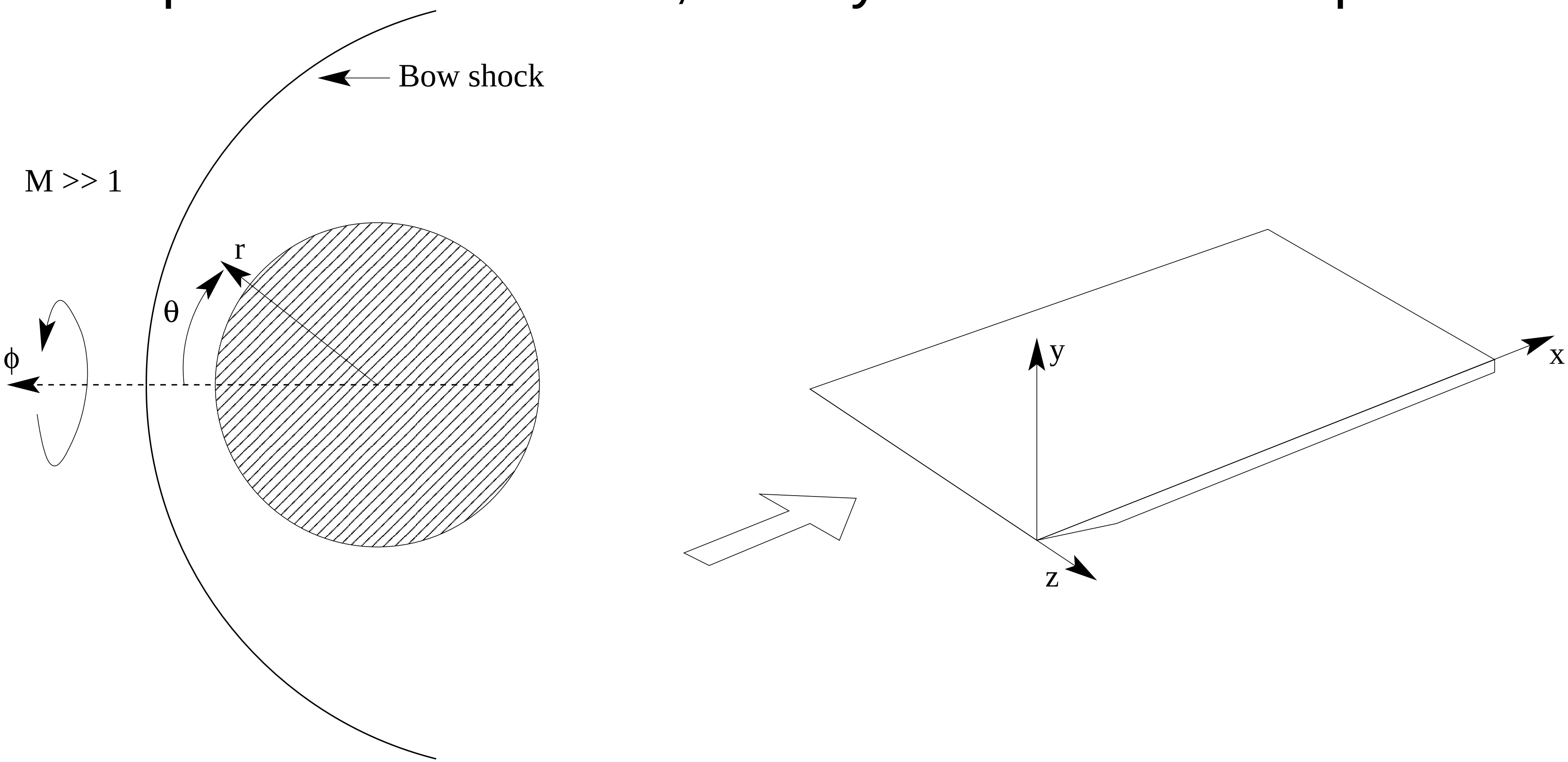
- Efficient and robust numerical determination of optimal perturbations in compressible flows.
- Formulation of the optimization problem in the discrete framework.
- Coupling conditions automatically recovered from the constrained optimization.
- Effect of energy norm choice at the outlet.

Tools

- Lagrange Multipliers technique.
- Iterative algorithm for the determination of optimal initial condition.

Problem formulation

- Geometry. Flat plate and sphere.
- Regimes. Compressible, sub/supersonic. Possibly reducing to incompressible regime for $M \to 0$.
- Equations. Linearized, steady Navier-Stokes equations.



Scaling (1/2)

- L_{ref} is a typical scale of the geometry (L for flat plate, R for sphere, etc.)
- $H_{\text{ref}} = \sqrt{\nu_{\text{ref}} L_{\text{ref}} / U_{\text{ref}}}$ is a typical boundary-layer scale in the wall-normal direction
  - Flat plate. $H_{\text{ref}} = l = \sqrt{\nu_\infty L / U_\infty}$; $\;\infty\; =$ freestream.
  - Sphere. $H_{\text{ref}} = \sqrt{\nu_{\text{ref}} R / U_{\text{ref}}}$; $\;_{\text{ref}}\; =$ edge-conditions at x_{ref}.
- $\epsilon = H_{\text{ref}} / L_{\text{ref}}$ is a small parameter.
  - Flat plate. $\epsilon = Re_L^{-1/2}$, $Re_L = U_\infty L / \nu_\infty$.
  - Sphere. $\epsilon = Re_{\text{ref}}^{-1/2}$, $Re_{\text{ref}} = U_{\text{ref}} R / \nu_{\text{ref}}$.

Scaling (2/2)

From previous works, disturbance expected as streamwise vortices. The natural scaling is therefore

- x normalized with L_{ref}, y and z scaled with ϵL_{ref}.
- u is scaled with U_{ref}, v and w with ϵU_{ref}.
- T with T_{ref} and p with $\epsilon^2 \rho_{\text{ref}} U_{\text{ref}}^2$. ρ eliminated through the state equation.

Due to the scaling, $(\cdot)_{xx} << 1$. The equations are parabolic!

By assuming perturbations in the form $q(x, y) \exp(\mathrm{i}\beta z)$ (flat plate – β spanwise wavenumber) and $q(x, y) \exp(\mathrm{i}m\phi)$ (sphere – m azimuthal index)...

Governing equations

$$(\boldsymbol{A}\mathbf{f})_x = (\boldsymbol{D}\mathbf{f}_y)_x + \boldsymbol{B}_0\mathbf{f} + \boldsymbol{B}_1\mathbf{f}_y + \boldsymbol{B}_2\mathbf{f}_{yy}$$

$\mathbf{f} = [u, v, w, T, p]^{\mathrm{T}}$; $\boldsymbol{A}$, $\boldsymbol{B}_0$, $\boldsymbol{B}_1$, $\boldsymbol{B}_2$, $\boldsymbol{D}$ 5×5 real matrices.

