Modal Deduction Systems for Quantum State Transformations

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We introduce two modal natural deduction systems, MSQS and MSpQS, which are suitable to represent and reason about transformations of quantum states in an abstract, qualitative, way. Our systems provide a modal framework for reasoning about operations on quantum states (unitary transformations and measurements) in terms of possible worlds (as abstractions of quantum states) and accessibility relations between these worlds. We give a Kripke–style semantics that formally describes quantum state transformations, and prove the soundness and completeness of our systems with respect to this semantics. We also prove a normalization result for MSQS and MSpQS, showing that all derivations can be reduced to a normal form that satisfies a subformula property and yields a syntactic proof of the consistency of our deduction systems.

Key words: Quantum computing, modal logics, natural deduction, labelled deduction, proof theory

1 INTRODUCTION

Quantum computing defines a computational paradigm that is based on a quantum model [4] rather than a classical one. The basic units of the quantum

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model are the *quantum bits*, or *qubits* for short (mathematically, normalized vectors of the Hilbert Space $\mathbb{C}^2$). Qubits represent informational units and can assume both classical values 0 and 1, and all their superpositional values. A *quantum state* is a generalization of the qubit: a generic quantum state is the representation of a quantum state of n qubits (mathematically, it is a normalized vector of the Hilbert space $\mathbb{C}^{2^n}$).⋆ In this paper, we are not interested in the structure of quantum states, but rather in the way quantum states are transformed. Hence, we will abstract away from the internals of quantum states and represent them in a generic way in order to describe how operations transform a state into another one.

It is possible to modify a quantum state in two ways: by applying a unitary transformation or by measuring. Unitary transformations (corresponding to the so-called unitary operators of the Hilbert space, such as $cnot$ or $Bell$) model the internal evolution of a quantum system, whereas measurements correspond to the results of the interaction between a quantum system and an observer. The outcome of an observation can be either the reduction to some quantum state or the reduction to a classical state, where we say that a state w is *classical* iff w is invariant with respect to measurement, i.e., each measurement of w has w as outcome. We call a measurement *total* when the outcome of the measurement is a classical state.

We propose to model measurement and unitary transformations by means of suitable modal operators. More specifically, the main contribution of this paper is the formalization of a *modal natural deduction system* [16, 19] in order to represent (in an abstract, qualitative, way) the fundamental operations on quantum states: unitary transformations and total measurements. We call this system MSQS. We also formalize a variant of this system, called MSpQS, to represent the case of generic (not necessarily total) measurements.

It is important to observe that our logical systems are not quantum logics. Since the work of Birkhoff and von Neumann in 1936 [5], various logics have been investigated as a means to formalize reasoning about propositions taking into account the principles of quantum theory, e.g., [8, 9]. In general, it is possible to view quantum logic as a logical axiomatization of quantum theory, which provides an adequate foundation for a theory of reversible quantum processes, e.g., [1, 2, 3, 13].

Our work moves from quite a different point of view: we do not aim at

⋆ Note that some works, including our own work [12] that we extend and generalize here, use the term *quantum register* in place of *quantum state*. We have adopted the latter here as it is becoming the more standard term in the literature.

proposing a general logical formalization of quantum theory, rather we describe how it is possible to use modal logic to reason in a simple way about quantum state transformations. Informally, in our proposal, a modal world represents (an abstraction of) a quantum state. The discrete temporal evolution of a quantum state is controlled and determined by a sequence of unitary transformations and measurements that can change the description of a quantum state into other descriptions. So, the evolution of a quantum state can be viewed as a graph, where the nodes are the (abstract) quantum states and the arrows represent quantum transformations. The arrows give us the so-called accessibility relations of Kripke models and two nodes linked by an arrow represent two related quantum states: the target node is obtained from the source node by means of the operation specified in the decoration of the arrow.

Modal logic, as a logic of possible worlds, is thus a natural way to represent this description of a quantum system: the worlds model the quantum states and the relations of accessibility between worlds model the dynamical behavior of the system, as a consequence of the application of measurements and unitary transformations. To emphasize this semantic view of modal logic, we give our deduction system in the style of *labelled deduction* [10, 18, 21], a framework for giving uniform presentations of different non-classical logics. The intuition behind labelled deduction is that the labelling (sometimes also called prefixing, annotating or subscripting) allows one to explicitly encode in the syntax additional information, of a semantic or proof-theoretical nature, that is otherwise implicit in the logic one wants to capture. Most notably, in the case of modal logic, this additional information comes from the underlying Kripke semantics: the labelled formula $x{:}A$ intuitively means that A holds at the world denoted by the label x within the underlying Kripke structure (i.e., model), and labels also allow one to specify at the syntactic level how the different worlds are related in the Kripke structures (e.g., the formula $x\mathsf{R}y$ specifies that the world denoted by y is accessible from that denoted by x).

We proceed as follows. In Section 2, we discuss the main ideas underlying our approach, in particular why modal logic provides a good instrument to describe qualitatively quantum processes and quantum system transformations. In Section 3, we define the labelled modal natural deduction system MSQS, which contains two modal operators suitable to represent and reason about unitary transformations and total measurements of quantum states. In Section 4, we give a possible worlds semantics that formally describes these quantum state transformations, and prove the soundness and completeness of

MSQS with respect to this semantics. In Section 5, we formalize MSpQS, a variant of MSQS that provides a modal system representing all the possible (thus not necessarily total) measurements. In Section 6, we prove a normalization result for MSQS and MSpQS, showing that derivations reduce to a normal form that satisfies a subformula property and from which we can prove the consistency of our deduction systems. We conclude in Section 7 with a brief summary and a discussion of future work.

2 WHY MODAL LOGIC?

2.1 A qualitative modal representation of quantum state transformations

In this section, we discuss in more detail the "philosophy" of our approach. As we remarked above, the logic that we give in this paper is not a quantum logic that formalizes reasoning about propositions taking into account the principles of quantum theory. Such a logic would require a semantics over sets of unit vectors in Hilbert Space, which would bring in the foreground the *quantitative* nature of the approach to reason about quantum state transformations. While such an approach would, of course, be interesting and useful, we follow here a quite radically different one that works at a higher abstraction level: we adopt a *qualitative* approach whose major contributions rely on the observations that quantum state transformations can be represented, in an abstract way, by means of modal operators and that we can give deduction systems that capture the properties of these transformations.

Modal logics are indeed a good and flexible instrument to describe qualitatively state transformations as they allow one to put the emphasis on the underlying "transition system" (the set of possible worlds of the Kripke semantics and the properties of the accessibility relations between them) rather than on the concrete meaning of the internal structures of possible worlds. Thanks to this abstract and adaptable nature, modal (and temporal) logics have often been employed to reason in a higher-level way on the properties of computational systems (e.g., [6, 11, 15]), and here we follow this path by taking into account computational systems that are simply quantum systems in which computational steps are given by the application of unitary operators and measurements. The systems MSQS and MSpQS that we introduce here are *pure* modal systems in which accessibility relations and modal operators reflect and model general properties of quantum state transformations; consequently, the proposed Kripke semantics abstracts away from the quantitative approaches that take into account Hilbert spaces and other traditional semantical instruments of quantum logics.[†]

The transition from a quantum state into another one by means of a unitary transformation or a measurement can then simply be viewed as the transformation of a current valid description of the state into another valid description. This is not a peculiar point of view as, for comparison, in quantum mechanics and in quantum computing we always work with mathematical objects (normalized vectors in a suitable Hilbert space) that are effectively descriptions of concrete physical systems, and the evolution of a quantum state can thus be viewed as a set of descriptions that hold in a discrete set of sequential instants. We can thus consider sequences of the form $s_0, s_1, \ldots$ where each s_i provides a full description of the quantum state in a specific instant of the transformations. Now, in the spirit of modal logic, the key point is not the description itself but rather how it can be reached from another description and what is the set of the descriptions reachable from it. Then, working with labeled expressions like $x{:}A$, where the label x denotes one of the s_i and the formula A is built by modal operators and propositional symbols, it is not actually crucial to say what propositional symbols stand for. In analogy, observe, for instance, that temporal logics (such as LTL [15]) developed to deal with concurrent systems do not possess any concurrent feature. Still, it is important to consider what modal formulas, modal worlds and the accessibility relation stand for.

Hence, to illustrate, and justify, our approach in more detail, let us consider the standard algebraic axiomatization of quantum mechanics and the Hilbert space formalism. For the sake of exemplification but without loss of generality, let us focus on the Hilbert space $\mathbb{C}^{2^2}$, the 4-dimensional space required to represent two qubit systems. Normalized vectors in $\mathbb{C}^{2^2}$ are mathematical representations of quantum states. Then, we can interpret $\mathbb{C}^{2^2}$ as a Kripke model

$$\langle W, R_1, \ldots, R_n, V \rangle \,,$$

where W is a set of possible worlds, $R_1, \ldots, R_n$ are accessibility relations, and V is an interpretation function, defined as follows. For the set of modal worlds, we can take the set of all normalized vectors in the space $\mathbb{C}^{2^2}$ (i.e., the vectors representing quantum states):

$$W = \{|\phi\rangle \mid ||\phi|| = 1 \text{ and } |\phi\rangle \in \mathbb{C}^{2^2}\}$$

[†] In other words, we can, intuitively, say that MSQS and MSpQS are (quite) standard modal systems with a "quantum flavor": the systems are able to capture modally (i.e., in terms of modal operators) the properties of quantum state transformations but abstracting away from any concrete quantitative information.

For example, $|00\rangle$ and $1/\sqrt{2}|01\rangle + 1/\sqrt{2}|11\rangle$ are (representations of) elements of the set W.

In order to give concrete examples, we fix a set of propositional symbols representing mathematical descriptions of normalized vectors in $\mathbb{C}^{2^2}$. Let us suppose to have a denumerable set $\mathcal{Q} \in \mathbb{C}^{2^2}$ of quantum states and a denumerable set $\mathcal{U}$ of names of unitary transforms in $\mathbb{C}^{2^2}$, where $\mathcal{U}$ contains, for each unitary transformation T, both the names T and T^{-1}, the name of the inverse. The set $Prop$ of propositional symbols is given by:

1. the set of strings of the shape $\lceil a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle \rceil$, where $a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$ is the representation of an element of $\mathcal{Q}$ with respect to the standard computational basis, and

2. the set of string of the shape $\lceil T_1(T_2(\cdots(T_n(a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle))\cdots))\rceil$, where $T_1, \ldots, T_n$ are in $\mathcal{U}$ and $a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$ is the representation of an element of $\mathcal{Q}$.

That is, we denote a generic propositional symbol by writing $\lceil|\phi\rangle\rceil$, where $|\phi\rangle$ is the representation of an element of $\mathcal{Q}$. For example, $\lceil|00\rangle\rceil$, $\lceil Bell|01\rangle\rceil$, $\lceil Bell^{-1}|11\rangle\rceil$ and $\lceil cnot|01\rangle\rceil$ are propositional symbols representing normalized vectors in $\mathbb{C}^{2^2}$, where $cnot$ and $Bell$ are names of standard unitary operators.

The interpretation function V is a map $W \to 2^{Prop}$ associating to each $w \in W$ all the propositional symbols describing equivalent quantum states. For example, $\lceil Bell|00\rangle\rceil$ and $\lceil 1/\sqrt{2}|00\rangle + 1/\sqrt{2}|11\rangle\rceil$ represent the same quantum state. Moreover, we can write valid equivalences between propositional symbols. So, for instance,

$$\lceil Bell|00\rangle\rceil \leftrightarrow \lceil 1/\sqrt{2}|00\rangle + 1/\sqrt{2}|11\rangle\rceil \tag{1}$$

is a valid equivalence that describes the effect of the application of a Bell circuit on a quantum state $\lceil|00\rangle\rceil$; hence, as for the previous example, the two propositional symbols are associated to the same world w.

For the accessibility relations, we can first of all consider a relation U to represent quantum state transformations by unitary operators. In addition to U, we can consider two different relations depending on the type of measurement that we wish to model:

1. If we focus only on total measurement, which completely reduces any quantum world to a classical one, then we can consider the set $\{U, M\}$ of accessibility relations, where U represents unitary quantum state transformations and M represents total measurement.

2. If we focus on generic, not necessarily total, measurements, then we can consider the set $\{U, P\}$ of accessibility relations, where P captures the transformation of a state into another one by means of a generic measurement.

An expression like (1) can be read in two different ways: we can read it simply as an equality between logical objects, but we can also stress the "operational information", i.e., the computational effect of the application of the $Bell$ operator to the unitary vector $|00\rangle$. It is this second reading that shows that modal logic provides a suitable and useful means for describing quantum computation, as we can read (1) modally as follows: from a world represented by $\lceil|00\rangle\rceil$ we can reach, by means of a unitary operator, a world represented by $\lceil 1/\sqrt{2}|00\rangle + 1/\sqrt{2}|11\rangle\rceil$ or, equivalently, by $\lceil Bell|00\rangle\rceil$. Let us now describe this modal representation in more detail by considering the cases of total and generic measurements, as well as unitary transformations.

2.2 Unitary transformations and total measurements

In the case of unitary transformations and total measurements, the underlying Kripke model is $\mathscr{M} = \langle W, U, M, V\rangle$; as we will see later, this is the semantics of the system MSQS. Let us consider again the example about the Bell circuit and let us discuss an admissible modal formula. The operation, i.e., the mathematical application of the circuit to the quantum state $|00\rangle$, can be expressed as a formula holding at some modal world w:

$$\models^{\mathscr{M},w} \lceil|00\rangle\rceil \supset \Diamond\lceil 1/\sqrt{2}|00\rangle + 1/\sqrt{2}|11\rangle\rceil$$

which, by the equivalence (1) of the two mathematical descriptions, is equivalent to

$$\models^{\mathscr{M},w} \lceil|00\rangle\rceil \supset \Diamond\lceil Bell|00\rangle\rceil$$

Here, we write $\models^{\mathscr{M},w}$ to denote truth at a world w in a model $\mathscr{M}$, and $\Diamond$ is the existential modal operator associated to the unitary transformation accessibility relation U, as formally defined in Section 3. Hence, if the formula $\lceil|00\rangle\rceil$ also holds at w, i.e.,

$$\models^{\mathscr{M},w} \lceil|00\rangle\rceil$$

then

$$\models^{\mathscr{M},w} \Diamond\lceil 1/\sqrt{2}|00\rangle + 1/\sqrt{2}|11\rangle\rceil$$

which tells us that there exists a world in $\mathscr{M}$, say z, accessible from w by a unitary transformation representing a Bell circuit application (namely, wUz holds in $\mathscr{M}$), and in which we have $\lceil 1/\sqrt{2}|00\rangle + 1/\sqrt{2}|11\rangle\rceil$, i.e.,

$$\models^{\mathscr{M},z} \lceil 1/\sqrt{2}|00\rangle + 1/\sqrt{2}|11\rangle\rceil$$

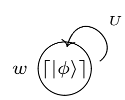
The Kripke semantics that we adopt for our modal systems allows us to abstract away from the Hilbert space formalism while preserving the capability of representing the properties of unitary transformations. Although we have not made use of them in this simple reasoning example, unitary transformations possess in fact a number of properties — namely, reflexivity, symmetry and transitivity — that we can capture both semantically in the Kripke models by imposing properties of the relation U, and syntactically by showing that the standard modal axioms T, 4 and B of the modal logic $S5$ are provable theorems of the modal system MSQS (and of MSpQS, which also includes unitary transformations). For example, we can easily express modally the unitariety of the identity operator Id, which acts as identity on quantum state descriptions, and have equivalences like $\lceil|\phi\rangle\rceil \leftrightarrow \lceil Id|\phi\rangle\rceil$ hold, so that we have the valid statement

$$\models^{\mathscr{M},w} \lceil|\phi\rangle\rceil \supset \Diamond\lceil Id|\phi\rangle\rceil$$

and, equivalently,

$$\models^{\mathscr{M},w} \lceil|\phi\rangle\rceil \supset \Diamond\lceil|\phi\rangle\rceil$$

This can be viewed, semantically, as the *reflexivity* of the accessibility relation U: each world $w \in W$ is accessible to itself by means of a unitary transformation, i.e., we have wUw. Graphically, we can represent this as



and our Kripke models will impose such a reflexivity. We can also formalize this syntactically by means of the standard modal axiom T, i.e., $A \supset \Diamond A$, or, dually, $\Box A \supset A$.

We can capture the other properties of U in a similar way: we can employ the standard modal axiom 4, $\Box A \supset \Box\Box A$, to capture the compositionality of unitary transformations (by means of *transitivity*), and the standard modal axiom B, $A \supset \Box\Diamond A$, to capture the reversible nature of unitary transformations (by means of *symmetry*). These three axioms T, 4 and B, in their labeled version, are provable theorems of MSQS (and of MSpQS) as explained in Section 3, where we will provide additional quantitative examples in order to further support the intuitive meaning of these formulas.

