A visual introduction to Tilting

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Overview

Quivers and representations

Definition

- A quiver Q is an oriented graph.
- We denote by Q_0 its vertices and by Q_1 its edges.
- The $\mathbb{C}$-vector space whose basis elements are all paths in Q is denoted by $\mathbb{C}Q$.

Example

$$Q = \qquad 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$$

$Q_0 = \{1, 2, 3\}$ and $Q_1 = \{\alpha, \beta\}$
$\mathbb{C}Q$ is a six-dimensional $\mathbb{C}$-vector space with basis

$$\mathbb{P} = \{e_1, e_2, e_3, \alpha, \beta, \beta\alpha\},$$

where e_1, e_2 and e_3 are lazy paths and $\beta\alpha$ is the long path going from vertex 1 to vertex 3.

Quivers and representations

Example

$$Q = \quad 1 \circlearrowright \gamma$$

$Q_0 = \{1\}$ and $Q_1 = \{\gamma\}$
$\mathbb{C}Q$ is an infinite-dimensional $\mathbb{C}$-vector space with basis

$$\mathbb{P} = \{e_1, \gamma^n : n \in \mathbb{N}\}.$$

- The examples suggest a further operation on the vector space of paths: concatenation of paths. When concatenation is not possible, we set it to be zero!
- This is a *multiplication* in the vector space $\mathbb{C}Q$. The sum of all the lazy paths acts as a multiplicative identity on any path.
- $\mathbb{C}Q$ has, thus, a ring structure. We call $\mathbb{C}Q$ the path algebra of Q.

Quivers and representations

Example

$$Q = \qquad 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$$

$\mathbb{C}Q$ is a finite-dimensional $\mathbb{C}$-vector space with basis $\mathbb{P} = \{e_1, e_2, e_3, \alpha, \beta, \beta\alpha\}$. Given two elements:

$$\Phi = \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 + \lambda_4 \alpha + \lambda_5 \beta + \lambda_6 \beta\alpha$$
$$\Psi = \mu_1 e_1 + \mu_2 e_2 + \mu_3 e_3 + \mu_4 \alpha + \mu_5 \beta + \mu_6 \beta\alpha$$

with λ_i, μ_i in $\mathbb{C}$, the multiplication $\Phi\Psi$ is defined distributively, multiplying the scalars and using the concatenation rules. For example:

$$e_1\alpha = 0, \quad \beta e_2 = \beta, \quad e_2 e_1 = 0 = e_1 e_2, \quad \beta\alpha = \beta\alpha.$$

Quivers and representations

Example

Exercise 1: Check that the path algebra $\mathbb{C}Q$ of the quiver

$$Q = \quad 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$$

is isomorphic to the ring $\begin{pmatrix} \mathbb{C} & 0 & 0 \\ \mathbb{C} & \mathbb{C} & 0 \\ \mathbb{C} & \mathbb{C} & \mathbb{C} \end{pmatrix}$.

Exercise 2: Check that the path algebra $\mathbb{C}Q$ of the quiver

$$Q = \quad 1 \circlearrowright \gamma$$

is isomorphic to the polynomial ring $\mathbb{C}[X]$.

Exercise 3: Check that the path algebra $\mathbb{C}Q$ of a quiver Q is a finite dimensional vector space if and only if Q has no loops.

Quivers and representations

Definition

A representation of a quiver Q is a pair $((V_i)_{i\in Q_0}, (f_\alpha)_{\alpha\in Q_1})$ where each V_i is a $\mathbb{C}$-vector space and for any arrow $\alpha : i \to j$, f_α is a linear map $V_i \to V_j$.

Example

$$Q = \quad 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$$

The following are examples of representations:

$$M := \quad \mathbb{C}^2 \xrightarrow{\begin{pmatrix} 1 & 0 \end{pmatrix}} \mathbb{C} \xrightarrow{0} 0$$

$$N := \quad \mathbb{C} \xrightarrow[\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}]{} \mathbb{C}^3 \xrightarrow[\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}]{} \mathbb{C}^2$$

Quivers and representations

Definition

A morphism between representations of a quiver Q

$$\phi : ((V_i)_{i \in Q_0}, (f_\alpha)_{\alpha \in Q_1}) \longrightarrow ((W_i)_{i \in Q_0}, (g_\alpha)_{\alpha \in Q_1})$$

is a family $(\phi_i)_{i \in Q_0}$ of linear maps $\phi_i : V_i \to W_i$ such that, for any arrow $\alpha : i \to j$ in Q_1, the diagram commutes

$$\begin{array}{ccc} V_i & \xrightarrow{f_\alpha} & V_j \\ \downarrow{\scriptstyle \phi_i} & & \downarrow{\scriptstyle \phi_j} \\ W_i & \xrightarrow{g_\alpha} & W_j \end{array}$$

The morphism ϕ is said to be an isomorphism if all the ϕ_i's are isomorphisms of vector spaces.