Boundary conditions

$$y = 0: \quad u = 0; v = 0; w = 0; T = 0$$
$$y \to \infty: \quad u \to 0; w \to 0; p \to 0; T \to 0$$

More compactly

$$(\boldsymbol{H}_1\mathbf{f})_x + \boldsymbol{H}_2\mathbf{f} = 0$$

with $\boldsymbol{H}_1 = \boldsymbol{A} - \boldsymbol{D}(\cdot)_y; \boldsymbol{H}_2 = -\boldsymbol{B}_0 - \boldsymbol{B}_1(\cdot)_y - \boldsymbol{B}_2(\cdot)_{yy}$

Objective function (1/2)

Caveat!

- Results depend on the choice of the objective function.
- Physics dominated by streamwise vortices.
- Common choices of the energy norms.
  - Inlet. $v_{\text{in}} \neq 0$ and $w_{\text{in}} \neq 0$ ($u_{\text{in}} = T_{\text{in}} = 0$).
  - Outlet. $v_{\text{out}} = 0$ and $w_{\text{out}} = 0$ ($u_{\text{in}} \neq 0; T_{\text{in}} \neq 0$).
- Blunt body. Largest transient growth close to the stagnation point.
  - Due to short x-interval, a streaks-dominated flow field might not be completely established.
  - Contribution of v_{out} and w_{out} could be non negligible.
  - Outlet norm. FEN vs. PEN

Objective function (2/2)

Mack's energy norm (derived for flat plate and temporal problem), after scaling and using state equation,

$$E_{\text{out}} = \int_0^\infty \left[\rho_{s\text{out}} (u_{\text{out}}^2 + v_{\text{out}}^2 + w_{\text{out}}^2) + \frac{p_{s\text{out}} T_{\text{out}}^2}{(\gamma - 1) T_{s\text{out}}^2 M^2} \right] dy$$

or in matrix form as $E_{\text{out}} = \int_0^\infty \left(\mathbf{f}_{\text{out}}^T \widetilde{\boldsymbol{M}}_{\text{out}} \mathbf{f}_{\text{out}} \right) dy$, with

$$\widetilde{\boldsymbol{M}}_{\text{out}} = \text{diag}\left(\rho_{s\text{out}}, \rho_{s\text{out}}, \rho_{s\text{out}}, \frac{p_{s\text{out}}}{(\gamma - 1) T_{s\text{out}}^2 M^2}, 0 \right).$$

Initial energy of the perturbation

$$E_{\text{in}} = \int_0^\infty \left[\rho_{s\text{in}} (v_{\text{in}}^2 + w_{\text{in}}^2) \right] dy \Rightarrow E_{\text{in}} = \int_0^\infty \left(\mathbf{f}_{\text{in}}^T \widetilde{\boldsymbol{M}}_{\text{in}} \mathbf{f}_{\text{in}} \right) dy$$

Constrained optimization (1/3)

Our constraints are the governing equations, boundary conditions and the normalization condition $E_{\text{in}} = E_0$.

After discretization ($\boldsymbol{M}_0 \Leftrightarrow \widetilde{\boldsymbol{M}}_{\text{in}}$ and $\boldsymbol{M}_N \Leftrightarrow \widetilde{\boldsymbol{M}}_{\text{out}}$),

- objective function $\mathcal{J} = \mathbf{f}_N^{\text{T}} \boldsymbol{M}_N \mathbf{f}_N$
- constraint $E_{\text{in}} = E_0 \Rightarrow \mathbf{f}_0^{\text{T}} \boldsymbol{M}_0 \mathbf{f}_0 = E_0$
- governing equations (BC included) $\boldsymbol{C}_{n+1} \mathbf{f}_{n+1} = \boldsymbol{B}_n \mathbf{f}_n$

The augmented functional $\mathcal{L}$ is

$$\mathcal{L}(\mathbf{f}_0, \ldots, \mathbf{f}_N) = \mathbf{f}_N^{\text{T}} \boldsymbol{M}_N \mathbf{f}_N + \lambda_0 [\mathbf{f}_0^{\text{T}} \boldsymbol{M}_0 \mathbf{f}_0 - E_0] + \sum_{n=0}^{N-1} \left[\mathbf{p}_n^{\text{T}} \left(\boldsymbol{C}_{n+1} \mathbf{f}_{n+1} - \boldsymbol{B}_n \mathbf{f}_n \right) \right]$$

with λ_0 and (vector) $\mathbf{p}_n$ Lagrangian multipliers.

Constrained optimization (2/3)

By adding and subtracting $\mathbf{p}_{n+1}^{\mathrm{T}} \boldsymbol{B}_{n+1} \mathbf{f}_{n+1}$ in the summation,

$$
\begin{aligned}
\sum_{n=0}^{N-1} \left[\mathbf{p}_n^{\mathrm{T}} \left(\boldsymbol{C}_{n+1}\mathbf{f}_{n+1} - \boldsymbol{B}_n \mathbf{f}_n\right)\right] \quad &= \quad \sum_{n=0}^{N-1} \left[\mathbf{p}_n^{\mathrm{T}} \boldsymbol{C}_{n+1}\mathbf{f}_{n+1} - \mathbf{p}_{n+1}^{\mathrm{T}} \boldsymbol{B}_{n+1}\mathbf{f}_{n+1}\right] + \\
& \quad\quad \sum_{n=0}^{N-1} \left[\mathbf{p}_{n+1}^{\mathrm{T}} \boldsymbol{B}_{n+1}\mathbf{f}_{n+1} - \mathbf{p}_n^{\mathrm{T}} \boldsymbol{B}_n \mathbf{f}_n\right] \\
&= \quad \sum_{n=0}^{N-1} \left[\mathbf{p}_n^{\mathrm{T}} \boldsymbol{C}_{n+1}\mathbf{f}_{n+1} - \mathbf{p}_{n+1}^{\mathrm{T}} \boldsymbol{B}_{n+1}\mathbf{f}_{n+1}\right] + \\
& \quad\quad \mathbf{p}_N^{\mathrm{T}} \boldsymbol{B}_N \mathbf{f}_N - \mathbf{p}_0^{\mathrm{T}} \boldsymbol{B}_0 \mathbf{f}_0,
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}(\mathbf{f}_0, \ldots, \mathbf{f}_N) \quad &= \quad \mathbf{f}_N^{\mathrm{T}} \boldsymbol{M}_N \mathbf{f}_N + \sum_{n=0}^{N-1} \left[\mathbf{p}_n^{\mathrm{T}} \boldsymbol{C}_{n+1}\mathbf{f}_{n+1} - \mathbf{p}_{n+1}^{\mathrm{T}} \boldsymbol{B}_{n+1}\mathbf{f}_{n+1}\right] + \\
& \quad\quad \mathbf{p}_N^{\mathrm{T}} \boldsymbol{B}_N \mathbf{f}_N - \mathbf{p}_0^{\mathrm{T}} \boldsymbol{B}_0 \mathbf{f}_0 + \lambda_0 [\mathbf{f}_0^{\mathrm{T}} \boldsymbol{M}_0 \mathbf{f}_0 - E_0].
\end{aligned}
$$