We can reason in a similar way for what concerns total measurement and make use of an existential modal operator $\blacklozenge$, with dual universal operator $\blacksquare$, associated to a total measurement accessibility relation M. Suppose, for example, that we have $\lceil a|00\rangle + b|10\rangle\rceil \in V(w)$ for a $w \in W$. Then

$$\models^{\mathscr{M},w} \lceil a|00\rangle + b|10\rangle\rceil \supset \blacklozenge\lceil|00\rangle\rceil$$

means that there exists a total measurement that reduces the state to the classical value $|00\rangle$, i.e., that there exist a modal world z in $\mathscr{M}$ such that wMz and $\lceil|00\rangle\rceil \in V(z)$.

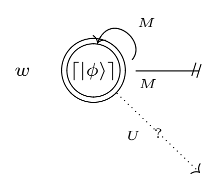
Similarly, the modal judgement

$$\models^{\mathscr{M},w} \lceil|00\rangle\rceil \supset \blacksquare\lceil|00\rangle\rceil$$

tells us that it is impossible to escape from a "classical world" (a world representing a state without superposition) by means of a total measurement. If we perform a total measurement, i.e., from the current world we access another one by means of the relation M, the reached world is terminal with respect to total measurement and it is possible to access a different world only by means of a unitary transformation. More generally, we have

$$\models^{\mathscr{M},w} \lceil|\phi\rangle\rceil \supset \blacksquare\lceil|\phi\rangle\rceil$$

and we can express graphically this "invariance" of a classical world with respect to total measurement by forcing the models to the particular shape



that depicts that with M we can go nowhere but to the same world (as represented by the interrupted outgoing arrow labeled by M and the double "termination" circle), which we can only leave by a possible unitary transformation U (whose possibility is represented by the dashed arrow and the question mark).

Total measurement imposes three properties on the corresponding accessibility relation M. To begin with, as for U, we can capture the compositionality of total measurement by means of the modal axiom 4, in this case $\blacksquare A \supset \blacksquare\blacksquare A$, and the *transitivity* of M. Second, given any quantum state it is always possible to perform a total measurement, which we can capture by means of the standard modal axiom D, i.e., $\blacksquare A \supset \blacklozenge A$, which formalizes the *seriality* of M (that from any world there exists at least one transition by means of total measurement). Finally, we have the standard modal axiom $\blacksquare(A \leftrightarrow \blacksquare A)$, which expresses that from each modal world w there always exists a transition to another world z interpreted in a classical state vector (if w is classical itself, then z is imposed to be equal to w), i.e., that it is always possible to completely reduce a quantum state to a classical one. This corresponds to the *shift-reflexivity* of the total measurement accessibility relation.

Also these three axioms, in their labeled version, are provable theorems of MSQS as explained in Section 3, where we will provide additional numerical examples.

2.3 Generic measurements

In the case of unitary transformations and generic measurements, the underlying Kripke model is $\mathscr{M} = \langle W, U, P, V \rangle$. Everything that we explained for U in the previous case still holds, while the generic measurement relation P behaves differently than the total measurement relation M. This is because, as we mentioned above, a generic measurement does not necessarily induce the reduction to a classical state, but rather it is possible to obtain another state with superposition. In fact, after a total measurement we always obtain a classical world and only classical worlds are invariant with respect to the relation M, while it is not possible to say the same in the generic case.

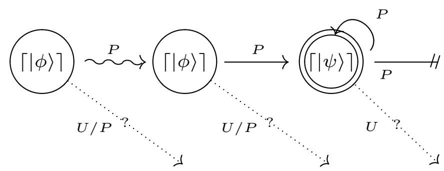
Let us associate the accessibility relation P to the dual modal operators $\boxdot$ and $\diamonddot$ that relate quantum (interpreted into superpositional vectors) worlds. This allows us, for example, to write

$$\vDash^{\mathscr{M},w} \lceil a|00\rangle + b|01\rangle \rceil \supset \diamonddot \lceil a|00\rangle + b|01\rangle \rceil$$

to express modally that there exists a generic measurement (actually, the measurement of the first qubit) that leaves unchanged the quantum state $|\psi\rangle = a|00\rangle + b|01\rangle$.

In the case of total measurements, we adopted models in which it is not possible to escape from a classical world by means of the accessibility relation M. This permitted us to adopt reflexive (i.e., self-referred, by means of measurement) worlds as classical states. In the case of generic, partial, measurements this is not a good choice: in fact, as we saw, a state with quantum superposition can be invariant with respect to a particular generic measurement. We thus opt for the following representation: if a state is invariant with respect to a specific partial measurement, we do not use the reflexive relation but we make the transition to a “duplicate” of the world, i.e., to a world that has a different name but is otherwise equivalent (i.e., it is interpreted by the function V into the same set of propositional symbols). Then, we assume that uPu if and only if u is a world representing a classical state.

The following example represents this graphically, where we we duplicate the state $|\phi\rangle$ into a copy-state (as displayed by the squiggly line), and only if the measurement is total (on $|\psi\rangle$) then we have a reflexive world:



Other properties of generic measurements are captured modally along the lines of what we saw before for U and M. For instance, the standard modal axiom $\boxdot A \supset \boxdot \boxdot A$ expresses the composability of generic measurements, and the standard modal axiom $\diamonddot(A \supset \boxdot A)$ expresses that also in the case of generic measurement it is always possible to perform a measurement with complete reduction to a classical state (i.e., total measurement is a particular case of the generic one). The labeled versions of these axioms are provable theorems of MSpQS and in Section 5 we give a numerical example.

2.4 Taking stock

We close this discussion with some further considerations. As classical, pure, modal systems, MSQS and MSpQS are able to represent and reason about transformations of quantum states in an abstract way, and thus provide the qualitative representation that we are aiming at here, but they possess no quantum features. This is obviously payed in terms of concreteness and expressiveness (in the sense that we do not represent the quantitative internals of quantum computations), but, in exchange, our systems enjoy important meta-theoretical properties such as normalization and the subformula property, which make them attractive from the proof-theoretical point of view, differently from some quantum logics.

Moreover, stressing the qualitative nature of our investigation, Kripke semantics is a good semantics for the description of quantum state transformations. Hence, modal logic is not incompatible with the Hilbert space formalism but it simply abstracts away from concrete descriptions and underlines other informations. Different semantics, such as the "classical" semantical approach of quantum logic closely related to the Hilbert space formalism, are interesting but completely different. They are oriented towards the *axiomatization* of quantum mechanics, and this requires a totally different approach in which modal logic, probably, would not be of much help. We leave a more detailed investigation of this for future work and instead now focus on our modal systems MSQS and MSpQS.

3 THE DEDUCTION SYSTEM MSQS

3.1 The language of MSQS

Our labelled modal natural deduction system MSQS, which formally represents unitary transformations and total measurements of quantum states, comprises of rules that derive formulas of two kinds: modal formulas and relational formulas. We thus define a modal language and a relational language.

The alphabet of the *relational language* consists of:

- the binary symbols U and M,
- a denumerable set $x_0, x_1, \ldots$ of *labels*.

Metavariables x, y, z, possibly annotated with subscripts and superscripts, range over the set of labels. For brevity, we will sometimes speak of a "world" x meaning that the label x stands for a world $\mathscr{I}(x)$, where $\mathscr{I}$ is an interpretation function mapping labels into worlds as formalized in Definition 2 below.

The set of *relational formulas* (*r-formulas* for short) is given by expressions of the form $x\mathsf{U}y$ and $x\mathsf{M}y$. We write $x\mathsf{R}y$ to denote a generic r-formula, with $\mathsf{R} \in \{\mathsf{U}, \mathsf{M}\}$.

The alphabet of the *modal language* consists of:

- a denumerable set $r, r_0, r_1, \ldots$ of *propositional symbols* (for instance, as discussed in Section 2),
- the standard *propositional connectives* $\perp$ and $\supset$,
- the unary *modal operators* $\Box$ and $\blacksquare$.

The set of *modal formulas* (*m-formulas* for short) is the least set that contains $\perp$ and the propositional symbols, and is closed under $\supset$ and the modal operators. Metavariables A, B, C, possibly indexed, range over modal formulas. Other connectives can be defined in the usual manner, e.g., $\neg A \equiv A \supset \perp$, $A \wedge B \equiv \neg(A \supset \neg B)$, $A \leftrightarrow B \equiv (A \supset B) \wedge (B \supset A)$, $\Diamond A \equiv \neg\Box\neg A$, $\blacklozenge A \equiv \neg\blacksquare\neg A$, etc.

Let us give, in a rather informal way, the intuitive meaning of the modal operators of our language:

- $\Box A$ means: A is true in each quantum state obtained by a unitary transformation.

- $\blacksquare A$ means: A is true in each quantum state obtained by a total measurement.

A *labelled formula* (*l-formula* for short) is an expression $x{:}A$, where x is a label and A is an m-formula. A *formula* is either an r-formula or an l-formula. The metavariable α, possibly indexed, ranges over formulas. We write $\alpha(x)$ to denote that the label x occurs in the formula α, so that $\alpha(y/x)$ denotes the substitution of the label y for all occurrences of x in α. Furthermore, we say that an l-formula $x{:}A$ is *atomic* when A is atomic, which is the case when A is a propositional symbol or $\bot$. Finally, we define the *grade* of an l-formula $x{:}A$, in symbols $grade(x{:}A)$, to be the number of times $\supset$, $\Box$ and $\blacksquare$ occur in A, so that $grade(x{:}A) = 0$ for an atomic A.

3.2 The rules of MSQS

Figure 1 shows the rules of MSQS, where the notion of *discharged/open assumption* is standard [16, 19], e.g., the formula $[x{:}A]$ is discharged in the rule $\supset I$:

Propositional rules: The rules $\supset I$, $\supset E$ and RAA are just the labelled version of the standard (e.g., [16, 19]) natural deduction rules for implication introduction and elimination and for *reductio ad absurdum*, where we enforce Prawitz's side condition that $A \neq \bot$. The "mixed" rule $\bot E$ allows us to derive a generic formula α whenever we have obtained a contradiction $\bot$ at a world x; in this case, we do not enforce the side condition that $A \neq \bot$ but allow the rule to derive $y{:}\bot$ for some y from $x{:}\bot$.[‡]

Modal rules: We give the rules for a generic modal operator $\bigstar$, with a corresponding generic relation R, since all the modal operators share the structure of these basic introduction/elimination rules; this holds because, for instance, we express $x{:}\Box A$ as the metalevel implication $x\mathsf{U}y \Longrightarrow y{:}A$ for an arbitrary (*fresh*) y accessible from x. In particular:

- if $\bigstar$ is $\Box$ then R is U,

[‡] See [21] for a detailed discussion of the rules for $\bot$, which in particular explains how, in order to maintain the duality of modal operators like $\Box$ and $\Diamond$, it must be possible to propagate a $\bot$ at a world x to any other different world y.

$$\dfrac{\begin{array}{c}[x{:}A]\\ \vdots \\ x{:}B\end{array}}{x{:}A \supset B} \supset I \qquad \dfrac{x{:}A \supset B \quad x{:}A}{x{:}B} \supset E \qquad \dfrac{\begin{array}{c}[x{:}\neg A]\\ \vdots \\ y{:}\bot\end{array}}{x{:}A} \; RAA$$

$$\dfrac{x{:}\bot}{\alpha} \; \bot E \qquad \dfrac{\begin{array}{c}[x\mathsf{R}y]\\ \vdots \\ y{:}A\end{array}}{x{:}\bigstar A} \; \bigstar I \qquad \dfrac{x{:}\bigstar A \quad x\mathsf{R}y}{y{:}A} \; \bigstar E$$

$$\dfrac{}{x\mathsf{U}x} \; \mathsf{U}\mathit{refl} \qquad \dfrac{x\mathsf{U}y}{y\mathsf{U}x} \; \mathsf{U}\mathit{symm} \qquad \dfrac{x\mathsf{U}y \quad y\mathsf{U}z}{x\mathsf{U}z} \; \mathsf{U}\mathit{trans} \qquad \dfrac{x\mathsf{M}y}{x\mathsf{U}y} \; \mathsf{U}I$$

$$\dfrac{\begin{array}{c}[x\mathsf{M}y]\\ \vdots \\ \alpha\end{array}}{\alpha} \; \mathsf{M}\mathit{ser} \qquad \dfrac{x\mathsf{M}y}{y\mathsf{M}y} \; \mathsf{M}\mathit{srefl}$$

$$\dfrac{\alpha(x) \quad x\mathsf{M}x \quad x\mathsf{M}y}{\alpha(y/x)} \; \mathsf{M}\mathit{sub1} \qquad \dfrac{\alpha(y) \quad x\mathsf{M}x \quad x\mathsf{M}y}{\alpha(x/y)} \; \mathsf{M}\mathit{sub2}$$

In RAA, $A \neq \bot$.
In $\bigstar I$, y is fresh: it is different from x and does not occur in any assumption on which $y{:}A$ depends other than $x\mathsf{R}y$.
In $\mathsf{M}\mathit{ser}$, y is fresh: it is different from x and does not occur in α nor in any assumption on which α depends other than $x\mathsf{M}y$.

FIGURE 1
The rules of MSQS

- if $\bigstar$ is $\blacksquare$ then R is M.

Other rules:

- In order to axiomatize $\Box$, we add rules $\mathsf{U}\mathit{refl}$, $\mathsf{U}\mathit{symm}$, and $\mathsf{U}\mathit{trans}$, formalizing that U is an equivalence relation.

- In order to axiomatize $\blacksquare$, we add rules formalizing the following properties:
  - If $x\mathsf{M}y$ then there is a specific unitary transformation (depending on x and y) that generates y from x: rule $\mathsf{U}I$.
  - The total measurement process is serial: rule $\mathsf{M}ser$ says that if from the assumption $x\mathsf{M}y$ we can derive α for a *fresh* y (i.e., y is different from x and does not occur in α nor in any assumption on which α depends other than $x\mathsf{M}y$), then we can discharge the assumption (since there always is some y such that $x\mathsf{M}y$) and conclude α.
  - The total measurement process is shift-reflexive: rule $\mathsf{M}srefl$.
  - Invariance with respect to classical worlds: rules $\mathsf{M}sub1$ and $\mathsf{M}sub2$ say that if $x\mathsf{M}x$ and $x\mathsf{M}y$, then y must be equal to x and so we can substitute the one for the other in any formula α.

We refer to the fresh y in $\bigstar I$ and $\mathsf{M}ser$ as the *parameter* of the rule.

3.3 Derivations and proofs

Definition 1 (Derivations and proofs)**.** *A* derivation *of a formula α from a set of formulas Γ in MSQS (an MSQS-derivation, for short, or just "derivation" when MSQS is clear from context or is not needed) is a tree formed using the rules in MSQS, ending with α and depending only on a finite subset of Γ. We write $\Gamma \vdash \alpha$ to denote that there exists an MSQS-derivation of α from Γ, and denote such a derivation Π graphically as*

$$\begin{array}{c}\Gamma\\ \Pi\\ \alpha\end{array}$$

A derivation in MSQS of α depending on the empty set is called a proof *of α and we then write $\vdash \alpha$ as an abbreviation of $\emptyset \vdash \alpha$ and say that α is a theorem of MSQS.*

For instance, the following labelled formula schemata, corresponding to standard modal axioms, are all provable in MSQS (where, in parentheses, we give the intuitive meaning of each formula in terms of quantum state transformations):

1. $x{:}\Box A \supset A$
   (the identity transformation is unitary).

2. $x{:}A \supset \Box\Diamond A$
   (each unitary transformation is invertible).

3. $x{:}\Box A \supset \Box\Box A$
   (unitary transformations are composable).

4. $x{:}\blacksquare A \supset \blacklozenge A$
   (it is always possible to perform a total measurement of a quantum state).

5. $x{:}\blacksquare(A \leftrightarrow \blacksquare A)$
   (it is always possible to perform a total measurement with a complete reduction of a quantum state to a classical one).

6. $x{:}\blacksquare A \supset \blacksquare\blacksquare A$
   (total measurements are composable).

Some concrete, numerical, examples explaining the intuitive meaning of these theorems are given in Section 4.2. Before doing that, we give some examples of derivations and formalize a Kripke semantics.

As examples of derivations, Figure 2 contains the proofs of the formulas 5 and 6, where, for simplicity, here and in the following (cf. Figure 5), we employ the rules for equivalence ($\leftrightarrow I$) and for negation ($\neg I$ and $\neg E$), which are derived from the propositional rules as is standard. For instance,

$$\dfrac{\begin{array}{c}[x{:}A]^1\\ \Pi\\ y{:}\bot\end{array}}{x{:}\neg A}\ \neg I^1 \qquad \text{abbreviates} \qquad \dfrac{\dfrac{\begin{array}{c}[x{:}A]^1\\ \Pi\\ y{:}\bot\end{array}}{x{:}\bot}\ \bot E}{x{:}A \supset \bot}\ \supset I^1$$

where, here and in the figure, we decorate, as is standard [16, 19], discharged formulas and the rule applications discharging them with the same numeric superscript.

