A visual introduction to Tilting

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Overview

1. Quivers and representations
2. Gabriel’s theorem
3. The Auslander-Reiten quiver of $A_3$
4. Some tilting representations and their endomorphism rings
5. Tilting representations and Happel’s theorem
Quivers and representations

Definition

- A quiver $Q$ is an oriented graph.
- We denote by $Q_0$ its vertices and by $Q_1$ its edges.
- The $\mathbb{C}$-vector space whose basis elements are all paths in $Q$ is denoted by $\mathbb{C}Q$.

Example

$$Q = \begin{array}{c} 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \end{array}$$

$Q_0 = \{1, 2, 3\}$ and $Q_1 = \{\alpha, \beta\}$

$\mathbb{C}Q$ is a six-dimensional $\mathbb{C}$-vector space with basis

$$\mathcal{P} = \{e_1, e_2, e_3, \alpha, \beta, \beta \alpha\},$$

where $e_1$, $e_2$ and $e_3$ are lazy paths and $\beta \alpha$ is the long path going from vertex 1 to vertex 3.
Quivers and representations

Example

\[ Q = \begin{array}{c}
1 \\
\gamma
\end{array} \]

\[ Q_0 = \{1\} \text{ and } Q_1 = \{\gamma\} \]

\(\mathbb{C}Q\) is an infinite-dimensional \(\mathbb{C}\)-vector space with basis

\[ \mathbb{P} = \{e_1, \gamma^n : n \in \mathbb{N}\}. \]

- The examples suggest a further operation on the vector space of paths: concatenation of paths. When concatenation is not possible, we set it to be zero!
- This is a multiplication in the vector space \(\mathbb{C}Q\). The sum of all the lazy paths acts as a multiplicative identity on any path.
- \(\mathbb{C}Q\) has, thus, a ring structure. We call \(\mathbb{C}Q\) the path algebra of \(Q\).
Quivers and representations

Example

\[ Q = \begin{array}{ccc}
1 & \alpha & 2 \\
& \beta & 3
\end{array} \]

\( \mathbb{C}Q \) is a finite-dimensional \( \mathbb{C} \)-vector space with basis 
\( P = \{ e_1, e_2, e_3, \alpha, \beta, \beta \alpha \} \). Given two elements:

\[ \Phi = \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 + \lambda_4 \alpha + \lambda_5 \beta + \lambda_6 \beta \alpha \]
\[ \Psi = \mu_1 e_1 + \mu_2 e_2 + \mu_3 e_3 + \mu_4 \alpha + \mu_5 \beta + \mu_6 \beta \alpha \]

with \( \lambda_i, \mu_i \) in \( \mathbb{C} \), the multiplication \( \Phi \Psi \) is defined distributively, multiplying the scalars and using the concatenation rules. For example:

\[ e_1 \alpha = 0, \quad \beta e_2 = \beta, \quad e_2 e_1 = 0 = e_1 e_2, \quad \beta \alpha = \beta \alpha. \]
Quivers and representations

Example

Exercise 1: Check that the path algebra $\mathbb{C}Q$ of the quiver

$$Q = \begin{array}{c}
1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3
\end{array}$$

is isomorphic to the ring

$$\begin{pmatrix}
\mathbb{C} & 0 & 0 \\
\mathbb{C} & \mathbb{C} & 0 \\
\mathbb{C} & \mathbb{C} & \mathbb{C}
\end{pmatrix}.$$

Exercise 2: Check that the path algebra $\mathbb{C}Q$ of the quiver

$$Q = \begin{array}{c}
1 \xleftarrow{\gamma} 2
\end{array}$$

is isomorphic to the polynomial ring $\mathbb{C}[X]$.

Exercise 3: Check that the path algebra $\mathbb{C}Q$ of a quiver $Q$ is a finite dimensional vector space if and only if $Q$ has no loops.
Quivers and representations

Definition

A representation of a quiver $Q$ is a pair $((V_i)_{i \in Q_0}, (f_\alpha)_{\alpha \in Q_1})$ where each $V_i$ is a $\mathbb{C}$-vector space and for any arrow $\alpha : i \to j$, $f_\alpha$ is a linear map $V_i \to V_j$.