Stationary condition

$$
\delta\mathcal{L} = 0 \Rightarrow \frac{\delta\mathcal{L}}{\delta\mathbf{f}_0}\delta\mathbf{f}_0 + \sum_{n=0}^{N-2}\left[\frac{\delta\mathcal{L}}{\delta\mathbf{f}_{n+1}}\delta\mathbf{f}_{n+1}\right] + \frac{\delta\mathcal{L}}{\delta\mathbf{f}_N}\delta\mathbf{f}_N = 0
$$

Constrained optimization (3/3)

$$\frac{\delta \mathcal{L}}{\delta \mathbf{f}_0} = -\mathbf{p}_0^{\mathrm{T}} \boldsymbol{B}_0 + 2\lambda_0 \mathbf{f}_0^{\mathrm{T}} \boldsymbol{M}_0 = 0$$

$$\frac{\delta \mathcal{L}}{\delta \mathbf{f}_{n+1}} = \mathbf{p}_n^{\mathrm{T}} \boldsymbol{C}_{n+1} - \mathbf{p}_{n+1}^{\mathrm{T}} \boldsymbol{B}_{n+1} = 0, \quad n = 0, \ldots, N-2$$

$$\frac{\delta \mathcal{L}}{\delta \mathbf{f}_N} = 2\mathbf{f}_N^{\mathrm{T}} \boldsymbol{M}_N + \mathbf{p}_N^{\mathrm{T}} \boldsymbol{B}_N = 0$$

Inlet conditions : $\mathbf{f}_{0j} = \begin{cases} \dfrac{(\mathbf{p}_0^T \boldsymbol{B}_0)_j}{2\lambda_0 \boldsymbol{M}_{0jj}} & \text{if} \quad \boldsymbol{M}_{0jj} \neq 0 \\ 0 & \text{if} \quad \boldsymbol{M}_{0jj} = 0 \end{cases}$

"Adjoint" equations : $\mathbf{p}_n^{\mathrm{T}} \boldsymbol{C}_{n+1} - \mathbf{p}_{n+1}^{\mathrm{T}} \boldsymbol{B}_{n+1} = 0$

Oulet conditions : $\boldsymbol{B}_N^{\mathrm{T}} \mathbf{p}_N = -2\boldsymbol{M}_N^{\mathrm{T}} \mathbf{f}_N$

An optimization algorithm

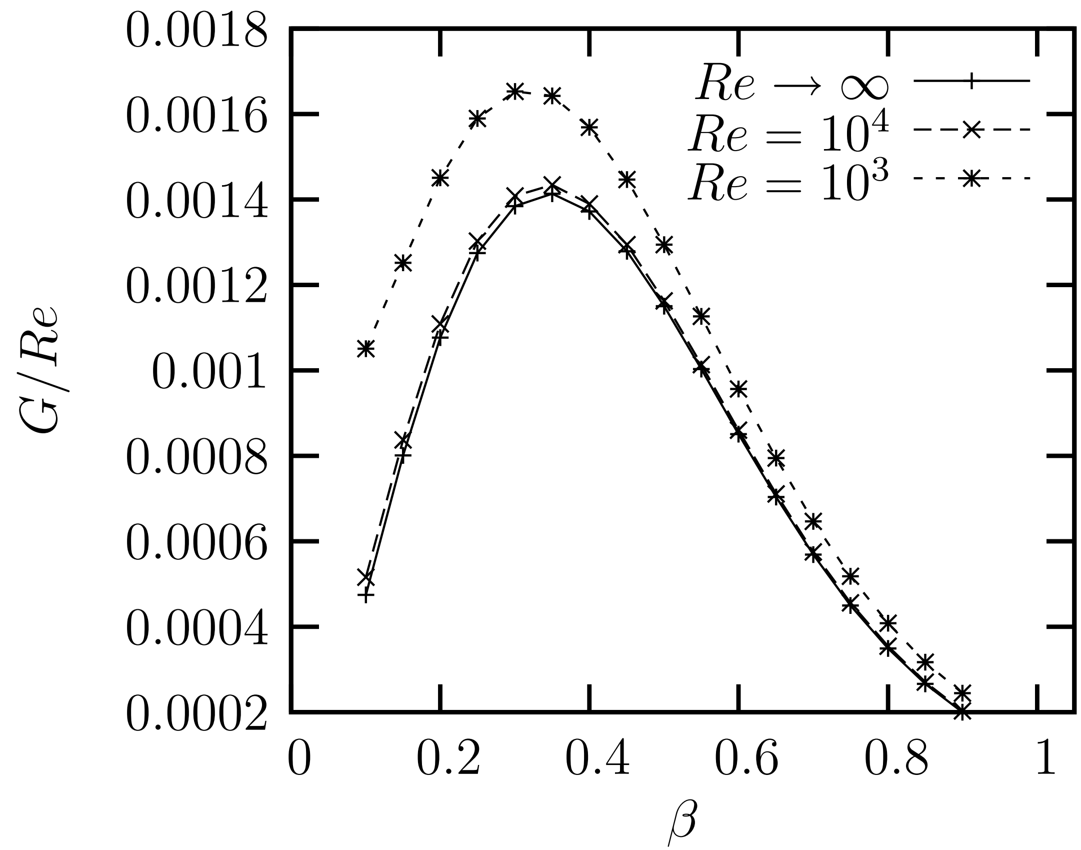
1. guessed initial condition $\mathbf{f}_{\text{in}}^{(0)}$
2. solution of forward problem with the IC $\mathbf{f}_{\text{in}}^{(n)}$
3. evaluation of objective function $\mathcal{J}^{(n)} = E_{\text{out}}^{(n)}$. If $|\mathcal{J}^{(n)}/\mathcal{J}^{(n-1)} - 1| < \epsilon_t$ optimization converged
4. if $|\mathcal{J}^{(n)}/\mathcal{J}^{(n-1)} - 1| > \epsilon_t$ outlet conditions provide the "initial" conditions for the backward problem at $x = x_{\text{out}}$
5. backward solution of the "adjoint" problem from $x = x_{\text{out}}$ to $x = x_{\text{in}}$
6. from the inlet conditions, update of the initial condition for the forward problem $\mathbf{f}_{\text{in}}^{(n+1)}$
7. repeat from step 2 on

Results

- Discretization.
  - 2nd-order backward finite differences in x and 4th-order finite differences in y.
  - Uneven grids in both x and y.
- Code verified against results by Tumin & Reshotko (2003, 2004) obtained with spectral collocation method.