We can similarly derive rules about r-formulas. For instance, we can derive a rule for the transitivity of M as shown at the top of the proof of the formula 6 in Figure 2:

$$\dfrac{x\mathsf{M}y \quad y\mathsf{M}z}{x\mathsf{M}z}\ \mathsf{M}\mathit{trans}$$

abbreviates

$$\dfrac{x\mathsf{M}y \quad \dfrac{x\mathsf{M}y}{y\mathsf{M}y}\ \mathsf{M}\mathit{srefl} \qquad y\mathsf{M}z}{x\mathsf{M}z}\ \mathsf{M}\mathit{sub1}$$

$$
\begin{prooftree}
\AxiomC{$[y{:}A]^2$}
\AxiomC{$[x\mathsf{M}y]^1$}
\RightLabel{$\mathsf{M}\mathit{srefl}$}
\UnaryInfC{$y\mathsf{M}y$}
\AxiomC{$[y\mathsf{M}z]^3$}
\RightLabel{$\mathsf{M}\mathit{sub1}$}
\TrinaryInfC{$z{:}A$}
\RightLabel{$\blacksquare I^3$}
\UnaryInfC{$y{:}\blacksquare A$}
\RightLabel{$\supset I^2$}
\UnaryInfC{$y{:}A \supset \blacksquare A$}
\AxiomC{$[y{:}\blacksquare A]^4$}
\AxiomC{$[x\mathsf{M}y]^1$}
\RightLabel{$\mathsf{M}\mathit{srefl}$}
\UnaryInfC{$y\mathsf{M}y$}
\RightLabel{$\blacksquare E$}
\BinaryInfC{$y{:}A$}
\RightLabel{$\supset I^4$}
\UnaryInfC{$y{:}\blacksquare A \supset A$}
\RightLabel{$\leftrightarrow I$}
\BinaryInfC{$y{:}A \leftrightarrow \blacksquare A$}
\RightLabel{$\blacksquare I^1$}
\UnaryInfC{$x{:}\blacksquare(A \leftrightarrow \blacksquare A)$}
\end{prooftree}
$$

$$
\begin{prooftree}
\AxiomC{$[x{:}\blacksquare A]^1$}
\AxiomC{$[x\mathsf{M}y]^2$}
\AxiomC{$[x\mathsf{M}y]^2$}
\RightLabel{$\mathsf{M}\mathit{srefl}$}
\UnaryInfC{$y\mathsf{M}y$}
\AxiomC{$[y\mathsf{M}z]^3$}
\RightLabel{$\mathsf{M}\mathit{sub1}$}
\TrinaryInfC{$x\mathsf{M}z$}
\RightLabel{$\blacksquare E$}
\BinaryInfC{$z{:}A$}
\RightLabel{$\blacksquare I^3$}
\UnaryInfC{$y{:}\blacksquare A$}
\RightLabel{$\blacksquare I^2$}
\UnaryInfC{$x{:}\blacksquare\blacksquare A$}
\RightLabel{$\supset I^1$}
\UnaryInfC{$x{:}\blacksquare A \supset \blacksquare\blacksquare A$}
\end{prooftree}
$$

FIGURE 2
Examples of proofs in MSQS

4 A SEMANTICS FOR UNITARY TRANSFORMATIONS AND TOTAL MEASUREMENTS

We give a semantics that formally describes unitary transformations and total measurements of quantum states in terms of accessibility relations between worlds, and then prove that MSQS is sound and complete with respect to this semantics. Together with the corresponding result for generic measurements in MSpQS described in Section 5, this means that our modal systems indeed provide a representation of quantum states and operations on them, which was one of the main goals of the paper.

4.1 Frames, models, structures and truth

In order to define truth of l-formulas, we first define frames (consisting of a set of possible worlds and accessibility relations), models (a frame plus an interpretation function mapping worlds into sets of formulas), and structures (a model plus an interpretation function mapping labels into possible worlds, so to be able to deal with l-formulas $x{:}A$ and not just m-formulas A).

Definition 2 (Frames, models, structures)**.** *A* frame *is a tuple* $\mathscr{F} = \langle W, U, M \rangle$*, where:*

- W *is a non-empty set of* worlds
  (representing abstractly the quantum states);
- $U \subseteq W \times W$ *is an equivalence relation*
  (vUw means that w is obtained by applying a unitary transformation to v; U is an equivalence relation since identity is a unitary transformation, each unitary transformation must be invertible, and unitary transformations are composable);
- $M \subseteq W \times W$
  (vMw means that w is obtained by means of a total measurement of v);

with the following properties:

(i) $\forall v, w.\ vMw \Longrightarrow vUw$

(ii) $\forall v.\ \exists w.\ vMw$

(iii) $\forall v, w.\ vMw \Longrightarrow wMw$

(iv) $\forall v, w.\ vMv\ \&\ vMw \Longrightarrow v = w$

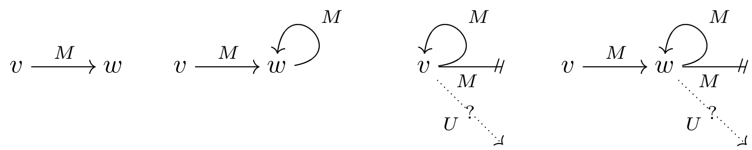
(i) *means that although it is not true that measurement is a unitary transformation, locally for each v, if vMw then there is a particular unitary transformation, depending on v and w, that generates w from v; the vice versa cannot hold, since in quantum theory measurements cannot be used to obtain the unitary evolution of a quantum system.* (ii) *means that each quantum state is totally measurable.* (iii) *and* (iv) *together mean that after a total measurement we obtain a classical world. Figure 3 shows properties* (ii)*,* (iii) *and* (iv)*, respectively, as well as the combination of* (iii) *and* (iv)*.*¶

A model *is a pair* $\mathscr{M} = \langle \mathscr{F}, V \rangle$*, where* $\mathscr{F}$ *is a frame and* $V : W \to 2^{Prop}$ *is an interpretation function mapping worlds into sets of formulas.*

A structure *is a pair* $\mathscr{S} = \langle \mathscr{M}, \mathscr{I} \rangle$*, where* $\mathscr{M}$ *is a model and* $\mathscr{I} : Var \to W$ *is an interpretation function mapping variables (labels) into worlds in* W*.*

We write R to denote a generic frame relation, i.e., $R \in \{U, M\}$, and, slightly abusing notation, we write $\mathscr{I}(\mathsf{R})$ to denote the corresponding R.

¶ Note that while (iv) says that v is invariant with respect to M, a unitary transformation U could still be applied to v (and hence the dotted arrow decorated with a "?" for U). Note also that here we use a more abstract graphical representation than the figures of Section 2 as here we are talking about the relations between worlds, and are abstracting away from the propositional symbols.



(ii) (iii) (iv) (iii) and (iv)

FIGURE 3
Some properties of the relation M

Given this semantics, we can define what it means for formulas to be true, and then prove the soundness and completeness of MSQS.

Definition 3 (Truth)**.** Truth *for an m-formula at a world* w *in a model* $\mathscr{M} = \langle W, U, M, V\rangle$ *is the smallest relation* $\vDash$ *satisfying:*

$$
\begin{array}{lll}
\vDash^{\mathscr{M},w} r & \textit{iff} & r \in V(w) \\
\vDash^{\mathscr{M},w} A \supset B & \textit{iff} & \vDash^{\mathscr{M},w} A \Longrightarrow \vDash^{\mathscr{M},w} B \\
\vDash^{\mathscr{M},w} \Box A & \textit{iff} & \forall w'.\ wUw' \Longrightarrow \vDash^{\mathscr{M},w'} A \\
\vDash^{\mathscr{M},w} \blacksquare A & \textit{iff} & \forall w'.\ wMw' \Longrightarrow \vDash^{\mathscr{M},w'} A
\end{array}
$$

Hence, $\nvDash^{\mathscr{M},w} \bot$ *for any* $\mathscr{M}$ *and* w*. For an m-formula* A*, we write* $\vDash^{\mathscr{M}} A$ *iff* $\vDash^{\mathscr{M},w} A$ *for all* w*.*

Truth *for a formula* α *in a structure* $\mathscr{S} = \langle \mathscr{M}, \mathscr{I}\rangle$ *is then the smallest relation* $\vDash$ *satisfying:*

$$
\begin{array}{lll}
\vDash^{\mathscr{M},\mathscr{I}} x\mathsf{M}y & \textit{iff} & \mathscr{I}(x) M \mathscr{I}(y) \\
\vDash^{\mathscr{M},\mathscr{I}} x\mathsf{U}y & \textit{iff} & \mathscr{I}(x) U \mathscr{I}(y) \\
\vDash^{\mathscr{M},\mathscr{I}} x{:}A & \textit{iff} & \vDash^{\mathscr{M},\mathscr{I}(x)} A
\end{array}
$$

Hence, $\vDash^{\mathscr{M},\mathscr{I}} x\mathsf{R}y$ *iff* $\mathscr{I}(x)\mathscr{I}(\mathsf{R})\mathscr{I}(y)$ *iff* $\mathscr{I}(x) R \mathscr{I}(y)$*. Moreover,* $\nvDash^{\mathscr{M},\mathscr{I}} x{:}\bot$ *for any* x*,* $\mathscr{M}$ *and* $\mathscr{I}$*.*

By extension, $\vDash^{\mathscr{M},\mathscr{I}} \Gamma$ *iff* $\vDash^{\mathscr{M},\mathscr{I}} \alpha$ *for all* α *in the set of formulas* Γ*. Thus, for a set of formulas* Γ *and a formula* α*,*

$$
\begin{array}{lll}
\Gamma \vDash \alpha & \text{iff} & \forall \mathscr{S}.\ \Gamma \vDash^{\mathscr{S}} \alpha \\
 & \text{iff} & \forall \mathscr{M}, \mathscr{I}.\ \Gamma \vDash^{\mathscr{M},\mathscr{I}} \alpha \\
 & \text{iff} & \forall \mathscr{M}, \mathscr{I}.\ \vDash^{\mathscr{M},\mathscr{I}} \Gamma \Longrightarrow \vDash^{\mathscr{M},\mathscr{I}} \alpha
\end{array}
$$

We omit $\mathscr{M}$ when it is not relevant and, for example, write $\Gamma \models^{\mathscr{I}} \alpha$ when $\models^{\mathscr{I}} \Gamma$ implies $\models^{\mathscr{I}} \alpha$.

4.2 Some concrete examples

Now that we have defined a Kripke semantics, we can give some concrete, numerical, examples explaining the intuitive meaning of the theorems stated in Section 3.3.

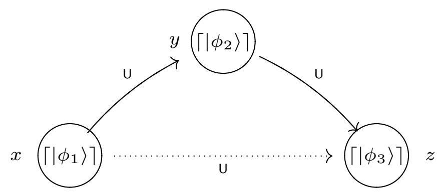
Let us thus consider a structure $\langle \mathscr{M}, \mathscr{I} \rangle$ where $\mathscr{M} = \langle W, U, M, V \rangle$ is the Kripke model defined in Section 2.2, and $\mathscr{I}$ is a generic interpretation function. With a little abuse of language for the sake of brevity, we identify, via the interpretation function $\mathscr{I}$, the labels with the corresponding concrete worlds of the semantics, and the syntactic relations U, M with their semantical counterparts U and M. Then we will write expressions like $\lceil |00\rangle \rceil \in V(x)$ in order to say that $\lceil |00\rangle \rceil \in V(\mathscr{I}(x))$. Moreover, under such a convention we will refer to labels as *worlds*.

Consider the formula 2, $x{:}A \supset \Box\Diamond A$, which states the symmetric property of the relation U and captures the reversible nature of unitary transformations. Roughly speaking, if we start from a normalized vector $|\phi\rangle$ and we apply a unitary operator, then we can always "come back" to the starting vector, as for a symmetric property. That is, A holds at x and for each world y accessible from x there is a world accessible from y such that A holds: that world is x itself. For example, suppose we have $x{:}\lceil |00\rangle \rceil \supset \Diamond \lceil T(|00\rangle) \rceil$ for a world x and a generic unitary transformation T; namely, given $x{:}\lceil |00\rangle \rceil$, there is a world y such that $x \mathsf{U} y$ and $y{:}\lceil T(|00\rangle) \rceil$. Since for each unitary transformation T there exists the inverse one T^{-1}, then there exists a world z such that $y \mathsf{U} z$ and $z{:}\lceil T^{-1}(T(|00\rangle)) \rceil$. But $\lceil T^{-1}(T(|00\rangle)) \rceil \leftrightarrow \lceil |00\rangle \rceil$, and thus z is x. Graphically, this amounts to the symmetry of unitary transformations U:

$$\begin{tikzcd} x \;\; \lceil |\phi\rangle \rceil \arrow[r, bend left, "\mathsf{U}"] & \lceil |\psi\rangle \rceil \;\; y \arrow[l, bend left, "\mathsf{U}"] \end{tikzcd}$$

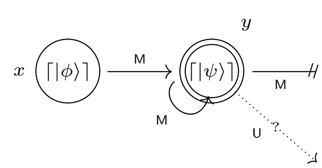
Formula 3, $x{:}\Box A \supset \Box\Box A$, models the transitivity of U, i.e., given two unitary transformations, there always exists the transformation defined by means of composition. As a concrete example, let us consider the following equivalences between propositional symbols: $\lceil cnot|11\rangle \rceil \leftrightarrow \lceil |10\rangle \rceil$ and $\lceil Bell|10\rangle \rceil \leftrightarrow \lceil 1/\sqrt{2}|00\rangle - 1/\sqrt{2}|11\rangle \rceil$. In terms of accessibility relation, given three modal worlds x, y and z such that $\lceil |11\rangle \rceil \in V(x)$, $\lceil |10\rangle \rceil \in V(y)$ and $\lceil 1/\sqrt{2}|00\rangle - 1/\sqrt{2}|11\rangle \rceil \in V(z)$, then we have $x \mathsf{U} y$ and $y \mathsf{U} z$ by means of transformations $Bell$ and $cnot$, respectively. The transformations can be expressed by the modal operator $\Diamond$ as $x{:}\lceil |11\rangle \rceil \supset \Diamond \lceil cnot|11\rangle \rceil$ and

$y{:}\lceil cnot|11\rangle\rceil \supset \Diamond\lceil Bell(cnot|11\rangle)\rceil$, where y is the world we access by the transformation, and $x{:}\lceil|11\rangle\rceil \supset \Diamond\lceil Bell(cnot|11\rangle)\rceil$. Summing up, graphically, this amounts to the transitivity of unitary transformations U:



Now, let us consider the formula 5, $x{:}\blacksquare(A \leftrightarrow \blacksquare A)$. If, for some world x, we have $x{:}\blacksquare(\lceil|01\rangle\rceil \leftrightarrow \blacksquare\lceil|01\rangle\rceil)$, then, for each world y reachable by means of total measurement (i.e., such that $x\mathsf{M}y$), we have that $y{:}\lceil|01\rangle\rceil \leftrightarrow \blacksquare\lceil|01\rangle\rceil$. In fact, it is immediate to check that after a total measurement we always obtain a "stable" state with respect to further measurements. The double implication here expresses the impossibility to escape from the description $\lceil|01\rangle\rceil$ by means of relation M, or, in other words, that we are in a classical state.

The following diagram graphically sums up the main characteristics of M:



(2)

4.3 Soundness and completeness

By adapting standard proofs to the case of labelled deduction (see, e.g., [10, 16, 18, 19, 21]), we can show:

Theorem 1 (Soundness and completeness of MSQS)**.** $\Gamma \vdash \alpha$ *iff* $\Gamma \vDash \alpha$. □

Theorem 1 follows from Theorems 2 and 3 below.

Theorem 2 (Soundness of MSQS)**.** $\Gamma \vdash \alpha$ *implies* $\Gamma \vDash \alpha$.

Proof. We let $\mathscr{M}$ be an arbitrary model and prove that if $\Gamma \vdash \alpha$ then $\Gamma \vDash^{\mathscr{I}} \alpha$ for any $\mathscr{I}$, i.e., $\vDash^{\mathscr{I}} \Gamma$ implies $\vDash^{\mathscr{I}} \alpha$ for any $\mathscr{I}$. The proof proceeds by induction on the structure of the derivation of α from Γ. The base case, where $\alpha \in \Gamma$, is trivial. There is one step case for each rule of MSQS.