Example

$Q = \begin{array}{ccc}
1 & \xrightarrow{\alpha} & 2 \\
& \xrightarrow{\beta} & 3
\end{array}$

The following are examples of representations:

$M := \begin{pmatrix}
1 & 0 \\
\end{pmatrix} : \mathbb{C}^2 \to \mathbb{C} \to 0$

$N := \begin{pmatrix}
1 \\
-1 \\
1
\end{pmatrix} : \mathbb{C} \to \mathbb{C}^3 \to \mathbb{C}^2$
Quivers and representations

Definition

A morphism between representations of a quiver $Q$

$$\phi : ((V_i)_{i \in Q_0}, (f_\alpha)_{\alpha \in Q_1}) \rightarrow ((W_i)_{i \in Q_0}, (g_\alpha)_{\alpha \in Q_1})$$

is a family $(\phi_i)_{i \in Q_0}$ of linear maps $\phi_i : V_i \rightarrow W_i$ such that, for any arrow $\alpha : i \rightarrow j$ in $Q_1$, the diagram commutes

$$\begin{array}{ccc}
V_i & \xrightarrow{f_\alpha} & V_j \\
\downarrow{\phi_i} & & \downarrow{\phi_j} \\
W_i & \xrightarrow{g_\alpha} & W_j
\end{array}$$

The morphism $\phi$ is said to be an **isomorphism** if all the $\phi_i$'s are isomorphisms of vector spaces.
This is a morphism between the representations $M$ and $N$. 

Quivers and representations

Example

$$Q = \begin{array}{ccc}
a & \xrightarrow{\alpha} & b & \xrightarrow{\beta} & c \\
\end{array}$$

$$\begin{array}{ccc}
\mathbb{C}^2 & \xrightarrow{\begin{pmatrix} 1 & 0 \end{pmatrix}} & \mathbb{C} & \xrightarrow{0} & \mathbb{C} \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{C} & \xrightarrow{\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}} & \mathbb{C}^3 & \xrightarrow{\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}} & \mathbb{C}^2 \\
\end{array}$$

This is a morphism between the representations $M$ and $N$. 

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Quivers and representations

**Definition**

A representation $M$ of a quiver $Q$ is said to be **indecomposable** if it is not isomorphic to the direct sum of two other representations.

**Example**

$Q = \begin{array}{c} a \xrightarrow{\alpha} b \xrightarrow{\beta} c \end{array}$, $M := \begin{array}{c} \begin{pmatrix} 1 & 0 \end{pmatrix} \end{array}$ is a decomposable representation as it can be written $P_1 \oplus P_2$, where

$P_1 := \begin{array}{c} \begin{pmatrix} 0 & 0 \end{pmatrix} \end{array}$

$P_2 := \begin{array}{c} \begin{pmatrix} 1 & 0 \end{pmatrix} \end{array}$

Throughout, we will work with quivers $Q$ that have **no loops** and our representations will be **finite dimensional**.
Gabriel’s theorem

How can we understand and classify (up to isomorphism) all the representations (and their morphisms) of a quiver $Q$?

**Theorem (Krull-Schmidt-Azumaya)**

*Every finite dimensional representation of a quiver decomposes uniquely as a direct sum of indecomposable representations.*

- We can, therefore, think of indecomposable representations as the atoms of the category of finite dimensional representations.
- There are also irreducible morphisms of representations, which provide a set of morphisms such that every other morphism can be built from them by forming compositions, linear combinations and matrices.
- A first problem is that there might be too many indecomposable representations.
Gabriel’s theorem

Definition

We say that a quiver $Q$ is of **finite representation type** if $Q$ has finitely many indecomposable representations (up to isomorphism).

- Gabriel’s theorem will say precisely which quivers have finite representation type.
- Among quivers of **infinite representation type**, there are two **subtypes**:
  - **Quivers of tame type**: Infinitely many indecomposable finite dimensional representations (up to isomorphism) but which are *possible to parametrise*;
  - **Quivers of wild type**: Infinitely many indecomposable finite dimensional representations (up to isomorphism) which *cannot be parametrised*. 
Gabriel’s theorem

Theorem

A quiver $Q$ is of finite representation type if and only if the underlying graph belongs to one of the following families of graphs:

- $A_n = 1 \rightarrow 2 \rightarrow 3 \rightarrow \cdots \rightarrow n-1 \rightarrow n$, $n \geq 1$
- $D_n = 1 \rightarrow 2 \rightarrow 3 \rightarrow \cdots \rightarrow n-2 \rightarrow n-1$, $n \geq 4$
- $E_6 = 1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5$
- $E_7 = 1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 6$
- $E_8 = 1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 6 \rightarrow 7$
Gabriel’s theorem

Example

How many indecomposable representations for each type?