- Inlet norm includes v_{in} and w_{in} only.
- Outlet norm.
  - Partial Energy Norm (PEN) u_{out} and T_{out} only.
  - Full Energy Norm (FEN) $u_{\text{out}}, v_{\text{out}}, w_{\text{out}}, T_{\text{out}}$.
  - FEN depends on Re, PEN is Re-independent.

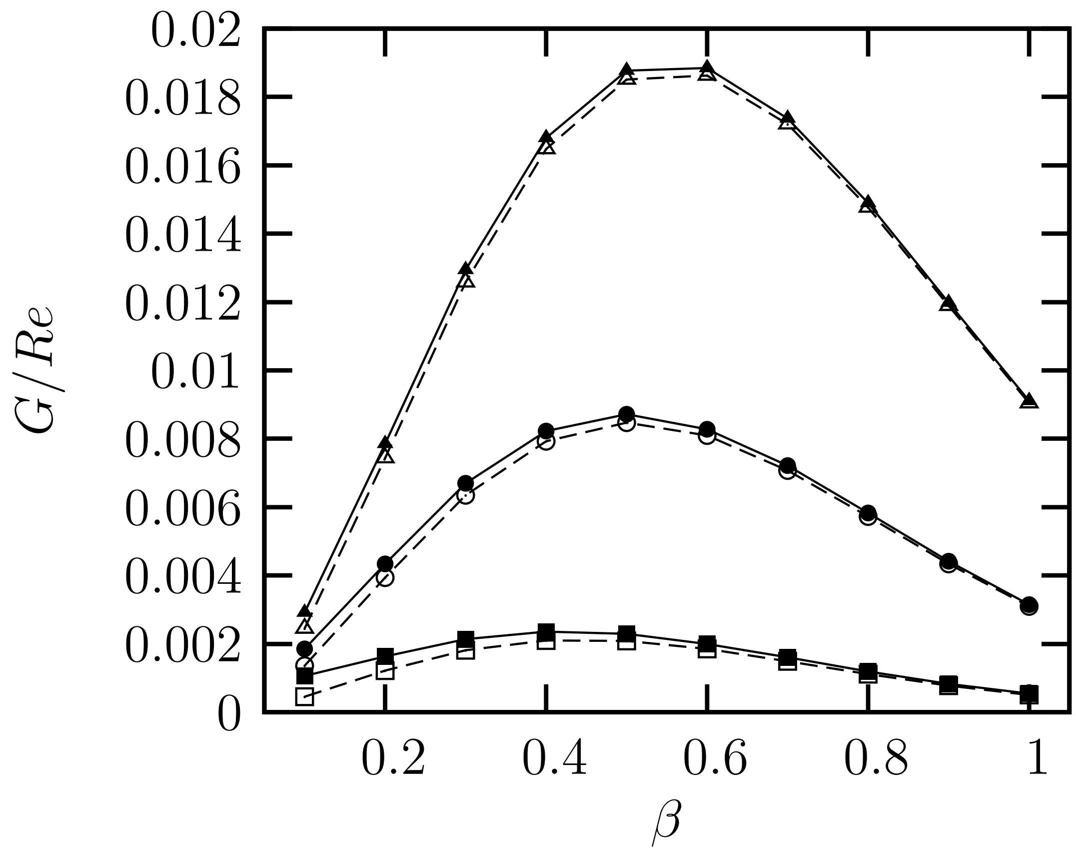
Results – Flat plate



Objective function G/Re: effect of Re and β for $M = 3$, $T_w/T_{\mathrm{ad}} = 1$, $x_{\mathrm{in}} = 0$ $x_{\mathrm{out}} = 1.0$, FEN.

$\Rightarrow$ Reynolds number effects only for $Re < 10^4$.

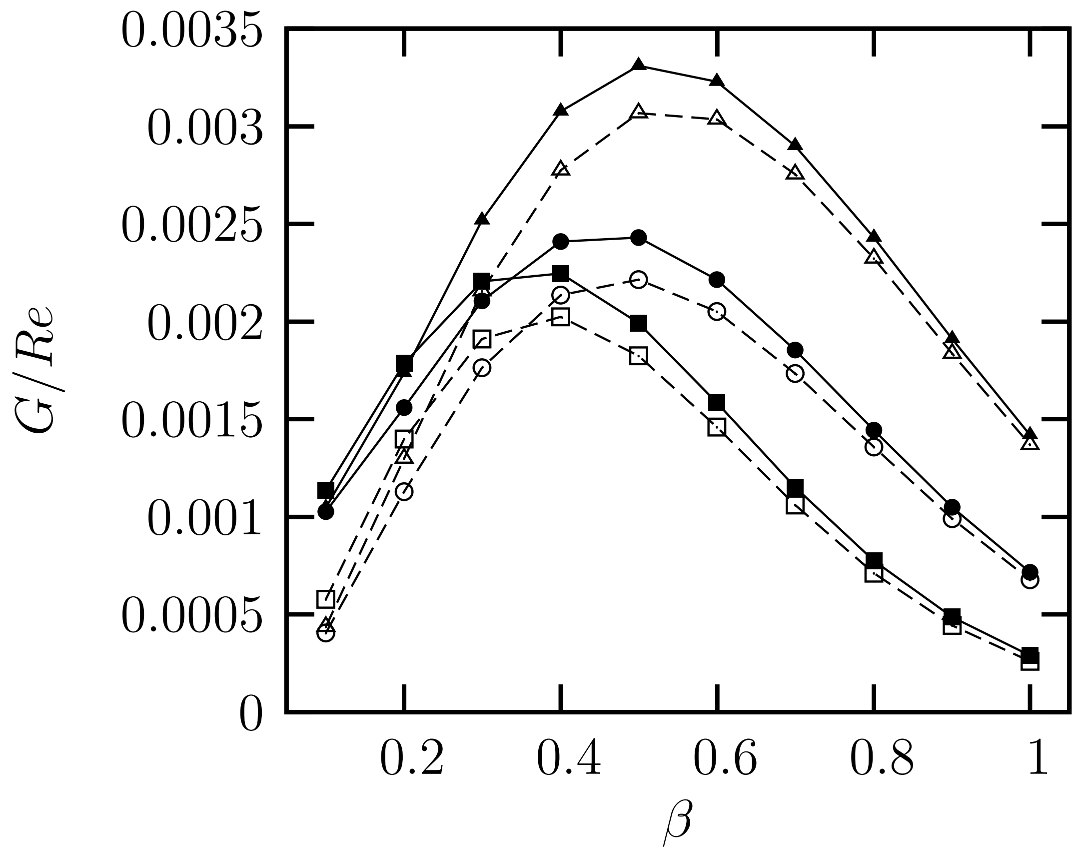
Results – Flat plate



Objective function G/Re: effect of β, T_w/T_{ad} and norm choice (PEN vs. FEN) for $M = 0.5$, $Re = 10^3$, $x_{in} = 0$ $x_{out} = 1.0$. □, $T_w/T_{ad} = 1.00$; ○, $T_w/T_{ad} = 0.50$; △, $T_w/T_{ad} = 0.25$.