Quivers and representations

Example

$$Q = \qquad a \xrightarrow{\alpha} b \xrightarrow{\beta} c$$

$$\begin{array}{ccccc}
\mathbb{C}^2 & \xrightarrow{\begin{pmatrix}1 & 0\end{pmatrix}} & \mathbb{C} & \xrightarrow{0} & 0 \\
{\scriptstyle \begin{pmatrix}1 & 0\end{pmatrix}}\Big\downarrow & & \Big\downarrow{\scriptstyle \begin{pmatrix}1\\-1\\1\end{pmatrix}} & & \Big\downarrow{\scriptstyle 0} \\
\mathbb{C} & \xrightarrow[\begin{pmatrix}1\\-1\\1\end{pmatrix}]{} & \mathbb{C}^3 & \xrightarrow[\begin{pmatrix}1 & 1 & 0\\0 & 1 & 1\end{pmatrix}]{} & \mathbb{C}^2
\end{array}$$

This is a morphism between the representations M and N.

Quivers and representations

Definition

A representation M of a quiver Q is said to be indecomposable if it is not isomorphic to the direct sum of two other representations.

Example

$Q = \quad a \xrightarrow{\alpha} b \xrightarrow{\beta} c\,, \quad M := \quad \mathbb{C}^2 \xrightarrow{\begin{pmatrix} 1 & 0 \end{pmatrix}} \mathbb{C} \xrightarrow{0} 0$ is a decomposable representation as it can be written $P_1 \oplus P_2$, where

$$P_1 := \quad \mathbb{C} \xrightarrow{0} 0 \xrightarrow{0} 0$$

$$P_2 := \quad \mathbb{C} \xrightarrow{1} \mathbb{C} \xrightarrow{0} 0$$

Throughout, we will work with quivers Q that have no loops and our representations will be finite dimensional.

Gabriel's theorem

How can we understand and classify (up to isomorphism) *all the representations* (and their morphisms) of a quiver Q?

Theorem (Krull-Schmidt-Azumaya)

Every finite dimensional representation of a quiver decomposes uniquely as a direct sum of indecomposable representations.

- We can, therefore, think of indecomposable representations as the atoms of the *category of finite dimensional representations*.
- There are also irreducible morphisms of representations, which provide a set of morphisms such that every other morphism can be *built from them* by forming compositions, linear combinations and matrices.
- A first problem is that there might be too many indecomposable representations.

Gabriel's theorem

Definition

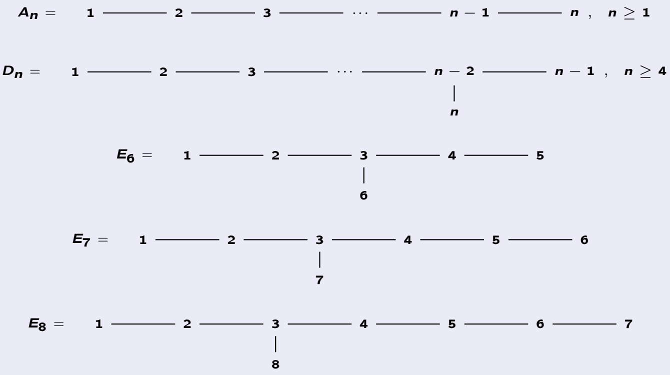
We say that a quiver Q is of finite representation type if Q has finitely many indecomposable representations (up to isomorphism).

- Gabriel's theorem will say precisely which quivers have finite representation type.
- Among quivers of infinite representation type, there are two *subtypes*:
  - Quivers of tame type: Infinitely many indecomposable finite dimensional representations (up to isomorphism) but which are *possible to parametrise*;
  - Quivers of wild type: Infinitely many indecomposable finite dimensional representations (up to isomorphism) which *cannot be parametrised*.

Gabriel's theorem

Theorem

A quiver Q is of finite representation type if and only if the underlying graph belongs to one of the following families of graphs:



Gabriel's theorem

Example

How many indecomposable representations for each type?

- Type A_n, $n \geq 1$: $n(n+1)/2$ indecomposable representations;
- Type D_n, $n \geq 4$: $n(n-1)$ indecomposable representations;
- Type E_6, 36 indecomposable representations;
- Type E_7, 63 indecomposable representations;
- Type E_8, 120 indecomposable representations.