- Type $A_n$, $n \geq 1$: $n(n + 1)/2$ indecomposable representations;
- Type $D_n$, $n \geq 4$: $n(n - 1)$ indecomposable representations;
- Type $E_6$, 36 indecomposable representations;
- Type $E_7$, 63 indecomposable representations;
- Type $E_8$, 120 indecomposable representations.

Example

- The quiver $1 \longrightarrow 2$ is of finite type.
- The quiver $1 \dashrightarrow 2$ is of tame type.
- The quiver $1 \rbrace\rbrace\rbrace\rbrace\longrightarrow 2$ is of wild type.
The Auslander-Reiten quiver of $A_3$

**Definition**

The **Auslander-Reiten quiver** of a quiver $Q$ is a quiver defined by:

- The vertices are the finite dimensional indecomposable representations of $Q$;
- The arrows are the irreducible morphisms between the indecomposable representations.

Consider the following quiver of type $A_3$,

\[ Q = 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3. \]

It is of finite representation type, by Gabriel’s theorem, and it has 6 indecomposable representations. We discuss its Auslander-Reiten quiver.
The Auslander-Reiten quiver of $A_3$

Indecomposable representations of $Q$:

- $P_1 := \mathbb{C} \rightarrow 0 \rightarrow 0$, sometimes denoted by $(1 0 0)$;
- $P_2 := \mathbb{C} \overset{1}{\rightarrow} \mathbb{C} \rightarrow 0$, sometimes denoted by $(1 1 0)$;
- $P_3 := \mathbb{C} \overset{1}{\rightarrow} \mathbb{C} \overset{1}{\rightarrow} \mathbb{C}$, sometimes denoted by $(1 1 1)$;
- $S_2 := 0 \rightarrow \mathbb{C} \rightarrow 0$, sometimes denoted by $(0 1 0)$;
- $I_2 := 0 \rightarrow \mathbb{C} \overset{1}{\rightarrow} \mathbb{C}$, sometimes denoted by $(0 1 1)$;
- $S_3 := 0 \rightarrow 0 \rightarrow \mathbb{C}$, sometimes denoted by $(0 0 1)$. 
The Auslander-Reiten quiver of $A_3$

Irreducible morphisms between representations of $Q$:
- An injective morphism from $P_1 = (1 \ 0 \ 0)$ to $P_2 = (1 \ 1 \ 0)$, defined by:

```
C → 0 → 0
|    |    |
1    ↓    ↓
|    |    |
C 1 → C → 0
```

Similar considerations give the following morphisms:
- An injective morphism from $P_2 = (1 \ 1 \ 0)$ to $P_3 = (1 \ 1 \ 1)$;
- A surjective morphism from $P_2 = (1 \ 1 \ 0)$ to $S_2 = (0 \ 1 \ 0)$;
- An injective morphism from $S_2 = (0 \ 1 \ 0)$ to $l_2 = (0 \ 1 \ 1)$;
- A surjective morphism from $P_3 = (1 \ 1 \ 1)$ to $l_2 = (0 \ 1 \ 1)$;
- A surjective morphism from $l_2 = (0 \ 1 \ 1)$ to $S_3 = (0 \ 0 \ 1)$. 
The Auslander-Reiten quiver of $A_3$

We are now ready to build the Auslander-Reiten quiver of $A_3$.