$\Rightarrow$ No remarkable norm effects; cold wall destabilizing factor.

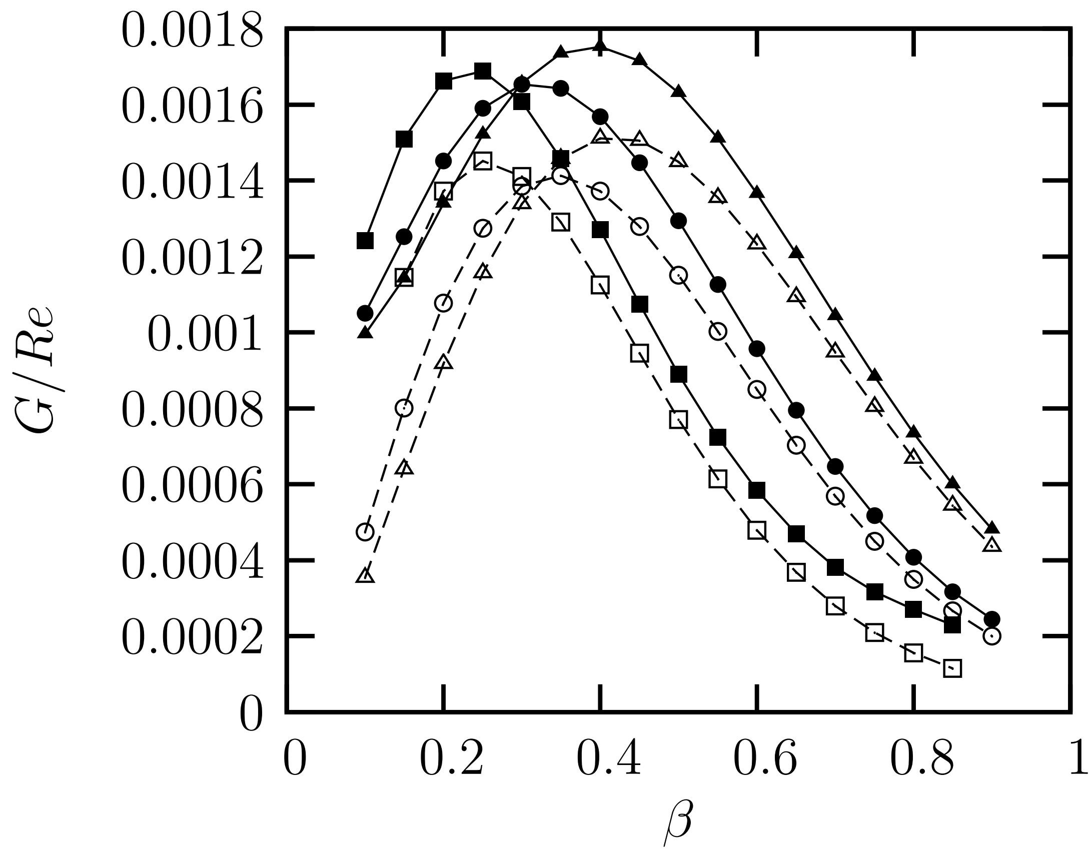
Results – Flat plate



Objective function G/Re: effect of β, $T_w/T_{\rm ad}$ and norm choice (PEN vs. FEN) for $M = 1.5$, $Re = 10^3$, $x_{\rm in} = 0$ $x_{\rm out} = 1.0$. $\square$, $T_w/T_{\rm ad} = 1.00$; $\circ$, $T_w/T_{\rm ad} = 0.50$; $\triangle$, $T_w/T_{\rm ad} = 0.25$.

$\Rightarrow$ Shift of the curves maximum, enhanced difference between norms ($T_w/T_{\rm ad} = 1.00$).

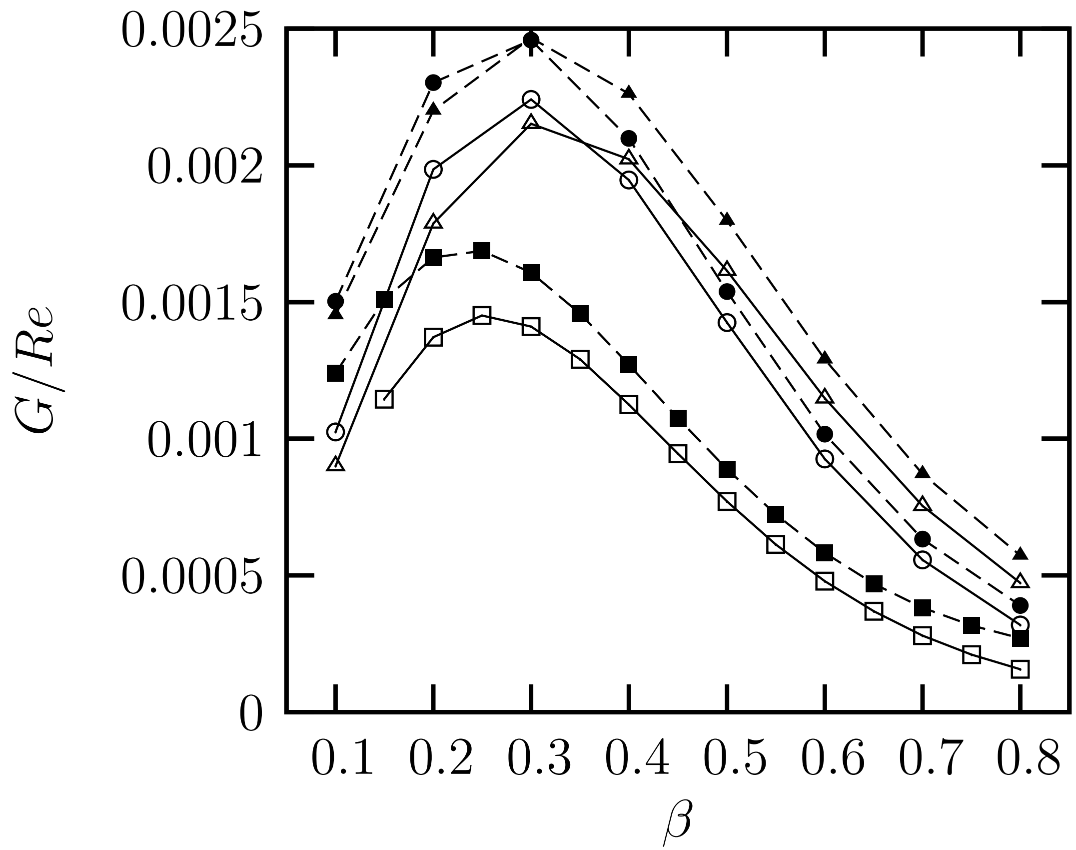
Results – Flat plate



Objective function G/Re: effect of β, T_w/T_{ad} and norm choice (PEN vs. FEN) for $M = 3$, $Re = 10^3$, $x_{\mathrm{in}} = 0$ $x_{\mathrm{out}} = 1.0$. □, $T_w/T_{\mathrm{ad}} = 1.00$; ○, $T_w/T_{\mathrm{ad}} = 0.50$; △, $T_w/T_{\mathrm{ad}} = 0.25$.