Consider an application of the rule RAA,

$$\dfrac{\begin{array}{c}[x{:}\neg A]\\ \vdots \\ y{:}\bot\end{array}}{x{:}A}\ RAA$$

where $\Gamma' \models y{:}\bot$ with $\Gamma' = \Gamma \cup \{x{:}\neg A\}$. By the induction hypothesis, $\Gamma' \vdash y{:}\bot$ implies $\Gamma' \models^{\mathscr{I}} y{:}\bot$ for any $\mathscr{I}$. We assume $\models^{\mathscr{I}} \Gamma$ and prove $\models^{\mathscr{I}} x{:}A$. Since $\not\models^{w} \bot$ for any world w, from the induction hypothesis we obtain $\not\models^{\mathscr{I}} \Gamma'$, and thus $\not\models^{\mathscr{I}} x{:}\neg A$, i.e., $\models^{\mathscr{I}} x{:}A$ and $\not\models^{\mathscr{I}} x{:}\bot$.

Consider an application of the rule $\bot E$,

$$\frac{x{:}\bot}{\alpha}\ \bot E$$

with $\Gamma \vdash x{:}\bot$. By the induction hypothesis, $\Gamma \vdash x{:}\bot$ implies $\Gamma \models^{\mathscr{I}} x{:}\bot$ for any $\mathscr{I}$. We assume $\models^{\mathscr{I}} \Gamma$ and prove $\models^{\mathscr{I}} \alpha$ for an arbitrary formula α. If $\models^{\mathscr{I}} \Gamma$ then $\models^{\mathscr{I}} x{:}\bot$ by the induction hypothesis, i.e., $\models^{\mathscr{I}(x)} \bot$. But since $\not\models^{w} \bot$ for any world w, then $\not\models^{\mathscr{I}} \Gamma$ and thus $\models^{\mathscr{I}} \alpha$ for any α.

Consider an application of the rule $\bigstar I$,

$$\dfrac{\begin{array}{c}[x\mathsf{R}y]\\ \vdots \\ y{:}A\end{array}}{x{:}\bigstar A}\ \bigstar I$$

where $\Gamma' \vdash y{:}A$ with y fresh and with $\Gamma' = \Gamma \cup \{x\mathsf{R}y\}$. By the induction hypothesis, for all interpretations $\mathscr{I}$, if $\models^{\mathscr{I}} \Gamma$ then $\models^{\mathscr{I}} y{:}A$. We let $\mathscr{I}$ be any interpretation such that $\models^{\mathscr{I}} \Gamma$, and show that $\models^{\mathscr{I}} x{:}\bigstar A$. Let w be any world such that $\mathscr{I}(x)\mathscr{I}(\mathsf{R})w$. Since $\mathscr{I}$ can be trivially extended to another interpretation (still called $\mathscr{I}$ for simplicity) by setting $\mathscr{I}(y) = w$, the induction hypothesis yields $\models^{\mathscr{I}} y{:}A$, i.e., $\models^{w} A$, and thus $\models^{\mathscr{I}(x)} \bigstar A$, i.e., $\models^{\mathscr{I}} x{:}\bigstar A$.

Consider an application of the rule $\bigstar E$,

$$\frac{x{:}\bigstar A \quad x\mathsf{R}y}{y{:}A}\ \bigstar E$$

with $\Gamma_1 \vdash x{:}\bigstar A$ and $\Gamma_2 \vdash x\mathsf{R}y$, and $\Gamma \supseteq \Gamma_1 \cup \Gamma_2$. We assume $\models^{\mathscr{I}} \Gamma$ and prove $\models^{\mathscr{I}} y{:}A$. By the induction hypothesis, for all interpretations $\mathscr{I}$, if $\models^{\mathscr{I}} \Gamma_1$ then $\models^{\mathscr{I}} x{:}\bigstar A$ and if $\models^{\mathscr{I}} \Gamma_2$ then $\mathscr{I}(x)\mathscr{I}(\mathsf{R})\mathscr{I}(y)$. If $\models^{\mathscr{I}} \Gamma$, then $\models^{\mathscr{I}} x{:}\bigstar A$ and $\mathscr{I}(x)\mathscr{I}(\mathsf{R})\mathscr{I}(y)$, and thus $\models^{\mathscr{I}} y{:}A$.

The rules U*refl*, $\mathsf{U}symm$, and $\mathsf{U}trans$ are sound by the properties of U.

The rule $\mathsf{U}I$ is sound by property (i) in Definition 2.

Consider an application of the rule $\mathsf{M}ser$,

$$\dfrac{\begin{array}{c}[x\mathsf{M}y]\\ \vdots\\ \alpha\end{array}}{\alpha}\ \mathsf{M}ser$$

with $\Gamma' = \Gamma \cup \{x\mathsf{M}y\}$, for y fresh. By the induction hypothesis, $\Gamma' \vdash \alpha$ implies $\Gamma' \models^{\mathscr{I}} \alpha$ for any $\mathscr{I}$. Let us suppose that there is an $\mathscr{I}'$ such that $\models^{\mathscr{I}'} \Gamma'$ and $\not\models^{\mathscr{I}'} \alpha$. Let us consider an $\mathscr{I}''$ such that $\mathscr{I}''(z) = \mathscr{I}'(z)$ for all z such that $z \neq y$ and $\mathscr{I}''(y)$ is the world w such that $\mathscr{I}''(y)Mw$, which exists by property (ii) in Definition 2. Since y does not occur in Γ nor in α, we then have that $\models^{\mathscr{I}''} \Gamma'$ and $\not\models^{\mathscr{I}''} \alpha$, contradicting the universality of the consequence of the induction hypothesis. Hence, $\mathsf{M}ser$ is sound.

The rule $\mathsf{M}\mathit{srefl}$ is sound by property (iii) in Definition 2.

Consider an application of the rule $\mathsf{M}\mathit{sub1}$,

$$\frac{\alpha(x) \quad x\mathsf{M}x \quad x\mathsf{M}y}{\alpha(y/x)}\ \mathsf{M}\mathit{sub1}$$

with $\Gamma_1 \vdash \alpha(x)$, $\Gamma_2 \vdash x\mathsf{M}x$, $\Gamma_3 \vdash x\mathsf{M}y$, and $\Gamma \supseteq \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$. We assume $\models^{\mathscr{I}} \Gamma$ and prove $\models^{\mathscr{I}} \alpha(y/x)$. By the induction hypothesis, $\Gamma_1 \vdash \alpha(x)$ implies $\Gamma_1 \models^{\mathscr{I}} \alpha(x)$, $\Gamma_2 \vdash x\mathsf{M}x$ implies if $\models^{\mathscr{I}} \Gamma_2$ then $\mathscr{I}(x)M\mathscr{I}(x)$, and $\Gamma_3 \vdash x\mathsf{M}y$ implies if $\models^{\mathscr{I}} \Gamma_3$ then $\mathscr{I}(x)M\mathscr{I}(y)$. By property *(iv)* in Definition 2, we then have $\mathscr{I}(x) = \mathscr{I}(y)$ and thus $\models^{\mathscr{I}} \alpha(y/x){:}A$. The case for rule $\mathsf{M}\mathit{sub2}$ follows analogously. □

To prove completeness (Theorem 3), we give some preliminary definitions and results. For simplicity, we split each set of formulas Γ into a pair (LF, RF) of the subsets of l-formulas and r-formulas of Γ, and then prove $(LF, RF) \models \alpha$ implies $(LF, RF) \vdash \alpha$. We call (LF, RF) a *context* and, slightly abusing notation, we write $\alpha \in (LF, RF)$ whenever $\alpha \in LF$ or $\alpha \in RF$, and write $x \in (LF, RF)$ whenever the label x occurs in some $\alpha \in (LF, RF)$. We say that a context (LF, RF) is *consistent* iff $(LF, RF) \nvdash x{:}\bot$ for every x, so that we have:

Fact 1. *If (LF, RF) is consistent, then for every x and every A, either $(LF \cup \{x{:}A\}, RF)$ is consistent or $(LF \cup \{x{:}\neg A\}, RF)$ is consistent.*

Let $\overline{(LF, RF)}$ be the *deductive closure* of (LF, RF) for r-formulas under the rules of MSQS, i.e.,

$$\overline{(LF, RF)} \equiv \{x\mathsf{R}y \mid (LF, RF) \vdash x\mathsf{R}y\}$$

for $\mathsf{R} \in \{\mathsf{U}, \mathsf{M}\}$. We say that a context (LF, RF) is *maximally consistent* iff

1. it is consistent,
2. it is deductively closed for r-formulas, i.e., $(LF, RF) = \overline{(LF, RF)}$, and
3. for every x and every A, either $x{:}A \in (LF, RF)$ or $x{:}\neg A \in (LF, RF)$.

Completeness follows by a Henkin–style proof, where a canonical structure

$$\mathscr{S}^c = \langle \mathscr{M}^c, \mathscr{I}^c \rangle = \langle W^c, U^c, M^c, V^c, \mathscr{I}^c \rangle$$

is built to show that $(LF, RF) \nvdash \alpha$ implies $(LF, RF) \not\models^{\mathscr{S}^c} \alpha$, i.e., $\models^{\mathscr{S}^c} (LF, RF)$ and $\not\models^{\mathscr{S}^c} \alpha$.

In standard proofs for unlabelled modal logics (e.g., [7]) and for other non-classical logics, the set W^c is obtained by progressively building maximally consistent sets of formulas, where consistency is locally checked within each set. In our case, given the presence of l-formulas and r-formulas, we modify the Lindenbaum lemma to extend (LF, RF) to one single maximally consistent context (LF^*, RF^*), where consistency is "globally" checked also against the additional assumptions in RF.[§] The elements of W^c are then built by partitioning LF^* and RF^* with respect to the labels, and the relations R between the worlds are defined by exploiting the information in RF^*.

In the Lindenbaum lemma for predicate logic, a maximally consistent and ω-complete set of formulas is inductively built by adding for every formula $\neg\forall x.A$ a *witness* to its truth, namely a formula $\neg A[c/x]$ for some new individual constant c. This ensures that the resulting set is ω-complete, i.e., that if, for every closed term t, $A[t/x]$ is contained in the set, then so is $\forall x.A$. A similar procedure applies here in the case of l-formulas of the form $x{:}\neg\bigstar A$. That is, together with $x{:}\neg\bigstar A$ we consistently add $y{:}\neg A$ and $x\mathsf{R}y$ for some new y, which acts as a *witness world* to the truth of $x{:}\neg\bigstar A$. This ensures that the maximally consistent context (LF^*, RF^*) is such that if $x\mathsf{R}z \in (LF^*, RF^*)$ implies $z{:}B \in (LF^*, RF^*)$ for every z, then $x{:}\bigstar B \in (LF^*, RF^*)$, as shown in Lemma 2 below. Note that in the standard completeness proof for unlabelled modal logics, one instead considers a canonical model $\mathscr{M}^c$ and

[§] We consider only consistent contexts. If (LF, RF) is inconsistent, then $LF, RF \vdash x{:}A$ for all $x{:}A$, and thus completeness immediately holds for l-formulas. Our language does not allow us to define inconsistency for a set of r-formulas, but, whenever (LF, RF) is inconsistent, the canonical model built in the following is nonetheless a counter-model to non-derivable r-formulas.

shows that if $w \in W^c$ and $\vDash^{\mathscr{M}^c,w} \neg\bigstar A$, then W^c also contains a world w' accessible from w that serves as a witness world to the truth of $\neg\bigstar A$ at w, i.e., $\vDash^{\mathscr{M}^c,w'} \neg A$.

Lemma 1. *Every consistent context* (LF, RF) *can be extended to a maximally consistent context* (LF^*, RF^*).

Proof. We first extend the language of MSQS with infinitely many new constants for witness worlds. Systematically let b range over labels, c range over the new constants for witness worlds, and a range over both. All these may be subscripted. Let $l_1, l_2, \ldots$ be an enumeration of all l-formulas in the extended language; when l_i is $a{:}A$, we write $\neg l_i$ for $a{:}\neg A$. Starting from $(LF_0, RF_0) = (LF, RF)$, we inductively build a sequence of consistent contexts by defining (LF_{i+1}, RF_{i+1}) to be:

- (LF_i, RF_i), if $(LF_i \cup \{l_{i+1}\}, RF_i)$ is inconsistent; else
- $(LF_i \cup \{l_{i+1}\}, RF_i)$, if l_{i+1} is not $a{:}\neg\bigstar A$; else
- $(LF_i \cup \{a{:}\neg\bigstar A, c{:}\neg A\}, RF_i \cup \{a\mathsf{R}c\})$ for a $c \notin (LF_i \cup \{a{:}\neg\bigstar A\}, RF_i)$, if l_{i+1} is $a{:}\neg\bigstar A$.

Every (LF_i, RF_i) is consistent. To show this, we show that if $(LF_i \cup \{a{:}\neg\bigstar A\}, RF_i)$ is consistent, then so is $(LF_i \cup \{a{:}\neg\bigstar A, c{:}\neg A\}, RF_i \cup \{a\mathsf{R}c\})$ for a $c \notin (LF_i \cup \{a{:}\neg\bigstar A\}, RF_i)$; the other cases follow by construction. We proceed by contraposition. Suppose that

$$(LF_i \cup \{a{:}\neg\bigstar A, c{:}\neg A\}, RF_i \cup \{a\mathsf{R}c\}) \vdash a_j{:}\bot$$

where $c \notin (LF_i \cup \{a{:}\neg\bigstar A\}, RF_i)$. Then, by RAA,

$$(LF_i \cup \{a{:}\neg\bigstar A\}, RF_i \cup \{a\mathsf{R}c\}) \vdash c{:}A\,,$$

and $\bigstar I$ yields

$$(LF_i \cup \{a{:}\neg\bigstar A\}, RF_i) \vdash a{:}\bigstar A\,.^{\|}$$

‖ Note that if $A = \bot$, then we cannot apply RAA. But in that case, if

$$(LF_i \cup \{a{:}\neg\bigstar\bot, c{:}\neg\bot\}, RF_i \cup \{a\mathsf{R}c\}) \vdash a_j{:}\bot$$

then also

$$(LF_i \cup \{a{:}\neg\bigstar\bot\}, RF_i \cup \{a\mathsf{R}c\}) \vdash a_j{:}\bot\,,$$

which can only be the case if either LF_i contains for some B both $a{:}\bigstar\neg B$ and $a{:}\bigstar B$, which give rise to a $\bot$ at c via $a\mathsf{R}c$, or LF_i contains $a{:}\bigstar A$, i.e., $a{:}\bigstar\bot$. In both such cases, it must be that $(LF_i \cup \{a{:}\neg\bigstar A\}, RF_i)$ is inconsistent, which contradicts the assumption.

Since also

$$(LF_i \cup \{a{:}\neg\bigstar A\}, RF_i) \vdash a{:}\neg\bigstar A\,,$$

by $\neg E$ we have

$$(LF_i \cup \{a{:}\neg\bigstar A\}, RF_i) \vdash a{:}\bot\,,$$

i.e., $(LF_i \cup \{a{:}\neg\bigstar A\}, RF_i)$ is inconsistent. Contradiction.

Now define

$$(LF^*, RF^*) = \overline{(\bigcup_{i\geq 0} LF_i, \bigcup_{i\geq 0} RF_i)}$$

We show that (LF^*, RF^*) is maximally consistent, by showing that it satisfies the three conditions in the definition of maximal consistency. For the first condition, note that if

$$(\bigcup_{i\geq 0} LF_i, \bigcup_{i\geq 0} RF_i)$$

is consistent, then so is

$$\overline{(\bigcup_{i\geq 0} LF_i, \bigcup_{i\geq 0} RF_i)}\,.$$

Now suppose that (LF^*, RF^*) is inconsistent. Then for some finite $(LF'$, $RF')$ included in (LF^*, RF^*) there exists an a such that $(LF', RF') \vdash a{:}\bot$. Every l-formula $l \in (LF', RF')$ is in some (LF_j, RF_j). For each $l \in (LF', RF')$, let i_l be the least j such that $l \in (LF_j, RF_j)$, and let $i = max\{i_l \mid l \in (LF', RF')\}$. Then $(LF', RF') \subseteq (LF_i, RF_i)$, and (LF_i, RF_i) is inconsistent, which is not the case.

The second condition is satisfied by definition of (LF^*, RF^*).

For the third condition, suppose that $l_{i+1} \notin (LF^*, RF^*)$. Then $l_{i+1} \notin (LF_{i+1}, RF_{i+1})$ and $(LF_i \cup \{l_{i+1}\}, RF_i)$ is inconsistent. Thus, by Fact 1, $(LF_i \cup \{\neg l_{i+1}\}, RF_i)$ is consistent, and $\neg l_{i+1}$ is consistently added to some (LF_j, RF_j) during the construction, and therefore $\neg l_{i+1} \in (LF^*, RF^*)$. $\square$

The following lemma states some properties of maximally consistent contexts.

Lemma 2. *Let (LF^*, RF^*) be a maximally consistent context. Then*

1. $(LF^*, RF^*) \vdash a_i \mathsf{R} a_j$ *iff* $a_i \mathsf{R} a_j \in (LF^*, RF^*)$.
2. $(LF^*, RF^*) \vdash a{:}A$ *iff* $a{:}A \in (LF^*, RF^*)$.
3. $a{:}B \supset C \in (LF^*, RF^*)$ *iff* $a{:}B \in (LF^*, RF^*)$ *implies* $a{:}C \in (LF^*, RF^*)$.