![Quiver Diagram]

- This quiver contains all the information about the category of finite dimensional representations of $Q$.
- The triples identifying the representations are called dimension vectors and they help us to keep in mind what the morphisms are.
Some tilting representations and their endomorphism rings

- Given finite dimensional representations $M$ and $N$ of a quiver $Q$, we denote by $\text{Hom}_Q(M, N)$ the set of morphisms of representations between $M$ and $N$.
- It is clear that $\text{Hom}_Q(M, N)$ is a $\mathbb{C}$-vector space.
- If $M = N$, we write $\text{End}_Q(M)$ for this space.
- $\text{End}_Q(M)$ has an additional operation: composition, which is distributive with respect to addition and commutes with scalar multiplication - i.e., $\text{End}_Q(M)$ has a ring structure. It is called the endomorphism ring of $M$. 
Some tilting representations and their endomorphism rings

Example (The tilting module $T = P_2 \oplus P_3 \oplus S_2$)

As before, let $Q = 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$. With the help of the Auslander-Reiten quiver, we can compute endomorphism rings of representations.

Let $T = P_2 \oplus P_3 \oplus S_2$. To compute $\text{End}_Q(T)$ we look at irreducible morphisms between the indecomposable summands of $T$. 
Some tilting representations and their endomorphism rings

Example (The tilting module $T = P_2 \oplus P_3 \oplus S_2$)

$T = P_2 \oplus P_3 \oplus S_2$

It turns out that $\text{End}_Q(T) \cong \mathbb{C}(1 \leftarrow 2 \rightarrow 3)$, where we identify the vertex 2 with the representation $P_2$ and the vertices 1 and 3 with the representations $P_3$ and $S_2$. 
Some tilting representations and their endomorphism rings

Example (The tilting module $T = P_2 \oplus P_3 \oplus S_2$)

The Auslander-Reiten quiver of $\text{End}_Q(T) \cong \mathbb{C}(1 \leftarrow 2 \rightarrow 3)$?

Can we relate it to the Auslander-Reiten quiver of Q?
Some tilting representations and their endomorphism rings

Example (The tilting module $V = I_2 \oplus P_3 \oplus S_2$)

$$V = I_2 \oplus P_3 \oplus S_2$$

It turns out that $\text{End}_Q(V) \cong \mathbb{C}(\xymatrix{1 & 2 & 3})$, where we identify the vertex 2 with the representation $I_2$ and the vertices 1 and 3 with the representations $P_3$ and $S_2$. 
Some tilting representations and their endomorphism rings

Example (The tilting module $V = l_2 \oplus P_3 \oplus S_2$)

The Auslander-Reiten quiver of $End_Q(V) \approx \mathbb{C}(1 \rightarrow 2 \leftarrow 3)$?

Can we relate it to the Auslander-Reiten quiver of $Q$?
Some tilting representations and their endomorphism rings

- The two representations $T$ and $V$ considered in the above examples are tilting representations.
- A tilting representation $M$ has good properties that allow to compare representations of $Q$ and representations of $\text{End}_Q(M)$.
- More precisely, it allows to compare the derived categories of representations of $Q$ and $\text{End}_Q(M)$ - denoted by $\mathcal{D}^b(Q)$ and $\mathcal{D}^b(\text{End}_Q(M))$, respectively.
- The Auslander-Reiten quiver of the derived category of a quiver $Q$ can be drawn by repetition of the Auslander-Reiten quiver of $Q$. 
The Auslander-Reiten quiver of $\mathcal{D}^b(Q)$ is obtained by repetition, where the colours represent:

- Auslander-Reiten quiver of $Q$ in position -1
- Auslander-Reiten quiver of $Q$ in position 0
- Auslander-Reiten quiver of $Q$ in position 1
- Auslander-Reiten quiver of $Q$ in position 2
Recall that $\text{End}_Q(T) \cong \mathbb{C}(Q')$, where $Q' = 1 \leftarrow 2 \rightarrow 3$. 

Example (Tilting representation $T = P_2 \oplus P_3 \oplus S_2$ over $Q = 1 \rightarrow 2 \rightarrow 3$)
**Tilting representations and Happel’s theorem**

**Example (Tilting representation $T = P_2 \oplus P_3 \oplus S_2$ over $Q = 1 \rightarrow 2 \rightarrow 3$)**

Recall that $\text{End}_Q(T) \cong \mathbb{C}(Q')$, where $Q' = 1 \leftarrow 2 \rightarrow 3$, and we know its Auslander-Reiten quiver.