Example

- The quiver $1 \longrightarrow 2$ is of finite type.
- The quiver $1 \rightrightarrows 2$ is of tame type.
- The quiver $1 \overset{\longrightarrow}{\rightrightarrows} 2$ is of wild type.

The Auslander-Reiten quiver of A_3

Definition

The Auslander-Reiten quiver of a quiver Q is a quiver defined by:

- The vertices are the finite dimensional indecomposable representations of Q;
- The arrows are the irreducible morphisms between the indecomposable representations.

Consider the following quiver of type A_3,

$$Q = 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3.$$

It is of finite representation type, by Gabriel's theorem, and it has 6 indecomposable representations. We discuss its Auslander-Reiten quiver.

The Auslander-Reiten quiver of A_3

Indecomposable representations of Q:

- $P_1 := \mathbb{C} \longrightarrow 0 \longrightarrow 0$, sometimes denoted by (1 0 0);
- $P_2 := \mathbb{C} \xrightarrow{1} \mathbb{C} \longrightarrow 0$, sometimes denoted by (1 1 0);
- $P_3 := \mathbb{C} \xrightarrow{1} \mathbb{C} \xrightarrow{1} \mathbb{C}$, sometimes denoted by (1 1 1);
- $S_2 := 0 \longrightarrow \mathbb{C} \longrightarrow 0$, sometimes denoted by (0 1 0);
- $I_2 := 0 \longrightarrow \mathbb{C} \xrightarrow{1} \mathbb{C}$, sometimes denoted by (0 1 1);
- $S_3 := 0 \longrightarrow 0 \longrightarrow \mathbb{C}$, sometimes denoted by (0 0 1).

The Auslander-Reiten quiver of A_3

Irreducible morphisms between representations of Q:

- An injective morphism from $P_1 = (1\ 0\ 0)$ to $P_2 = (1\ 1\ 0)$, defined by:

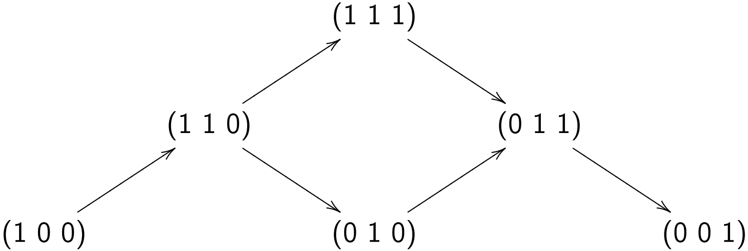
$$\begin{array}{ccccc} \mathbb{C} & \longrightarrow & 0 & \longrightarrow & 0 \\ {\scriptstyle 1}\downarrow & & \downarrow & & \downarrow \\ \mathbb{C} & \xrightarrow[1]{} & \mathbb{C} & \longrightarrow & 0 \end{array}$$

Similar considerations give the following morphisms:

- An injective morphism from $P_2 = (1\ 1\ 0)$ to $P_3 = (1\ 1\ 1)$;
- A surjective morphism from $P_2 = (1\ 1\ 0)$ to $S_2 = (0\ 1\ 0)$;
- An injective morphism from $S_2 = (0\ 1\ 0)$ to $I_2 = (0\ 1\ 1)$;
- A surjective morphism from $P_3 = (1\ 1\ 1)$ to $I_2 = (0\ 1\ 1)$;
- A surjective morphism from $I_2 = (0\ 1\ 1)$ to $S_3 = (0\ 0\ 1)$.

The Auslander-Reiten quiver of A_3

We are now ready to build the Auslander-Reiten quiver of A_3.



- This quiver contains all the information about the *category of finite dimensional representations of* Q.
- The triples identifying the representations are called dimension vectors and they help us to keep in mind what the morphisms are.

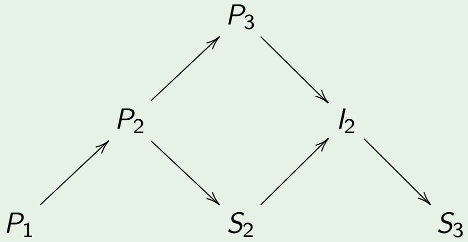
Some tilting representations and their endomorphism rings

- Given finite dimensional representations M and N of a quiver Q, we denote by $Hom_Q(M, N)$ the set of morphisms of representations between M and N.
- It is clear that $Hom_Q(M, N)$ is a $\mathbb{C}$-vector space.
- If $M = N$, we write $End_Q(M)$ for this space.
- $End_Q(M)$ has an additional operation: composition, which is distributive with respect to addition and commutes with scalar multiplication - i.e., $End_Q(M)$ has a ring structure. It is called the endomorphism ring of M.