4. $a_i{:}\bigstar B \in (LF^*, RF^*)$ *iff* $a_i \mathsf{R} a_j \in (LF^*, RF^*)$ *implies* $a_j{:}B \in (LF^*, RF^*)$ *for all* a_j*.*

Proof. *1* and *2* follow immediately by definition. We only treat *4* as *3* follows analogously. For the left-to-right direction, suppose that $a_i{:}\bigstar B \in (LF^*, RF^*)$. Then, by (ii), $(LF^*, RF^*) \vdash a_i{:}\bigstar B$, and, by $\bigstar E$, we have $(LF^*, RF^*) \vdash a_i \mathsf{R} a_j$ implies $(LF^*, RF^*) \vdash a_j{:}B$ for all a_j. By *1* and *2*, conclude $a_i \mathsf{R} a_j \in (LF^*, RF^*)$ implies $a_j{:}B \in (LF^*, RF^*)$ for all a_j. For the converse, suppose that $a_i{:}\bigstar B \notin (LF^*, RF^*)$. Then $a_i{:}\neg\bigstar B \in (LF^*, RF^*)$, and, by the construction of (LF^*, RF^*), there exists an a_j such that $a_i \mathsf{R} a_j \in (LF^*, RF^*)$ and $a_j{:}B \notin (LF^*, RF^*)$. $\square$

We can now define the canonical structure

$$\mathscr{S}^c = \langle \mathscr{M}^c, \mathscr{I}^c \rangle = \langle W^c, U^c, M^c, V^c, \mathscr{I}^c \rangle .$$

Definition 4. *Given a maximally consistent context* (LF^*, RF^*)*, we define the* canonical structure $\mathscr{S}^c$ *as follows:*

- $W^c = \{a \mid a \in (LF^*, RF^*)\}$,
- $(a_i, a_j) \in U^c$ *iff* $a_i \mathsf{U} a_j \in (LF^*, RF^*)$,
- $(a_i, a_j) \in M^c$ *iff* $a_i \mathsf{M} a_j \in (LF^*, RF^*)$,
- $V^c(r) = a$ *iff* $a{:}r \in (LF^*, RF^*)$,
- $\mathscr{I}^c(a) = a$.

Note that the standard definition of R^c adopted for unlabelled modal logics, i.e.

$$(a_i, a_j) \in R^c \text{ iff } \{A \mid \Box A \in a_i\} \subseteq a_j ,$$

is not applicable in our setting, since $\{A \mid \Box A \in a_i\} \subseteq a_j$ does *not* imply $\vdash a_i \mathsf{R} a_j$. We would therefore be unable to prove completeness for r-formulas, since there would be cases, e.g., when $RF = \{\}$, where $\nvdash a_i \mathsf{R} a_j$ but $(a_i, a_j) \in R^c$ and thus $\models^{\mathscr{S}^c} a_i \mathsf{R} a_j$. Hence, we instead define $(a_i, a_j) \in R^c$ iff $a_i \mathsf{R} a_j \in (LF^*, RF^*)$; note that therefore $a_i \mathsf{R} a_j \in (LF^*, RF^*)$ implies $\{A \mid \Box A \in a_i\} \subseteq a_j$. As a further comparison with the standard definition, note that in the canonical model the label a can be identified with the set of formulas $\{A \mid a{:}A \in (LF^*, RF^*)\}$. Moreover, we immediately have:

Fact 2. $a_i \mathsf{R} a_j \in (LF^*, RF^*)$ *iff* $(LF^*, RF^*) \models^{\mathscr{S}^c} a_i \mathsf{R} a_j$.

The deductive closure of (LF^*, RF^*) for r-formulas ensures not only completeness for r-formulas, as shown in Theorem 3 below, but also that the conditions on R^c are satisfied, so that $\mathscr{S}^c$ is really a structure for MSQS. More concretely:

- U^c is an equivalence relation by construction and rules $\mathsf{U}\mathit{refl}$, $\mathsf{U}\mathit{symm}$, and $\mathsf{U}\mathit{trans}$. For instance, for transitivity, consider an arbitrary context (LF, RF) from which we build $\mathscr{S}^c$. Assume $(a_i, a_j) \in U^c$ and $(a_j, a_k) \in U^c$. Then $a_i \mathsf{U} a_j \in (LF^*, RF^*)$ and $a_j \mathsf{U} a_k \in (LF^*, RF^*)$. Since (LF^*, RF^*) is deductively closed, by *1* in Lemma 2 and rule $\mathsf{U}\mathit{trans}$, we have $a_i \mathsf{U} a_k \in (LF^*, RF^*)$. Thus, $(a_i, u_k) \in U^c$ and U^c is indeed transitive.

- $\forall v, w \in W^c.\ vMw \Longrightarrow vUw$ holds by construction and rule $\mathsf{U}I$.

- $\forall v \in W^c.\ \exists w \in W^c.\ vMw$ holds by construction and rule $\mathsf{M}\mathit{ser}$. For the sake of contradiction, consider an arbitrary a_i and a variable a'_j that do not satisfy the property. Define $(LF', RF') = (LF^*, RF^*) \cup \{a_i \mathsf{M} a'_j\}$. Then it cannot be the case that $(LF', RF') \vdash \alpha$, for otherwise $(LF^*, RF^*) \vdash \alpha$ would be derivable by an application of the rule $\mathsf{M}\mathit{ser}$. Thus, $(LF', RF') \nvdash \alpha$. But then (LF', RF') must be in the chain of contexts built in Lemma 2. So, by the maximality of (LF^*, RF^*), we have that $(LF', RF') = (LF^*, RF^*)$, contradicting our assumption. Hence, for some a_j, the r-formula $a_i \mathsf{M} a_j$ is in (LF^*, RF^*), which is what we had to show.

- $\forall v, w \in W^c.\ vMw \Longrightarrow wMw$ holds by construction and rule $\mathsf{M}\mathit{srefl}$.

- $\forall v, w \in W^c.\ vMv\ \&\ vMw \Longrightarrow v = w$ holds by construction and rules $\mathsf{M}\mathit{sub1}$ and $\mathsf{M}\mathit{sub2}$ since v is a classical world. Consider an arbitrary context (LF, RF) from which we build $\mathscr{S}^c$ and assume $(a_i, a_i) \in M^c$ and $(a_i, a_j) \in M^c$. Then $a_i \mathsf{M} a_i \in (LF^*, RF^*)$ and $a_i \mathsf{M} a_j \in (LF^*, RF^*)$. Thus, for each $a_i{:}A \in (LF^*, RF^*)$, we also have $a_j{:}A \in (LF^*, RF^*)$; otherwise, since (LF^*, RF^*) is deductively closed, we would have $a_j{:}\neg A \in (LF^*, RF^*)$ and also $a_j{:}A \in (LF^*, RF^*)$ by *1* in Lemma 2 and rule $\mathsf{M}\mathit{sub1}$, and thus a contradiction. Similarly, if $a_j{:}A \in (LF^*, RF^*)$ then $a_i{:}A \in (LF^*, RF^*)$ by rule $\mathsf{M}\mathit{sub2}$. Hence, for each m-formula A, we have that $a_i{:}A \in (LF^*, RF^*)$ iff $a_j{:}A \in (LF^*, RF^*)$, which means that a_i and a_j are equal with respect to m-formulas.

Under the same assumptions, we can similarly show that a_i and a_j are equal with respect to r-formulas, i.e., that whenever (LF^*, RF^*) contains an r-formula that includes a_i then it also contains the same r-formula with a_j substituted for a_i, and vice versa. To this end, we must consider eight different cases corresponding to eight different r-formulas.

1. If $a_k \mathsf{U} a_i \in (LF^*, RF^*)$ for some a_k, then from the assumption that $a_i \mathsf{M} a_j \in (LF^*, RF^*)$ we have $a_i \mathsf{U} a_j \in (LF^*, RF^*)$, by *1* in Lemma 2 and rule $\mathsf{U}I$. Therefore, $a_k \mathsf{U} a_j \in (LF^*, RF^*)$ by rule $\mathsf{U}trans$.
2. We can reason similarly for $a_j \mathsf{U} a_k \in (LF^*, RF^*)$ and also apply rules $\mathsf{U}I$ and $\mathsf{U}trans$ to conclude that then also $a_i \mathsf{U} a_k \in (LF^*, RF^*)$.
3. If $a_i \mathsf{U} a_k \in (LF^*, RF^*)$ for some a_k, then from the assumption that $a_i \mathsf{M} a_j \in (LF^*, RF^*)$ we have $a_i \mathsf{U} a_j \in (LF^*, RF^*)$, by *1* in Lemma 2 and rule $\mathsf{U}I$, and thus $a_j \mathsf{U} a_i \in (LF^*, RF^*)$, by rule $\mathsf{U}symm$. Therefore, $a_j \mathsf{U} a_k \in (LF^*, RF^*)$ by rule $\mathsf{U}trans$.
4. We can reason similarly for $a_k \mathsf{U} a_j \in (LF^*, RF^*)$ and also apply rules $\mathsf{U}I$, $\mathsf{U}symm$, and $\mathsf{U}trans$ to conclude that then also $a_k \mathsf{U} a_i \in (LF^*, RF^*)$.
5. If $a_k \mathsf{M} a_i \in (LF^*, RF^*)$ for some a_k, then from the assumption that $a_i \mathsf{M} a_j \in (LF^*, RF^*)$ we have $a_k \mathsf{M} a_j \in (LF^*, RF^*)$, by *1* in Lemma 2 and the derived rule $\mathsf{M}trans$.
6. We can reason similarly for $a_j \mathsf{M} a_k \in (LF^*, RF^*)$ and also apply rule $\mathsf{M}trans$ to conclude that then also $a_i \mathsf{U} a_k \in (LF^*, RF^*)$.
7. If $a_i \mathsf{M} a_k \in (LF^*, RF^*)$ for some a_k, then from the assumptions that $a_i \mathsf{M} a_i \in (LF^*, RF^*)$ and $a_i \mathsf{M} a_j \in (LF^*, RF^*)$ we have $a_j \mathsf{M} a_k \in (LF^*, RF^*)$, by *1* in Lemma 2 and rule $\mathsf{M}sub1$.
8. We can reason similarly for $a_k \mathsf{M} a_j \in (LF^*, RF^*)$ and apply rule $\mathsf{M}sub2$ to conclude that then also $a_k \mathsf{M} a_i \in (LF^*, RF^*)$.

Hence, a_i and a_j are equal also with respect to r-formulas, and thus $a_i = a_j$ whenever $(a_i, a_i) \in M^c$ and $(a_i, a_j) \in M^c$, which is what we had to show.

By Lemma 2 and Fact 2, it follows that:

Lemma 3. $a{:}A \in (LF^*, RF^*)$ *iff* $(LF^*, RF^*) \vDash^{\mathscr{S}^c} a{:}A$.

Proof. We proceed by induction on the grade of $a{:}A$, and we treat only the step case where $a{:}A$ is $a_i{:}\bigstar B$; the other cases follow analogously. For the left-to-right direction, assume $a_i{:}\bigstar B \in (LF^*, RF^*)$. Then, by Lemma 2, $a_i \mathsf{R} a_j \in (LF^*, RF^*)$ implies $a_j{:}B \in (LF^*, RF^*)$, for all a_j. Fact 2 and the induction hypothesis yield that $(LF^*, RF^*) \vDash^{\mathscr{S}^c} a_j{:}B$ for all a_j such that $(LF^*, RF^*) \vDash^{\mathscr{S}^c} a_i \mathsf{R} a_j$, i.e. $(LF^*, RF^*) \vDash^{\mathscr{S}^c} a_i{:}\bigstar B$ by Definition 3. For the converse, assume $a_i{:}\neg\bigstar B \in (LF^*, RF^*)$. Then, by Lemma 2, $a_i \mathsf{R} a_j \in (LF^*, RF^*)$ and $a_j{:}\neg B \in (LF^*, RF^*)$, for some a_j. Fact 2 and the induction hypothesis yield $(LF^*, RF^*) \vDash^{\mathscr{S}^c} a_i \mathsf{R} a_j$ and $(LF^*, RF^*) \vDash^{\mathscr{S}^c} a_j{:}\neg B$, i.e., $(LF^*, RF^*) \vDash^{\mathscr{S}^c} a_i{:}\neg\bigstar B$ by Definition 3. □

We can now finally show:

Theorem 3 (Completeness of MSQS)**.** $\Gamma \vDash \alpha$ *implies* $\Gamma \vdash \alpha$.

Proof. If $(LF, RF) \nvdash b_i \mathsf{R} b_j$, then $b_i \mathsf{R} b_j \notin (LF^*, RF^*)$, and thus, by Fact 2, $(LF^*, RF^*) \nvDash^{\mathscr{S}^c} b_i \mathsf{R} b_j$.

If $(LF, RF) \nvdash b{:}A$, then $(LF \cup \{b{:}\neg A\}, RF)$ is consistent; otherwise there exists a b_i such that $(LF \cup \{b{:}\neg A\}, RF) \vdash b_i{:}\bot$, and then $(LF, RF) \vdash b{:}A$. Therefore, by Lemma 1, $(LF \cup \{b{:}\neg A\}, RF)$ is included in a maximally consistent context $((LF \cup \{b{:}\neg A\})^*, RF^*)$. Then, by Lemma 3, $((LF \cup \{b{:}\neg A\})^*, RF^*) \vDash^{\mathscr{S}^C} b{:}\neg A$, i.e., $((LF \cup \{b{:}\neg A\})^*, RF^*) \nvDash^{\mathscr{S}^c} b{:}A$, and thus $(LF, RF) \nvDash^{\mathscr{S}^c} b{:}A$. □

5 GENERIC MEASUREMENTS

5.1 The rules of MSpQS

In quantum computing, not all measurements are required to be total: think, for example, of the case of observing only the first qubit of a quantum state. To this end, in this section, we propose MSpQS, a variant of MSQS that provides a modal system representing all the partial (thus not necessarily total) measurements. We obtain MSpQS from MSQS by means of the following changes:

- The alphabet of the modal language contains the unary modal operator $\boxdot$ instead of $\blacksquare$, with corresponding $\diamond$, where $\boxdot A$ intuitively means that A is true in each quantum state obtained by a measurement.

- The set of relational formulas contains expressions of the form $x \mathsf{P} y$ instead of $x \mathsf{M} y$, and we now write $x \mathsf{R} y$ to denote a generic r-formula, with $\mathsf{R} \in \{\mathsf{U}, \mathsf{P}\}$.

- The rules of MSpQS are given in Figure 4. In particular, $\bigstar$ is either $\Box$ (as before) or $\boxdot$, for which then R is P, and whose properties are formalized by the following additional rules:
  - If $x\mathsf{P}y$ then there is a specific unitary transformation (depending on x and y) that generates y from x: rule $\mathsf{PU}I$.
  - The measurement process is transitive: rule $\mathsf{P}trans$.
  - There are (always reachable) classical worlds: $class$ says that y is a classical world reachable from world x by a measurement.
  - Invariance with respect to classical worlds for measurement: rules $\mathsf{P}sub1$ and $\mathsf{P}sub2$.

We refer to the fresh y in $\bigstar I$ and $class$ as the *parameter* of the rule.

Some concrete, numerical, examples explaining the intuitive meaning of these theorems are given in Section 5.3. Before doing that, we consider derivations and show the soundness and completeness of the system.

Derivations and proofs in MSpQS are defined as for MSQS. For instance, in addition to the formulas for $\Box$ already listed for MSQS, the following labelled formula schemata, corresponding to standard modal axioms, are all provable in MSpQS (as shown, e.g., for formula 3 in Figure 5):

1. $x{:} \boxdot A \supset \diamonddot A$
   (it is always possible to perform a measurement of a quantum state).
2. $x{:} \boxdot A \supset \boxdot \boxdot A$
   (measurements are composable).
3. $x{:}\diamonddot(A \supset \boxdot A)$, i.e., $x{:}\neg \boxdot \neg(A \supset \boxdot A)$
   (it is always possible to perform a measurement with a complete reduction of a quantum state to a classical one).

5.2 The semantics of MSpQS

The semantics of MSpQS is also obtained by simple changes with respect to the definitions of Section 4. A *frame* is a tuple $\mathscr{F} = \langle W, U, P\rangle$, where $P \subseteq W \times W$ and vPw means that w is obtained by means of a measurement of v, with the following properties:

(i) $\forall v, w.\ vPw \Longrightarrow vUw$
(as for (i) in Section 4).