$\Rightarrow$ Up to 17% difference for low values of β.

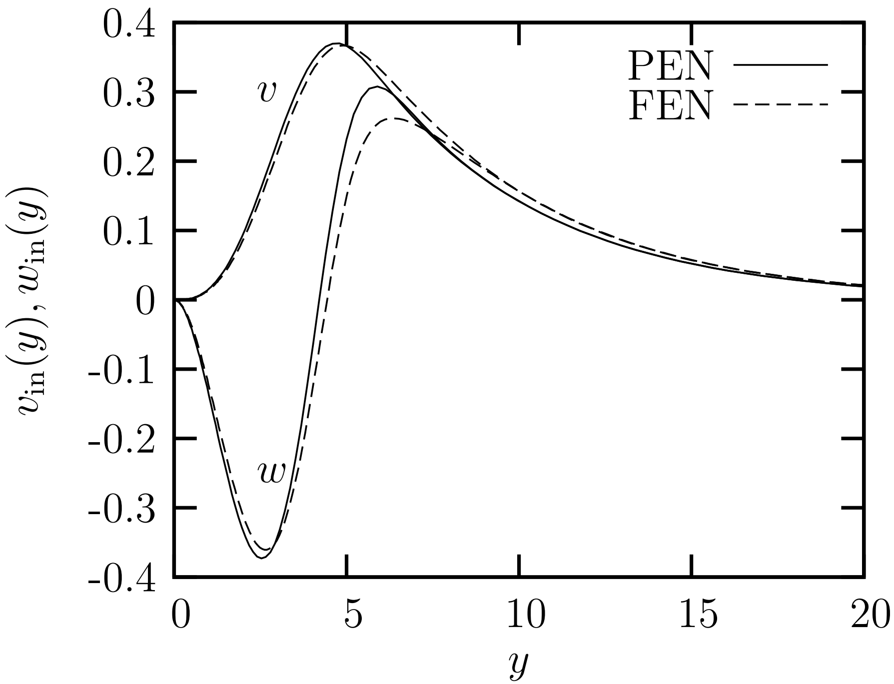
Results – Flat plate

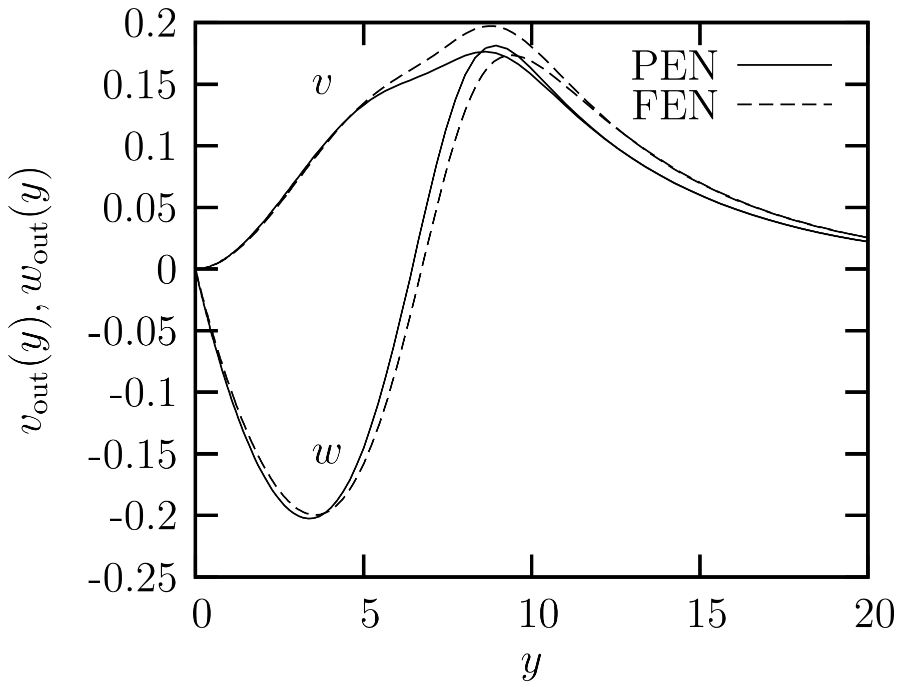


Objective function G/Re: effect of x_{in} and β and norm choice (PEN vs. FEN) for $M = 3$, $T_w/T_{\text{ad}} = 1$, $x_{\text{out}} = 1.0$. □, $x_{\text{in}} = 0.0$; ○, $x_{\text{in}} = 0.2$; △, $x_{\text{in}} = 0.4$.

⇒ Up to 60% difference for $x_{\text{in}} = 0.4$ and $\beta = 0.1$.