If we draw its *repetition quiver*, then we get
Tilting representations and Happel’s theorem

Example (Tilting representation $T = P_2 \oplus P_3 \oplus S_2$ over $Q = 1 \rightarrow 2 \rightarrow 3$)

Recall that $\text{End}_Q(T) \cong \mathbb{C}(Q')$, where $Q' = 1 \leftarrow 2 \rightarrow 3$, and we know its Auslander-Reiten quiver. If we draw its repetition quiver, then we get the same quiver!, i.e., the derived categories $D^b(Q)$ and $D^b(Q')$ are equivalent.
Recall that $\text{End}_Q(T) \cong \mathbb{C}(Q'')$, where $Q'' = 1 \rightarrow 2 \leftarrow 3$. 

Example (Tilting representation $V = l_2 \oplus P_3 \oplus S_2$ over $Q = 1 \rightarrow 2 \rightarrow 3$)
Recall that $\text{End}_Q(T) \cong \mathbb{C}(Q'')$, where $Q'' = 1 \rightarrow 2 \leftarrow 3$, and we know its Auslander-Reiten quiver.
If we draw its repetition quiver, then we get
Recall that $\text{End}_Q(T) \cong \mathbb{C}(Q'')$, where $Q'' = 1 \to 2 \leftarrow 3$, and we know its Auslander-Reiten quiver.

If we draw its repetition quiver, then we get the same quiver!, i.e., the derived categories $D^b(Q)$ and $D^b(Q'')$ are equivalent.
Tilting representations and Happel’s theorem

**Definition (Tilting representation of a quiver)**

A finite dimensional representation $T$ of a quiver $Q$ is said to be tilting if

1. $\text{Ext}^1_Q(T, T) = 0$, i.e., every short exact sequence of representations of the form $0 \to T \to M \to T \to 0$ splits.

2. The number of indecomposable summands of $T$ equals the number of vertices in $Q$.

**Theorem (Happel, 1989)**

Let $T$ be a tilting representation of a quiver $Q$. Then $\mathcal{D}^b(Q)$ is equivalent to $\mathcal{D}^b(\text{End}_Q(T))$.

Note that $\text{End}_Q(T)$ is not always of the form $\mathbb{C}Q'$ for some quiver $Q'$. 
Tilting representations and Happel’s theorem

Example (The tilting representation \( W = P_1 \oplus P_3 \oplus S_3 \))

\[
W = P_1 \oplus P_3 \oplus S_3 \quad \text{over} \quad Q = \begin{array}{c}
1 \\
\rightarrow \\
2 \\
\rightarrow \\
3
\end{array}
\]

To understand \( \text{End}_Q(T) \), identify \( P_1 \), \( P_3 \) and \( S_3 \) with the vertices of a quiver but remember that the composition \( P_1 \rightarrow P_3 \rightarrow S_3 \) is a morphism between the representations \( P_1 = (1 \ 0 \ 0) \) and \( S_3 = (0 \ 0 \ 1) \),
Example (The tilting representation $W = P_1 \oplus P_3 \oplus S_3$)

$W = P_1 \oplus P_3 \oplus S_3$ over $Q = 1 \rightarrow 2 \rightarrow 3$

To understand $\text{End}_Q(T)$, identify $P_1$, $P_3$ and $S_3$ with the vertices of a quiver but remember that the composition $P_1 \rightarrow P_3 \rightarrow S_3$ is a morphism between the representations $P_1 = (1 \ 0 \ 0)$ and $S_3 = (0 \ 0 \ 1)$, i.e., it is the zero morphism. Thus, $\text{End}_Q(W) \cong \mathbb{C}(1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3) / < \beta \alpha >$, where $< \beta \alpha >$ is the ideal generated by the path $\beta \alpha$. 
Tilting representations and Happel’s theorem

Example (The tilting representation $W = P_1 \oplus P_3 \oplus S_3$)

Still, Happel’s theorem applies, and $D^b(Q) \cong D^b(\text{End}_Q(W))$, with

$$\text{End}_Q(W) \cong \mathbb{C}(1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3) / \langle \beta \alpha \rangle.$$ 

Representations of $\text{End}_Q(W)$ are representations $((M_i)_{i \in Q_0}, (f_\gamma)_{\gamma \in Q_1})$ of $Q$ satisfying the relation $f_\beta f_\alpha = 0$. 
References