$\supset I,\ \supset E,\ RAA,\ \bot E,\ \bigstar I,\ \bigstar E,\ \mathsf{U}\mathit{refl},\ \mathsf{U}\mathit{symm},\ \mathsf{U}\mathit{trans},$

$$\dfrac{x\mathsf{P}y}{x\mathsf{U}y}\ \mathsf{PU}I \qquad \dfrac{x\mathsf{P}y \quad y\mathsf{P}z}{x\mathsf{P}z}\ \mathsf{P}\mathit{trans} \qquad \begin{array}{c}[x\mathsf{P}y]\,[y\mathsf{P}y]\\ \vdots \\ \dfrac{\alpha}{\alpha}\ \mathit{class}\end{array}$$

$$\dfrac{\alpha(x) \quad x\mathsf{P}x \quad x\mathsf{P}y}{\alpha(y/x)}\ \mathsf{P}\mathit{sub1} \qquad \dfrac{\alpha(y) \quad x\mathsf{P}x \quad x\mathsf{P}y}{\alpha(x/y)}\ \mathsf{P}\mathit{sub2}$$

In $\bigstar I$, y is fresh: it is different from x and does not occur in any assumption on which $y{:}A$ depends other than $x\mathsf{R}y$.
In class, y is fresh: it is different from x and does not occur in α nor in any assumption on which α depends other than $x\mathsf{P}y$ and $y\mathsf{P}y$.

FIGURE 4
The rules of MSpQS

$$\dfrac{\dfrac{\dfrac{\dfrac{[x{:}\boxdot\neg(A\supset\boxdot A)]^2 \quad [x\mathsf{P}y]^1}{y{:}\neg(A\supset\boxdot A)}\ \boxdot E \qquad \dfrac{\dfrac{\dfrac{[y{:}A]^3 \quad [y\mathsf{P}y]^1 \quad [y\mathsf{P}z]^4}{z{:}A}\ \mathsf{P}\mathit{sub1}}{y{:}\boxdot A}\ \boxdot I^4}{y{:}A\supset\boxdot A}\ \supset I^3}{y{:}\bot}\ \neg E}{x{:}\neg\boxdot\neg(A\supset\boxdot A)}\ \neg I^2}{x{:}\neg\boxdot\neg(A\supset\boxdot A)}\ \mathit{class}^1$$

FIGURE 5
An example proof in MSpQS

(ii) $\forall v, w', w''.\ vPw' \ \&\ w'Pw'' \Longrightarrow vPw''$
(measurements are composable).

(iii) $\forall v.\ \exists w.\ vPw \ \&\ wPw$

(each quantum state v can be reduced to a classical one w by means of a measurement).

(iv) $\forall v, w.\ vPv \ \&\ vPw \Longrightarrow v = w$
(each measurement of a classical state v has v as outcome).

Models and *structures* are defined as before, with $\mathscr{I}(\mathsf{P}) = P$, while the *truth* relation now comprises the clauses

$$\begin{array}{lll} \vDash^{\mathscr{M},w} \boxdot A & \text{iff} & \forall w'.\ wPw' \Longrightarrow \vDash^{\mathscr{M},w'} A \\ \vDash^{\mathscr{M},\mathscr{I}} x\mathsf{P}y & \text{iff} & \mathscr{I}(x)P\mathscr{I}(y) \end{array}$$

Finally, MSpQS is also sound and complete.

Theorem 4 (Soundness and completeness of MSpQS)**.** $\Gamma \vdash \alpha$ *iff* $\Gamma \vDash \alpha$. $\square$

We can reason similarly to what we did for MSQS to show the soundness and completeness of MSpQS with respect to the corresponding semantics: Theorem 4 follows from Theorems 5 and 6 below.

Theorem 5 (Soundness of MSpQS)**.** $\Gamma \vdash \alpha$ *implies* $\Gamma \vDash \alpha$.

Proof. We let $\mathscr{M}$ be an arbitrary model and prove that if $\Gamma \vdash \alpha$ then $\vDash^{\mathscr{I}} \Gamma$ implies $\vDash^{\mathscr{I}} \alpha$ for any $\mathscr{I}$. The proof proceeds by induction on the structure of the derivation of α from Γ. The base case, where $\alpha \in \Gamma$, is trivial. There is one step case for each rule of MSpQS, where the soundness of the rules $\supset I$, $\supset E$, RAA, $\bot E$, $\mathsf{U}\mathit{refl}$, $\mathsf{U}\mathit{symm}$, $\mathsf{U}\mathit{trans}$ follows exactly like in the proof of Theorem 2.

Also the soundness of the rules $\bigstar I$ and $\bigstar E$ follows exactly like in the proof of Theorem 2, with the only difference that when $\bigstar$ is $\boxdot$ then R is P.

The rules $\mathsf{PU}I$ and $\mathsf{P}\mathit{trans}$ are sound by properties (i) and (ii) in the definition of the semantics for MSpQS, respectively.

The soundness of the rule class follows like for the soundness of the rule $\mathsf{M}\mathit{ser}$ in the proof of Theorem 2, this time exploiting property (iii) in the definition of the semantics for MSpQS.

The soundness of the rules $\mathsf{P}\mathit{sub1}$ and $\mathsf{P}\mathit{sub2}$ follows like for the soundness of the rules $\mathsf{M}\mathit{sub1}$ and $\mathsf{M}\mathit{sub2}$ in the proof of Theorem 2, this time exploiting property (iv) in the definition of the semantics for MSpQS. $\square$

To prove completeness (Theorem 3), we proceed like for MSQS, mutatis mutandis in the construction of the canonical model. In particular, given a maximally consistent context (LF^*, RF^*), we define the canonical structure $\mathscr{S}^c = \langle W^c, U^c, P^c, V^c, \mathscr{I}^c\rangle$ by setting

- $(a_i, a_j) \in P^c$ iff $a_i \mathsf{P} a_j \in (LF^*, RF^*)$.

To show that the conditions on R^c are satisfied, so that $\mathscr{S}^c$ is really a structure for MSpQS, we reuse the results proved for MSQS and in addition show the following:

- $\forall v, w \in W^c.\ vPw \Longrightarrow vUw$ holds by construction and rule $\mathsf{PU}I$.
- $\forall v, w', w'' \in W^c.\ vPw'\ \&\ w'Pw'' \Longrightarrow vPw''$ holds by construction and rule $\mathsf{P}trans$.
- $\forall v \in W^c.\ \exists w \in W^c.\ vPw\ \&\ wPw$ holds by construction and rule $class$. For the sake of contradiction, consider an arbitrary a_i and a variable a'_j that do not satisfy the property. Define $(LF', RF') = (LF^*, RF^*) \cup \{a_i \mathsf{P} a'_j, a'_j \mathsf{P} a'_j\}$. Then it cannot be the case that $(LF', RF') \vdash \alpha$, for otherwise $(LF^*, RF^*) \vdash \alpha$ would be derivable by an application of the rule $class$. Thus, $(LF', RF') \nvdash \alpha$. But then (LF', RF') must be in the chain of contexts built in Lemma 2. So, by the maximality of (LF^*, RF^*), we have that $(LF', RF') = (LF^*, RF^*)$, contradicting our assumption. Hence, for some a_j, the r-formulas $a_i \mathsf{P} a_j$ and $a_j \mathsf{P} a_j$ are both in (LF^*, RF^*), which is what we had to show.
- $\forall v, w \in W^c.\ vPv\ \&\ vPw \Longrightarrow v = w$ holds by construction and rules $\mathsf{P}sub1$ and $\mathsf{P}sub2$ since v is a classical world. Consider an arbitrary context (LF, RF) from which we build $\mathscr{S}^c$ and assume $(a_i, a_i) \in P^c$ and $(a_i, a_j) \in P^c$. Then $a_i \mathsf{P} a_i \in (LF^*, RF^*)$ and $a_i \mathsf{P} a_j \in (LF^*, RF^*)$. Thus, for each $a_i{:}A \in (LF^*, RF^*)$, we also have $a_j{:}A \in (LF^*, RF^*)$; otherwise, since (LF^*, RF^*) is deductively closed, we would have $a_j{:}\neg A \in (LF^*, RF^*)$ and also $a_j{:}A \in (LF^*, RF^*)$ by *1* in Lemma 2 and rule $\mathsf{P}sub1$, and thus a contradiction. Similarly, if $a_j{:}A \in (LF^*, RF^*)$ then $a_i{:}A \in (LF^*, RF^*)$ by rule $\mathsf{P}sub2$. Hence, for each m-formula A, we have that $a_i{:}A \in (LF^*, RF^*)$ iff $a_j{:}A \in (LF^*, RF^*)$, which means that a_i and a_j are equal with respect to m-formulas.

  Under the same assumptions, we can similarly show that a_i and a_j are equal with respect to r-formulas, i.e., that whenever (LF^*, RF^*) contains an r-formula that includes a_i then it also contains the same r-formula with a_j substituted for a_i, and vice versa. To this end, we must consider eight different cases corresponding to eight different r-formulas.

1. If $a_k \mathsf{U} a_i \in (LF^*, RF^*)$ for some a_k, then from the assumption that $a_i \mathsf{P} a_j \in (LF^*, RF^*)$ we have $a_i \mathsf{U} a_j \in (LF^*, RF^*)$, by *1* in Lemma 2 and rule $\mathsf{PU}I$. Therefore, $a_k \mathsf{U} a_j \in (LF^*, RF^*)$ by rule $\mathsf{U}trans$.

2. We can reason similarly for $a_j \mathsf{U} a_k \in (LF^*, RF^*)$ and also apply rules $\mathsf{PU}I$ and $\mathsf{U}trans$ to conclude that then also $a_i \mathsf{U} a_k \in (LF^*, RF^*)$.

3. If $a_i \mathsf{U} a_k \in (LF^*, RF^*)$ for some a_k, then from the assumption that $a_i \mathsf{P} a_j \in (LF^*, RF^*)$ we have $a_i \mathsf{U} a_j \in (LF^*, RF^*)$, by *1* in Lemma 2 and rule $\mathsf{PU}I$, and thus $a_j \mathsf{U} a_i \in (LF^*, RF^*)$, by rule $\mathsf{U}symm$. Therefore, $a_j \mathsf{U} a_k \in (LF^*, RF^*)$ by rule $\mathsf{U}trans$.

4. We can reason similarly for $a_k \mathsf{U} a_j \in (LF^*, RF^*)$ and also apply rules $\mathsf{PU}I$, $\mathsf{U}symm$, and $\mathsf{U}trans$ to conclude that then also $a_k \mathsf{U} a_i \in (LF^*, RF^*)$.

5. If $a_k \mathsf{P} a_i \in (LF^*, RF^*)$ for some a_k, then from the assumption that $a_i \mathsf{P} a_j \in (LF^*, RF^*)$ we have $a_k \mathsf{P} a_j \in (LF^*, RF^*)$, by *1* in Lemma 2 and the rule $\mathsf{P}trans$.

6. We can reason similarly for $a_j \mathsf{P} a_k \in (LF^*, RF^*)$ and also apply rule $\mathsf{P}trans$ to conclude that then also $a_i \mathsf{U} a_k \in (LF^*, RF^*)$.

7. If $a_i \mathsf{P} a_k \in (LF^*, RF^*)$ for some a_k, then from the assumptions that $a_i \mathsf{P} a_i \in (LF^*, RF^*)$ and $a_i \mathsf{P} a_j \in (LF^*, RF^*)$ we have $a_j \mathsf{P} a_k \in (LF^*, RF^*)$, by *1* in Lemma 2 and rule $\mathsf{P}sub1$.

8. We can reason similarly for $a_k \mathsf{P} a_j \in (LF^*, RF^*)$ and apply rule $\mathsf{P}sub2$ to conclude that then also $a_k \mathsf{P} a_i \in (LF^*, RF^*)$.

Hence, a_i and a_j are equal also with respect to r-formulas, and thus $a_i = a_j$ whenever $(a_i, a_i) \in P^c$ and $(a_i, a_j) \in P^c$, which is what we had to show.

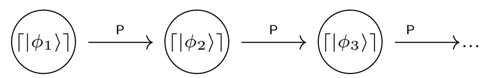
Proceeding like for MSQS, we then have:

Theorem 6 (Completeness of MSpQS)**.** $\Gamma \vDash \alpha$ *implies* $\Gamma \vdash \alpha$. $\square$

5.3 Some concrete examples

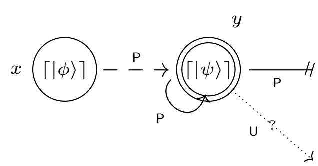
We give some concrete, numerical, examples explaining the intuitive meaning of the theorems stated in Section 5.1. As done in examples 4.2, let us consider a structure $\langle \mathscr{M}, \mathscr{I} \rangle$ where $\mathscr{M} = \langle W, U, P, V \rangle$ is the Kripke model defined in Section 2.3, and $\mathscr{I}$ is a generic interpretation function. With a little abuse of language for the sake of brevity, we identify, via the interpretation function $\mathscr{I}$, the labels with the corresponding concrete worlds of the semantics, and the syntactic relations U, P with their semantical counterparts U and P. Under such a convention we will refer to labels as *worlds*.

Theorems 1 and 2 are exactly axioms D and 4 for $\boxdot$, as explained in Section 2 for relations U and M, respectively. As we remarked above, axiom D expresses the seriality of U, which suggests models like



in the sense that it is always possible to make a partial measurement of a state.

Let us consider theorem 3, $x{:}\diamonddot(A \supset \boxdot A)$, i.e., $x{:}\neg\boxdot\neg(A \supset \boxdot A)$, and let x and y be two worlds such that $x\mathsf{P}y$ with $\lceil 1/\sqrt{2}|00\rangle + 1/\sqrt{2}|01\rangle\rceil \in V(x)$ and $\lceil|00\rangle\rceil \in V(y)$. Then we have $x{:}\lceil 1/\sqrt{2}|00\rangle + 1/\sqrt{2}|01\rangle\rceil \supset \diamonddot\lceil|00\rangle\rceil$, $y{:}\lceil|00\rangle\rceil \supset \boxdot\lceil|00\rangle\rceil$ and finally $\diamonddot(\lceil|\phi\rangle\rceil \supset \boxdot\lceil|\phi\rangle\rceil)$. More generally, we have that for each world x there is a classical reachable world y. Graphically:



where the dashed line denotes that y is that particular classical world reachable from x; this is in contrast to the similar figure (2) for total measurements where the line is solid meaning that after a total measurement we always obtain a state that is stable with respect to further measurements (i.e., all worlds reachable by a total measurement are classical).

6 NORMALIZATION

In this section, we show that each derivation of an l-formula in MSQS and MSpQS can be reduced to a normal form that does not contain unnecessary detours and satisfies a subformula property, from which we then obtain syntactic proofs of the consistency of both MSQS and MSpQS. We first consider MSQS and then discuss the extensions and changes needed in the case of MSpQS.

6.1 Normalization for MSQS

We begin by proving a useful lemma about parameters, i.e., as we mentioned above, the fresh variables used in the applications of $\bigstar I$ and $\mathsf{M}ser$. By extension, we speak of a parameter y of a derivation if y is the parameter of some application of $\bigstar I$ or $\mathsf{M}ser$ in the derivation.

Lemma 4 (Parameter condition)**.** *Let Π be an MSQS-derivation of $x{:}A$ from a set Γ of assumptions. Then we can build an MSQS-derivation Π' of $x{:}A$ from Γ such that:*

- *each parameter is the parameter of exactly one application of $\bigstar I$ or $\mathsf{M}ser$, and*
- *the parameter of any application of $\bigstar I$ or $\mathsf{M}ser$ occurs only in the sub-derivation above that application of the rule.*

Proof. The lemma follows quite straightforwardly by induction on the derivation of $\Gamma \vdash x{:}A$, where the proof essentially boils down to a systematic renaming of the parameters. $\square$

In the remainder of the paper, we thus assume that all the derivations satisfy the parameter condition.

To show normalization, we follow, where possible, standard presentations such as [16, 17, 19, 20]. We begin by introducing some restrictions to simplify the development; in particular, we restrict applications of RAA and $\bot E$ to the case where the conclusion $x{:}A$ is atomic, i.e., A is atomic.[#] Moreover, we also restrict applications of $\mathsf{M}sub1$, $\mathsf{M}sub2$ and $\mathsf{M}ser$ to atomic conclusions.

Lemma 5. *If $\Gamma \vdash \alpha$ in MSQS, then there is an MSQS-derivation of α from Γ where the conclusions of applications of RAA, $\bot E$, $\mathsf{M}sub1$, $\mathsf{M}sub2$, and $\mathsf{M}ser$ are atomic.*

Note that we do not need to consider derivations of r-formulas, e.g., by $\bot E$, since in MSQS we only have atomic r-formulas by definition; the same holds for MSpQS. We can then prove Lemma 5 as follows.