Results – Flat plate

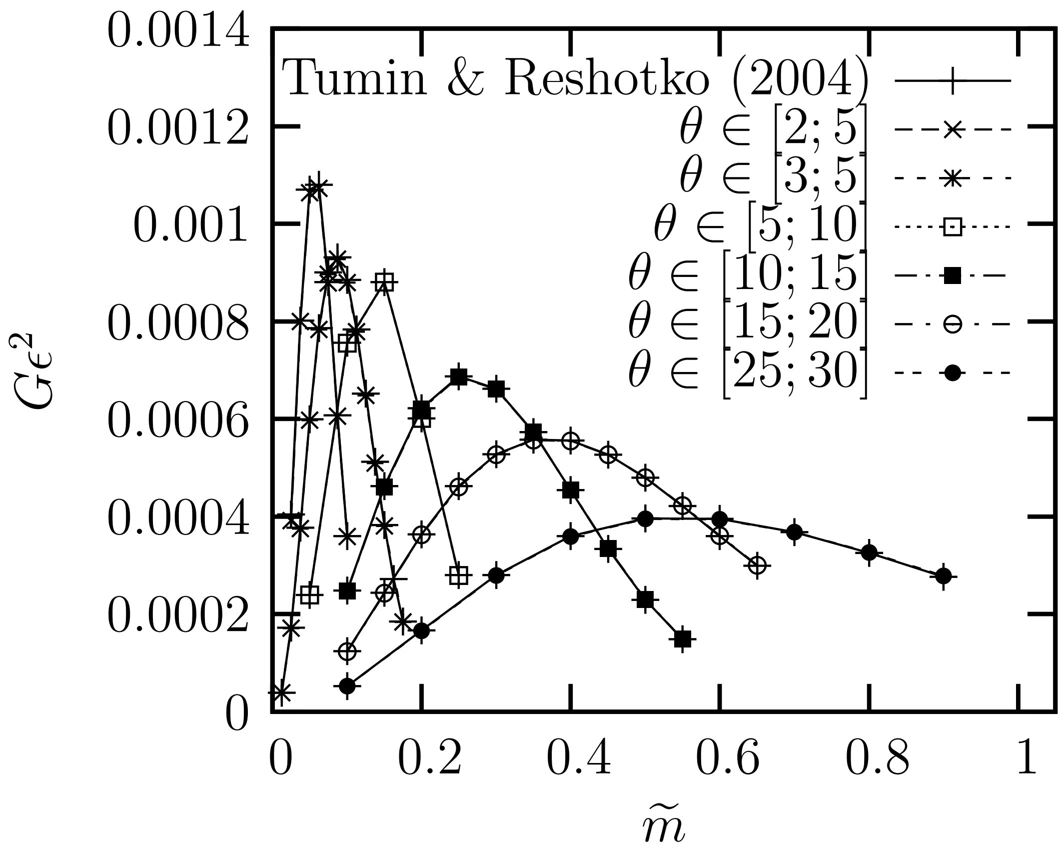




Inlet and outlet profiles: effect of norm choice (PEN vs. FEN) for $M = 3.0$, $Re = 10^3$, $x_{\rm in} = 0.4$, $x_{\rm out} = 1.0$ and $\beta = 0.1$.

$\Rightarrow$ No significant changes in $v_{\rm in}$, some discrepancies in $w_{\rm in}$; larger effects on $v_{\rm out}$, rather than on $w_{\rm out}$. No significant effects on $u_{\rm out}$ and $T_{\rm out}$.

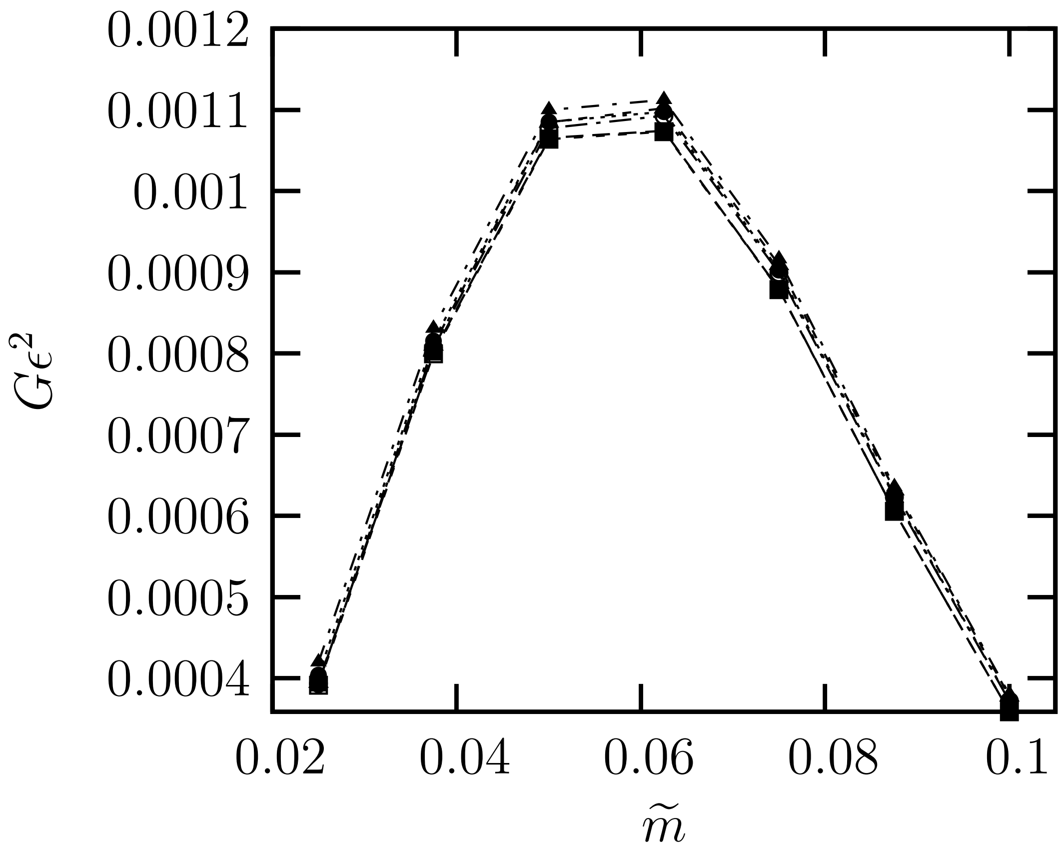
Results – Sphere



Objective function $G\epsilon^2$: effect of interval location and $\widetilde{m} = m\epsilon$ for $\theta_{\mathrm{ref}} = 30.0$ deg, $T_w/T_{\mathrm{ad}} = 0.5$, $\epsilon = 10^{-3}$. PEN.

$\Rightarrow$ Largest gain for small $\theta_{\mathrm{out}} - \theta_{\mathrm{in}}$; strongest transient growth close to the stagnation point.

Results – Sphere



Objective function $G\epsilon^2$: effect of ϵ, energy norm (PEN vs. FEN) and $\widetilde{m} = m\epsilon$ for $\theta_{\rm in} = 2.0$ deg, $\theta_{\rm out} = 5.0$ deg, $\theta_{\rm ref} = 30.0$ deg, $T_w/T_{\rm ad} = 0.5$. $\square$, $\epsilon = 1 \cdot 10^{-3}$; $\circ$, $\epsilon = 2 \cdot 10^{-3}$; $\triangle$, $\epsilon = 3 \cdot 10^{-3}$.

$\Rightarrow$ Maximum appreciable difference within 1%. Effect increases with ϵ.

Conclusions

- √ Efficient and robust numerical method for computing compressible optimal perturbations on flat plate and sphere.
- √ Adjoint-based optimization technique in the discrete framework and automatic in/out-let conditions.
- √ Analysis including full energy norm at the outlet.
- √ Flat plate. For $Re = 10^3$, significant difference in G/Re (up to 62%) between PEN and FEN. Effect of M and $x_{\rm in}$. No effect in subsonic basic flow. If $Re > 10^4$, $v_{\rm out}$ and $w_{\rm out}$ do not play significant role.
- √ Sphere. Largest $G\epsilon^2$ close to the stagnation point and for small range of θ. No significant role played by $v_{\rm out}$ and $w_{\rm out}$ in the interesting range of parameters.