[#] When presenting classical first-order logic, Prawitz [16] first introduces a natural deduction system consisting of an elimination rule for $\bot$ and introduction and elimination rules for all the other connectives, and then, to show normalization, restricts his attention to the functionally complete $\bot$, $\wedge$, $\supset$, $\forall$ fragment, where RAA is restricted to atomic conclusions (that are also different from $\bot$). In this way, he avoids having to treat the rules for $\vee$ and $\exists$, which behave 'badly' for normalization. Here, since we have already focused on the functionally complete $\bot$, $\supset$, $\bigstar$ system, we do not need further restrictions than the ones on RAA and $\bot E$ (where, however, we allow the atomic conclusion A to be falsum itself, albeit labelled differently), as well as on $\mathsf{M}sub1$, $\mathsf{M}sub2$, and $\mathsf{M}ser$.

Proof. We first show that any application of RAA with a non-atomic conclusion can be replaced with a derivation in which RAA is applied only to l-formulas of smaller grade. Note that we only show the part of the derivation where the transformation, denoted by $\rightsquigarrow$, actually takes place; the missing parts remain unchanged. There are two possible cases, depending on whether the conclusion is $x{:}B \supset C$ or $x{:}\bigstar B$.

Case 1: We distinguish two subcases, depending on whether C is $\bot$ or not. If $C \neq \bot$, then

$$\begin{array}{c}[x{:}(B \supset C) \supset \bot]^1\\ \Pi\\ \dfrac{y{:}\bot}{x{:}B \supset C}\ RAA^1\end{array} \quad\rightsquigarrow\quad \begin{array}{c} \dfrac{\dfrac{[x{:}C \supset \bot]^2 \quad \dfrac{[x{:}B \supset C]^1 \quad [x{:}B]^3}{x{:}C} \supset E}{x{:}\bot} \supset E}{x{:}(B \supset C) \supset \bot} \supset I^1 \\ \Pi \\ \dfrac{\dfrac{y{:}\bot}{x{:}C}\ RAA^2}{x{:}B \supset C} \supset I^3\end{array}$$

If $C = \bot$, then

$$\begin{array}{c}[x{:}(B \supset \bot) \supset \bot]^1\\ \Pi\\ \dfrac{y{:}\bot}{x{:}B \supset \bot}\ RAA^1\end{array} \quad\rightsquigarrow\quad \begin{array}{c} \dfrac{\dfrac{[x{:}B \supset \bot]^1 \quad [x{:}B]^2}{x{:}\bot} \supset E}{x{:}(B \supset \bot) \supset \bot} \supset I^1 \\ \Pi \\ \dfrac{\dfrac{y{:}\bot}{x{:}\bot}\ \bot E}{x{:}B \supset \bot} \supset I^2\end{array}$$

Case 2: We distinguish two subcases, depending on whether B is $\bot$ or not. If $B \neq \bot$, then

$$\begin{array}{c}[x{:}\bigstar B \supset \bot]^1\\ \Pi\\ \dfrac{y{:}\bot}{x{:}\bigstar B}\ RAA^1\end{array} \quad\rightsquigarrow\quad \begin{array}{c} \dfrac{\dfrac{\dfrac{[y{:}B \supset \bot]^2 \quad \dfrac{[x{:}\bigstar B]^1 \quad [x\mathsf{R}y]^3}{y{:}B} \bigstar E}{y{:}\bot} \supset E}{x{:}\bot}\ \bot E}{x{:}\bigstar B \supset \bot} \supset I^1 \\ \Pi \\ \dfrac{\dfrac{y{:}\bot}{y{:}B}\ RAA^2}{x{:}\bigstar B} \bigstar I^3\end{array}$$

where, if necessary, we follow Lemma 4 to rename the parameters in the derivation to avoid possible clashes due to the new application of $\bigstar I$.

If $B = \bot$, then

$$\dfrac{\begin{array}{c}[x{:}\bigstar\bot \supset \bot]^1\\ \Pi\\ y{:}\bot\end{array}}{x{:}\bigstar\bot}\ RAA^1 \quad\rightsquigarrow\quad \dfrac{\begin{array}{c}\dfrac{\dfrac{\dfrac{[x{:}\bigstar\bot]^1 \quad [x\mathsf{R}y]^2}{y{:}\bot}\supset E}{x{:}\bot}\ \bot E}{x{:}\bigstar\bot \supset \bot}\supset I^1\\ \Pi\\ y{:}\bot\end{array}}{x{:}\bigstar B}\ \bigstar I^2$$

We proceed analogously for $\bot E$: we show that any application of $\bot E$ with a non-atomic conclusion can be replaced with a derivation in which $\bot E$ is applied only to l-formulas of smaller grade. Hence, there are again two possible cases, depending on whether the conclusion is $x{:}B \supset C$ or $x{:}\bigstar B$.
Case 1:

$$\dfrac{\begin{array}{c}\Pi\\ y{:}\bot\end{array}}{x{:}B \supset C}\ \bot E \quad\rightsquigarrow\quad \dfrac{\dfrac{\begin{array}{c}\Pi\\ y{:}\bot\end{array}}{x{:}C}\ \bot E}{x{:}B \supset C}\supset I$$

Case 2:

$$\dfrac{\begin{array}{c}\Pi\\ y{:}\bot\end{array}}{x{:}\bigstar B}\ \bot E \quad\rightsquigarrow\quad \dfrac{\dfrac{\begin{array}{c}\Pi\\ y{:}\bot\end{array}}{z{:}B}\ \bot E}{x{:}\bigstar B}\ \bigstar I$$

Applications of $\mathsf{M}sub1$ and $\mathsf{M}sub2$ can be reduced to atomic formulas as follows, where we now consider the two subcases for $\square$ and $\blacksquare$ explicitly:

$$\dfrac{\begin{array}{ccc}\Pi_1 & \Pi_2 & \Pi_3\\ x{:}A\supset B & x\mathsf{M}x & x\mathsf{M}y\end{array}}{y{:}A\supset B}\,\mathsf{M}\mathit{sub1} \qquad \rightsquigarrow$$

$$\dfrac{\dfrac{\begin{array}{c}\Pi_1\\ x{:}A\supset B\end{array} \quad \dfrac{[y{:}A]^1 \quad \begin{array}{c}\Pi_2\\ x\mathsf{M}x\end{array} \quad \begin{array}{c}\Pi_3\\ x\mathsf{M}y\end{array}}{x{:}A}\,\mathsf{M}\mathit{sub2}}{x{:}B}\supset E \qquad \begin{array}{c}\Pi_2\\ x\mathsf{M}x\end{array}\quad\begin{array}{c}\Pi_3\\ x\mathsf{M}y\end{array}}{\dfrac{y{:}B}{y{:}A\supset B}\supset I^1}\,\mathsf{M}\mathit{sub1}$$

$$\dfrac{\begin{array}{ccc}\Pi_1 & \Pi_2 & \Pi_3\\ y{:}A\supset B & x\mathsf{M}x & x\mathsf{M}y\end{array}}{x{:}A\supset B}\,\mathsf{M}\mathit{sub2} \qquad \rightsquigarrow$$

$$\dfrac{\dfrac{\begin{array}{c}\Pi_1\\ y{:}A\supset B\end{array} \quad \dfrac{[x{:}A]^1 \quad \begin{array}{c}\Pi_2\\ x\mathsf{M}x\end{array} \quad \begin{array}{c}\Pi_3\\ x\mathsf{M}y\end{array}}{y{:}A}\,\mathsf{M}\mathit{sub1}}{y{:}B}\supset E \qquad \begin{array}{c}\Pi_2\\ x\mathsf{M}x\end{array}\quad\begin{array}{c}\Pi_3\\ x\mathsf{M}y\end{array}}{\dfrac{x{:}B}{x{:}A\supset B}\supset I^1}\,\mathsf{M}\mathit{sub2}$$

$$\dfrac{\begin{array}{ccc}\Pi_1 & \Pi_2 & \Pi_3\\ x{:}\Box A & x\mathsf{M}x & x\mathsf{M}y\end{array}}{y{:}\Box A}\,\mathsf{M}\mathit{sub1} \quad \rightsquigarrow \quad \dfrac{\dfrac{\begin{array}{c}\Pi_1\\ x{:}\Box A\end{array} \quad \dfrac{[y\mathsf{U}z]^1 \quad \begin{array}{c}\Pi_2\\ x\mathsf{M}x\end{array} \quad \begin{array}{c}\Pi_3\\ x\mathsf{M}y\end{array}}{x\mathsf{U}z}\,\mathsf{M}\mathit{sub2}}{z{:}A}\,\Box E}{y{:}\Box A}\,\Box I^1$$

$$\dfrac{\begin{array}{ccc}\Pi_1 & \Pi_2 & \Pi_3\\ y{:}\Box A & x\mathsf{M}x & x\mathsf{M}y\end{array}}{x{:}\Box A}\,\mathsf{M}\mathit{sub2} \quad \rightsquigarrow \quad \dfrac{\dfrac{\begin{array}{c}\Pi_1\\ y{:}\Box A\end{array} \quad \dfrac{[x\mathsf{U}z]^1 \quad \begin{array}{c}\Pi_2\\ x\mathsf{M}x\end{array} \quad \begin{array}{c}\Pi_3\\ x\mathsf{M}y\end{array}}{y\mathsf{U}z}\,\mathsf{M}\mathit{sub1}}{z{:}A}\,\Box E}{x{:}\Box A}\,\Box I^1$$

$$\dfrac{\begin{matrix}\Pi_1\\ x{:}\blacksquare A\end{matrix} \quad \begin{matrix}\Pi_2\\ x\mathsf{M}x\end{matrix} \quad \begin{matrix}\Pi_3\\ x\mathsf{M}y\end{matrix}}{y{:}\blacksquare A}\,\mathsf{M}\mathit{sub1} \quad\rightsquigarrow\quad \dfrac{\dfrac{\begin{matrix}\Pi_1\\ x{:}\blacksquare A\end{matrix} \quad \dfrac{[y\mathsf{M}z]^1 \quad \begin{matrix}\Pi_2\\ x\mathsf{M}x\end{matrix} \quad \begin{matrix}\Pi_3\\ x\mathsf{M}y\end{matrix}}{x\mathsf{M}z}\,\mathsf{M}\mathit{sub2}}{z{:}A}\,\blacksquare E}{y{:}\blacksquare A}\,\blacksquare I^1$$

$$\dfrac{\begin{matrix}\Pi_1\\ y{:}\blacksquare A\end{matrix} \quad \begin{matrix}\Pi_2\\ x\mathsf{M}x\end{matrix} \quad \begin{matrix}\Pi_3\\ x\mathsf{M}y\end{matrix}}{x{:}\blacksquare A}\,\mathsf{M}\mathit{sub2} \quad\rightsquigarrow\quad \dfrac{\dfrac{\begin{matrix}\Pi_1\\ y{:}\blacksquare A\end{matrix} \quad \dfrac{[x\mathsf{M}z]^1 \quad \begin{matrix}\Pi_2\\ x\mathsf{M}x\end{matrix} \quad \begin{matrix}\Pi_3\\ x\mathsf{M}y\end{matrix}}{y\mathsf{M}z}\,\mathsf{M}\mathit{sub1}}{z{:}A}\,\blacksquare E}{x{:}\blacksquare A}\,\blacksquare I^1$$

We proceed in the same way for the $\mathsf{M}ser$ rule.
Case 1:

$$\dfrac{\begin{matrix}[x\mathsf{M}y]^1\\ \Pi\\ u{:}B \supset C\end{matrix}}{u{:}B \supset C}\,\mathsf{M}ser^1 \quad\rightsquigarrow\quad \dfrac{\dfrac{\dfrac{\begin{matrix}[x\mathsf{M}y]^1\\ \Pi\\ u{:}B \supset C\end{matrix} \quad [u{:}B]^2}{u{:}C}\,\supset E}{u{:}C}\,\mathsf{M}ser^1}{u{:}B \supset C}\,\supset I^2$$

Case 2:

$$\dfrac{\begin{matrix}[x\mathsf{M}y]^1\\ \Pi\\ u{:}\bigstar A\end{matrix}}{x{:}\bigstar A}\,\mathsf{M}ser^1 \quad\rightsquigarrow\quad \dfrac{\dfrac{\dfrac{\begin{matrix}[x\mathsf{M}y]^1\\ \Pi\\ u{:}\bigstar A\end{matrix} \quad [u\mathsf{R}w]^2}{w{:}A}\,\bigstar E}{w{:}A}\,\mathsf{M}ser^1}{u{:}\bigstar A}\,\bigstar I^2$$

where we choose the parameter w so to allow for the application of $\bigstar I$.

By iterating these transformations, we transform an arbitrary MSQS-derivation $\Gamma \vdash \alpha$ into an MSQS-derivation of α from Γ where the conclusions of applications of RAA, $\bot E$, $\mathsf{M}\mathit{sub1}$, $\mathsf{M}\mathit{sub2}$, and $\mathsf{M}ser$ are atomic. $\square$

An immediate consequence of this lemma is the equivalence of the restricted and the unrestricted natural deduction systems. In the rest of this section, we will therefore assume applications of RAA, $\bot E$, $\mathsf{M}\mathit{sub1}$, $\mathsf{M}\mathit{sub2}$, and $\mathsf{M}ser$ to be restricted in this way.

In a generic derivation, we can have a detour caused by the application of an elimination rule immediately below the application of the corresponding introduction rule. That is, if an l-formula is introduced and then immediately eliminated, then we can avoid introducing it in the first place; recall that in MSQS we only have atomic r-formulas by definition, so we do not need to consider the detours that would arise from non-atomic r-formulas. Formally, since the same formula may appear several times in a derivation, we distinguish these different formula occurrences to define:

Definition 5. *An l-formula occurrence $x{:}A$ is a* cut *in an MSQS-derivation when it is both the conclusion of an introduction rule and the major premise of an elimination rule. We call $x{:}A$ the* cut-formula *of the cut.*

An MSQS-derivation is in normal form *(is a* normal MSQS-derivation*) iff it contains no cut-formulas.*

Like for any "good" deduction system, we prove a normalization result that shows how to transform (in an effective way) each MSQS-derivation into a normal one. In order to remove cut-formulas, we introduce the notion of *contraction*, where the contraction relation $\triangleright$ is defined as follows:

$$\dfrac{\dfrac{\begin{array}{c}[x{:}A]\\ \Pi_1\\ x{:}B\end{array}}{x{:}A \supset B}\supset I \qquad \begin{array}{c}\Pi_2\\ x{:}A\end{array}}{x{:}B}\supset E \quad \triangleright \quad \begin{array}{c}\Pi_2\\ x{:}A\\ \Pi_1\\ x{:}B\end{array} \tag{$\triangleright_{\supset}$}$$

$$\dfrac{\dfrac{\begin{array}{c}[x\mathsf{R}y]\\ \Pi_1\\ y{:}A\end{array}}{x{:}\bigstar A}\bigstar I \qquad \begin{array}{c}\Pi_2\\ x\mathsf{R}z\end{array}}{z{:}A}\bigstar E \quad \triangleright \quad \begin{array}{c}\Pi_2\\ x\mathsf{R}z\\ \Pi_1[z/y]\\ z{:}A\end{array} \tag{$\triangleright_{\bigstar}$}$$

where $\Pi'[z/y]$ is obtained from Π by systematically substituting z for y. Note that the correctness of the contractions, and also of the substitution $\Pi'[z/y]$, is guaranteed by the assumption that all the derivations satisfy the parameter condition of Lemma 4. Note also that it suffices to consider the generic modal operator $\bigstar$ since the two modal operators $\square$ and $\blacksquare$ do not interfere (nor do the corresponding contractions).

Cuts are removed from a derivation by finitely many applications of contractions. Context closure of the contraction relation leads to the formal definition of the notions of *reduction* and *normalization*.

Definition 6. *We say that an MSQS-derivation Π_1* immediately reduces *to an MSQS-derivation Π_2, in symbols $\Pi_1 \succ \Pi_2$, iff there exist MSQS-derivations Π_3 and Π_4 such that $\Pi_3 \triangleright \Pi_4$ and Π_2 is obtained by replacing Π_3 with Π_4 in Π_1 .*

Hence, if Π is a normal MSQS-derivation (i.e., it contains no cut-formulas), there is no Π' such that $\Pi \succ \Pi'$.

Definition 7. *Writing $\succeq$ for the reflexive and transitive closure of $\succ$, we say that an MSQS-derivation Π* normalizes *to another MSQS-derivation Π' if $\Pi \succeq \Pi'$ and Π' is in normal form.*

Definition 8. *We define the* rank *of an l-formula as $rank(x{:}A) = rank(A)$ where*

- $rank(A) = 0$ *if A is atomic;*
- $rank(A \supset B) = max\{rank(A), rank(B)\} + 1$*;*
- $rank(\bigstar A) = rank(A) + 1$.

Then, for Π a derivation in MSQS,

- *a* maximal cut-formula *in Π is a cut-formula in Π with maximal rank;*
- $d = max\{rank(x{:}A) \mid x{:}A$ *is a cut-formula in* $\Pi\}$*, where* $\max\{\} = 0$*;*
- $cr(\Pi) = (d, n)$ *is the* cut rank *of Π, where n is the number of maximal cut-formulas in Π and where $cr(\Pi) = (0, 0)$ when Π has no cuts.*

The ordering on cut ranks is lexicographic: $(d, n) < (d', n')$ iff $d < d'$ or both $d = d'$ and $n < n'$. To prove our normalization result, we will systematically lower the cut rank of a derivation until all cuts have been eliminated. Before we do that, we prove a useful lemma:

Lemma 6. *Let Π be an MSQS-derivation with a cut at the bottom, and let this cut have rank q while all the other cuts in Π have rank $< q$. Then the contraction of Π at this lowest cut yields a derivation with only cuts of rank $< q$.*

Proof. Consider all the possible cuts at the bottom of Π and check the ranks of the cuts after the contraction. The proof follows since the two contractions ($\triangleright_{\supset}$) and ($\triangleright_{\bigstar}$) explicitly give formulas with lower rank, while nothing happens in Π_1 and Π_2, so all the cuts in the derivation resulting from the contraction have rank $< q$. □

Lemma 7. *Let Π be an MSQS-derivation. If $cr(\Pi) > (0, 0)$, then there is an MSQS-derivation Π' with $\Pi \triangleright \Pi'$ and $cr(\Pi') < cr(\Pi)$.*

Proof. Select a maximal cut-formula in Π such that all cuts above it have lower rank. Apply the appropriate contraction to this maximal cut. Then the part of Π ending in the conclusion of the cut is replaced, by Lemma 6, by a sub-derivation in which all cut-formulas have lower rank. If the maximal cut-formula was the only one, then d is lowered by 1, otherwise n is lowered by 1 and d remains unchanged. In both cases, $cr(\Pi)$ gets smaller. (Note that in the first case n may become much larger, but that does not matter in the lexicographic order.) □

We are now in a position to give our desired normalization results.

Theorem 7. *Every MSQS-derivation of $x{:}A$ from Γ reduces to an MSQS-derivation in normal form.*

Proof. By Lemma 7, the cut rank can be lowered to $(0, 0)$ in a finite number of steps, and therefore the last derivation in the reduction sequence has no more cuts. □

Normal MSQS-derivations possess a well-defined structure that has several desirable properties. Specifically, by analyzing the structure of a normal MSQS-derivation, we can characterize its form: we can identify particular sequences of formulas, and show that in these sequences there is an ordering on inferences. By exploiting this ordering, we can then show a subformula property for MSQS.

Definition 9. *A* thread *in an MSQS-derivation Π is a sequence of formulas $\alpha_1, \dots, \alpha_n$ such that (i) α_1 is an assumption of Π, (ii) α_i stands immediately above α_{i+1}, for $1 \leq i < n-1$, and (iii) α_n is the conclusion of Π.*

We further characterize a thread in terms of the formulas occurring in it: an l-formula-thread *is a thread where $\alpha_1, \dots, \alpha_n$ are all l-formulas, and an* r-formula-thread *is a thread where $\alpha_1, \dots, \alpha_n$ are all r-formulas.*

A track *in an MSQS-derivation Π is an initial part of a thread in Π which stops either at the first minor premise of an elimination rule in the thread or at the conclusion of the thread. We call* main track *a track that is also a thread and ends at the conclusion of the derivation.*

Definition 10. *B is a* subformula *of A iff (i) A is B; or (ii) A is $A_1 \supset A_2$ and B is a subformula of A_1 or A_2; or (iii) A is $\bigstar A_1$ and B is a subformula of A_1. We say that $y{:}B$ is a* labelled subformula *(or, slightly abusing notation, simply "subformula") of $x{:}A$ iff B is a subformula of A.*

One interesting property of normal MSQS-derivations, which can be read off from their structure, is that tracks in a normal MSQS-derivation have a standard form:

Lemma 8. *Let Π be a normal MSQS-derivation, and let t be a track α_1, …, α_n in Π. Then t contains a subsequence of formulas $\alpha_i, \dots, \alpha_k$, called the* minimal part*, which separates two possibly empty parts of t, called the* elimination part *and the* introduction part *of t, where:*

- *each formula α_j in the elimination part, i.e., for $j < i$, is an l-formula and is the major premise of an application of an elimination rule and contains α_{j+1} as a subformula;*

- *each formula α_s in the minimal part except the last one is the premise of an application of RAA, $\perp E$, M$sub1$, M$sub2$, Mser, U$refl$, U$symm$, U$trans$, UI, or M$srefl$;*

- *each formula α_j in the introduction part except the last one, i.e., for $k < j < n$, is an l-formula, is a premise of an introduction rule, and is a subformula of α_{j+1};*

- Π *has at least one main track, ending in the conclusion.*

The lemma follows quite straightforwardly by observing that in a track in a normal MSQS-derivation no introduction rule application can precede an application of an elimination rule; hence, if the first rule is an elimination, then all eliminations come first.

From these considerations, we can derive some other properties of normal tracks. For example, we can observe that if a thread t has an r-formula as top formula, then t is an *r-formula–thread* and if we extract a track t' from t, then we have empty elimination and introduction parts. Moreover, let $\alpha_1, \dots, \alpha_n$ be a thread and let $\alpha_1, \dots, \alpha_i$ be l-formulas; if α_{i+1} is an r-formula, then all α_j, for $i < j \leq n$, are r-formulas.

We can further observe that a "mixed" track (i.e., a track consisting of l-formulas and r-formulas) has the following structure: an introduction part of l-formulas; a minimal part in which an r-formula is introduced by an application of $\perp E$ and a (possibly empty) sequence of applications of RAA, M$sub1$, M$sub2$, Mser, U$refl$, U$symm$, U$trans$, UI, M$srefl$; and an empty introduction part.

The above results allow us to show that normal derivations in MSQS satisfy the following subformula property.

Definition 11. *Given an* MSQS*-derivation* Π *of* $x{:}A$ *from a set* Γ *of assumptions, let* $\mathcal{S}$ *be the set of subformulas of the formulas in* $\{C \mid z{:}C \in \Gamma \cup \{x{:}A\}$ *for some* $z\}$*, i.e.,* $\mathcal{S}$ *is the set consisting of the subformulas of the assumptions* Γ *and of the conclusion* $x{:}A$*.*

We say that Π *satisfies the* subformula property *iff for each l-formula occurrence* $y{:}B$ *in the derivation (i)* $B \in \mathcal{S}$*; or (ii)* B *is an assumption* $D \supset \bot$ *discharged by an application of* RAA*, where* $D \in \mathcal{S}$*; or (iii)* B *is an occurrence of* $\bot$ *obtained by* $\supset E$ *from an assumption* $D \supset \bot$ *discharged by an application of* RAA*, where* $D \in \mathcal{S}$*; or (iv)* B *is an occurrence of* $\bot$ *obtained by an application of* $\bot E$*.*

In other words, we define an MSQS-derivation to have the subformula property iff for all $y{:}B$ in the derivation, either B is a subformula of the assumptions or of the conclusion of the derivation, or B is the negation of such a subformula and is discharged by RAA, or B is an occurrence of $\bot$ immediately below the negation of a subformula, or B is an occurrence of $\bot$ immediately below another occurrence of $\bot$ that is labelled differently.

Theorem 8. *Every normal derivation of* $x{:}A$ *from* Γ *in* MSQS *satisfies the subformula property.*

Proof. We introduce an ordering of the tracks in a normal MSQS-derivation depending on their distance from the main track: the *order* of a track is $o(t_m) = 0$ for a main track t_m, and $o(t) = o(t')$ +1 if the end formula of a generic track t is a minor premise belonging to a major premise in t'.

Consider now an l-formula occurrence $y{:}B$ in a normal derivation Π of $x{:}A$ from Γ in MSQS. If $y{:}B$ occurs in the elimination part of its track t, then it is a subformula of the assumptions at the top of t. If not, then it is a subformula of the l-formula $z{:}C$ at the end of t. Hence, $z{:}C$ is a subformula of an l-formula $w{:}D$ of a track t_1 with $o(t_1) < o(t)$. Repeating the argument, we find that $y{:}B$ is a subformula of an assumption in Γ or of the conclusion $x{:}A$. This closes the case for all assumptions, so let us now consider the other formulas.

If $y{:}B$ is a subformula of a discharged assumption, then it must be a subformula of the resulting implicational l-formula in the case of an application of $\supset I$, or of the resulting l-formula in the case of an application of RAA, or (and these are the only exceptions) it is itself discharged by an application of RAA or it is $z{:}\bot$ (for some z) immediately following such an assumption or an application of $\bot E$. $\square$

Corollary 1. MSQS *is consistent.*

Proof. Suppose, for the sake of contradiction, that $\vdash x{:}\bot$ in MSQS. Then there is a normal derivation ending in $\vdash x{:}\bot$ with all assumptions discharged. There is a track through the conclusion; in this track there are no introduction rules, so the top assumption is not discharged. Contradiction. $\square$

6.2 Normalization for MSpQS

We can again simplify the development by restricting applications of RAA and $\bot E$ to the case where the conclusion $x{:}A$ is atomic, and we can also restrict applications of $\mathsf{P}sub1$, $\mathsf{P}sub2$, and $class$ to atomic conclusions, where, as for MSQS, we do not need to consider derivations of r-formulas, e.g., by $\bot E$, since also in MSpQS we only have atomic r-formulas by definition.

Lemma 9. *If* $\Gamma \vdash \alpha$ *in* MSpQS*, then there is an* MSpQS*-derivation of* α *from* Γ *where the conclusions of applications of* RAA*,* $\bot E$*,* $\mathsf{P}sub1$*,* $\mathsf{P}sub2$*, and* $class$ *are atomic.*

Recalling that the grade of an l-formula $x{:}A$ is the number of times $\supset$ and $\bigstar$ occur in A, where $\bigstar$ is either $\Box$ or $\boxdot$ for MSpQS, we can prove the lemma and thus the equivalence of the restricted and the unrestricted system MSpQS as follows.

Proof. By considering the same transformations that we employed for MSQS in Lemma 5, we can replace applications of RAA and $\bot E$ with non-atomic conclusions with derivations in which RAA and $\bot E$ are applied only to l-formulas of smaller grade.

Applications of $\mathsf{P}sub1$ and $\mathsf{P}sub2$ can be reduced to atomic formulas as follows, where we consider only the two subcases for $\Box$ and $\boxdot$, as the subcases for $\supset$ follow like those in Lemma 5:

$$\dfrac{\begin{matrix}\Pi_1\\ x{:}\Box A\end{matrix}\quad \begin{matrix}\Pi_2\\ x\mathsf{P}x\end{matrix}\quad \begin{matrix}\Pi_3\\ x\mathsf{P}y\end{matrix}}{y{:}\Box A}\,\mathsf{P}\mathit{sub1} \quad\rightsquigarrow\quad \dfrac{\dfrac{\begin{matrix}\Pi_1\\ x{:}\Box A\end{matrix}\quad \dfrac{[y\mathsf{U}z]^1\quad \begin{matrix}\Pi_2\\ x\mathsf{P}x\end{matrix}\quad \begin{matrix}\Pi_3\\ x\mathsf{P}y\end{matrix}}{x\mathsf{U}z}\,\mathsf{P}\mathit{sub2}}{z{:}A}\,\Box E}{y{:}\Box A}\,\Box I^1$$

$$\dfrac{\begin{matrix}\Pi_1\\ y{:}\Box A\end{matrix}\quad \begin{matrix}\Pi_2\\ x\mathsf{P}x\end{matrix}\quad \begin{matrix}\Pi_3\\ x\mathsf{P}y\end{matrix}}{x{:}\Box A}\,\mathsf{P}\mathit{sub2} \quad\rightsquigarrow\quad \dfrac{\dfrac{\begin{matrix}\Pi_1\\ y{:}\Box A\end{matrix}\quad \dfrac{[x\mathsf{U}z]^1\quad \begin{matrix}\Pi_2\\ x\mathsf{P}x\end{matrix}\quad \begin{matrix}\Pi_3\\ x\mathsf{P}y\end{matrix}}{y\mathsf{U}z}\,\mathsf{P}\mathit{sub1}}{z{:}A}\,\Box E}{x{:}\Box A}\,\Box I^1$$

$$\dfrac{\begin{matrix}\Pi_1\\ x{:}\boxdot A\end{matrix}\quad \begin{matrix}\Pi_2\\ x\mathsf{P}x\end{matrix}\quad \begin{matrix}\Pi_3\\ x\mathsf{P}y\end{matrix}}{y{:}\boxdot A}\,\mathsf{P}\mathit{sub1} \quad\rightsquigarrow\quad \dfrac{\dfrac{\begin{matrix}\Pi_1\\ x{:}\boxdot A\end{matrix}\quad \dfrac{[y\mathsf{P}z]^1\quad \begin{matrix}\Pi_2\\ x\mathsf{P}x\end{matrix}\quad \begin{matrix}\Pi_3\\ x\mathsf{P}y\end{matrix}}{x\mathsf{P}z}\,\mathsf{P}\mathit{sub2}}{z{:}A}\,\boxdot E}{y{:}\boxdot A}\,\boxdot I^1$$

$$\dfrac{\begin{matrix}\Pi_1\\ y{:}\boxdot A\end{matrix}\quad \begin{matrix}\Pi_2\\ x\mathsf{P}x\end{matrix}\quad \begin{matrix}\Pi_3\\ x\mathsf{P}y\end{matrix}}{x{:}\boxdot A}\,\mathsf{P}\mathit{sub2} \quad\rightsquigarrow\quad \dfrac{\dfrac{\begin{matrix}\Pi_1\\ y{:}\boxdot A\end{matrix}\quad \dfrac{[x\mathsf{P}z]^1\quad \begin{matrix}\Pi_2\\ x\mathsf{P}x\end{matrix}\quad \begin{matrix}\Pi_3\\ x\mathsf{P}y\end{matrix}}{y\mathsf{P}z}\,\mathsf{P}\mathit{sub1}}{z{:}A}\,\boxdot E}{x{:}\boxdot A}\,\boxdot I^1$$

For $class$, similarly to $\mathsf{M}ser$ in Lemma 5, we have

$$\dfrac{\begin{matrix}[x\mathsf{P}y]^1\ [y\mathsf{P}y]^1\\ \Pi\\ u{:}B\supset C\end{matrix}}{u{:}B\supset C}\,class^1 \quad\rightsquigarrow\quad \dfrac{\dfrac{\dfrac{\begin{matrix}[x\mathsf{P}x]^1\ [y\mathsf{P}y]^1\\ \Pi\\ u{:}B\supset C\end{matrix}\qquad [u{:}B]^2}{u{:}C}\,\supset E}{u{:}C}\,class^1}{u{:}B\supset C}\,\supset I^2$$

$$\dfrac{\begin{matrix}[x\mathsf{P}y]^1\ [y\mathsf{P}y]^1\\ \Pi\\ u{:}\bigstar A\end{matrix}}{u{:}\bigstar A}\,class^1 \quad\rightsquigarrow\quad \dfrac{\dfrac{\dfrac{\begin{matrix}[x\mathsf{P}y]^1\ [y\mathsf{P}y]^1\\ \Pi\\ u{:}\bigstar A\end{matrix}\qquad [u\mathsf{R}w]^2}{w{:}A}\,\bigstar E}{w{:}A}\,class^1}{w{:}A}\,\bigstar E^2$$

where we choose the parameter w so to allow for the application of $\bigstar I$.

By iterating these transformations, we transform an arbitrary MSpQS-derivation $\Gamma \vdash \alpha$ into an MSpQS-derivation of α from Γ where the conclusions of applications of RAA, $\bot E$, P$sub1$, P$sub2$ and $class$ are atomic. $\square$

The contractions that remove cut-formulas from a derivation in MSpQS are the same as the ones for MSQS, where in this case $\bigstar$ stands for $\square$ and $\boxdot$. Hence, proceeding as in the previous section, mutatis mutandis, we obtain a normalization result for MSpQS and the corresponding consequences.

Theorem 9. *Every MSpQS-derivation of $x{:}A$ from Γ reduces to an MSpQS-derivation in normal form.* $\square$

Theorem 10. *Every normal derivation of $x{:}A$ from Γ in MSpQS satisfies the subformula property.* $\square$

Corollary 2. *MSpQS is consistent.* $\square$

7 CONCLUSIONS AND FUTURE WORK

We have shown that our modal natural deduction systems MSQS and MSpQS provide suitable qualitative representations of quantum state transformations. We have also studied the proof theory of our systems showing that all derivations can be reduced to a normal form that satisfies a subformula property and yields a syntactic proof of the consistency of our systems. As future work, we plan to further investigate the proof theory of our systems, focusing in particular on (un)decidability, in view of a possible mechanization of reasoning in MSQS and MSpQS (e.g., encoding them into a logical framework [14]). Moreover, and perhaps more importantly, we are also working at suitable extensions of our approach in order to represent and reason about quantum entanglement phenomena and further quantum notions.

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