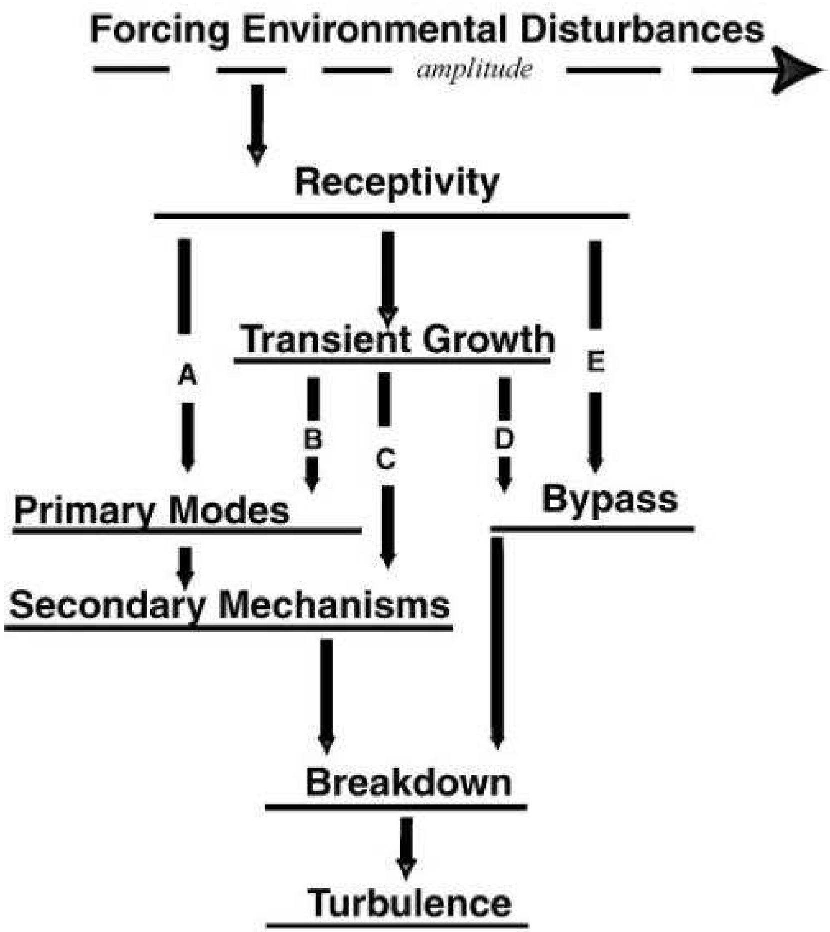
The End!

Are we missing something?

- At *non-infinitesimal* level of disturbance streaks are observed on a flat plate, instead of Tollmien–Schlichting waves.
- Linear Stability Theory (classical modal approach) fails even for the simplest geometries (Hagen-Poiseuille pipe flow, predicted stability vs. $Re_{\mathrm{crex}} \approx 2300$)!
- Certain transitional phenomena have no explanation yet, e.g. the "blunt body paradox" on spherical fore-bodies at super/hypersonic speeds.

There must exist another mechanism, not related to the eigenvalue analysis: transient growth.

Alternative paths of BL transition

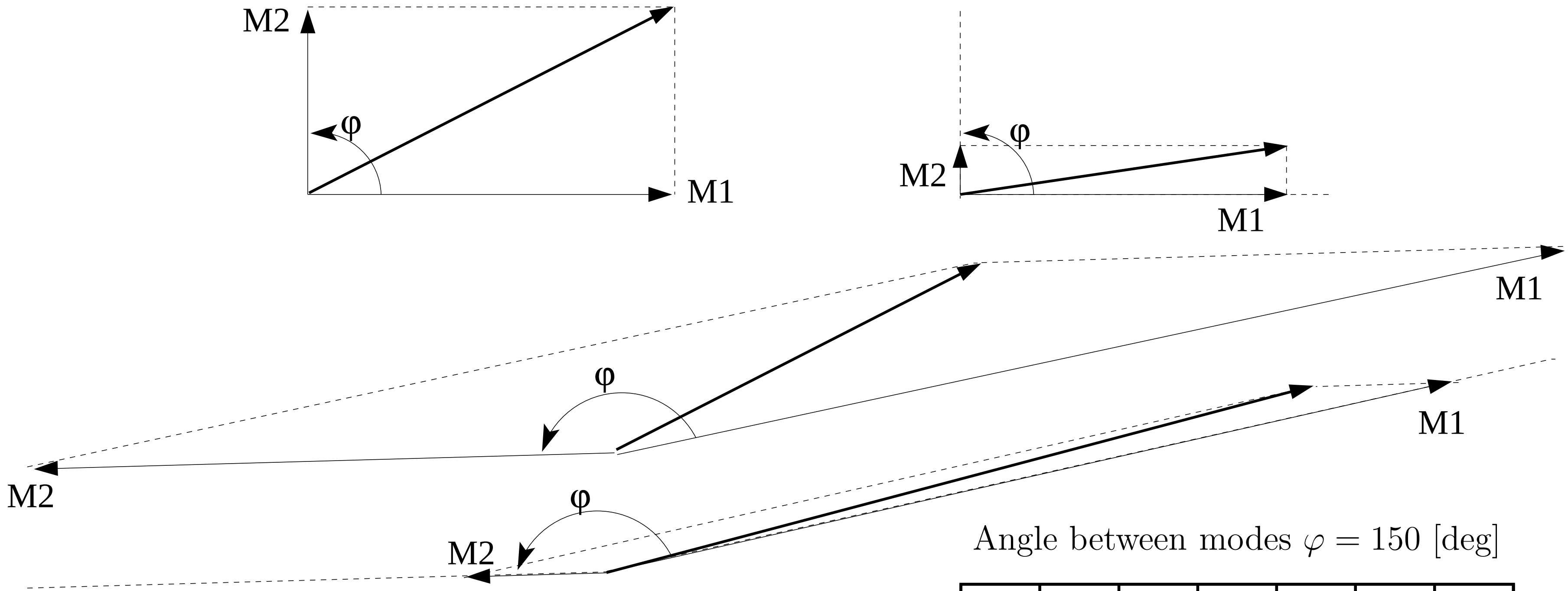


M. V. Morkovin, E. Reshotko, and T. Herbert, (1994),“Transition in open flow systems – A re-assessment”, *Bull. Am. Phys. Soc.* **39**, 1882.

“At the present time, no mathematical model exists that can predict the transition Reynolds number on a flat plate”!

Saric et al., *Annu. Rev. Fluid Mech.* 2002. **34**:291–319

Transient growth



Non-normality of the operator. For most flows the linear stability equations are not self-adjoint (the eigenfunctions are not orthogonal)