Some tilting representations and their endomorphism rings

Example (The tilting module $T = P_2 \oplus P_3 \oplus S_2$)

As before, let $Q = 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$. With the help of the Auslander-Reiten quiver, we can compute endomorphism rings of representations.

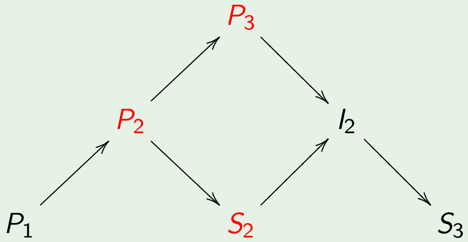


Let $T = P_2 \oplus P_3 \oplus S_2$. To compute $End_Q(T)$ we look at irreducible morphisms between the indecomposable summands of T.

Some tilting representations and their endomorphism rings

Example (The tilting module $T = P_2 \oplus P_3 \oplus S_2$)

$T = P_2 \oplus P_3 \oplus S_2$

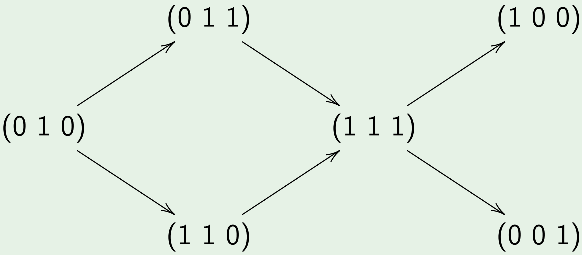


It turns out that $End_Q(T) \cong \mathbb{C}(\ 1 \longleftarrow 2 \longrightarrow 3\ )$, where we *identify* the vertex 2 with the representation P_2 and the vertices 1 and 3 with the representations P_3 and S_2.

Some tilting representations and their endomorphism rings

Example (The tilting module $T = P_2 \oplus P_3 \oplus S_2$)

The Auslander-Reiten quiver of $End_Q(T) \cong \mathbb{C}(\ 1 \longleftarrow 2 \longrightarrow 3\ )$?

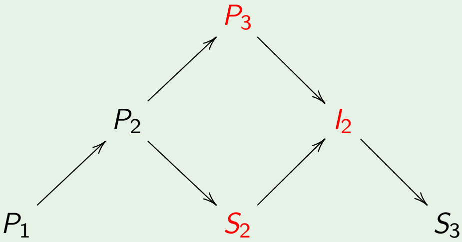


Can we relate it to the Auslander-Reiten quiver of Q?

Some tilting representations and their endomorphism rings

Example (The tilting module $V = I_2 \oplus P_3 \oplus S_2$)

$V = I_2 \oplus P_3 \oplus S_2$

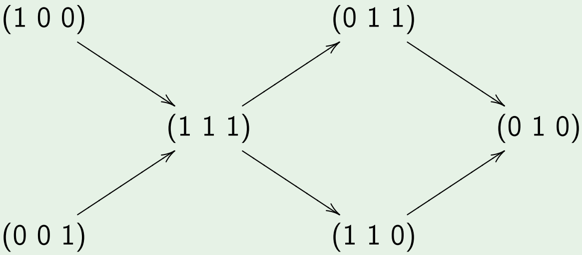


It turns out that $End_Q(V) \cong \mathbb{C}(\ 1 \longrightarrow 2 \longleftarrow 3\ )$, where we *identify* the vertex 2 with the representation I_2 and the vertices 1 and 3 with the representations P_3 and S_2.

Some tilting representations and their endomorphism rings

Example (The tilting module $V = I_2 \oplus P_3 \oplus S_2$)

The Auslander-Reiten quiver of $End_Q(V) \cong \mathbb{C}(1 \longrightarrow 2 \longleftarrow 3)$?



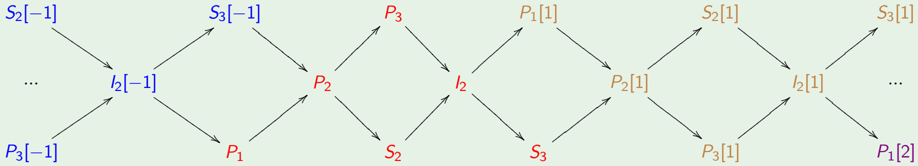
Can we relate it to the Auslander-Reiten quiver of Q?

Some tilting representations and their endomorphism rings

- The two representations T and V considered in the above examples are tilting representations.
- A tilting representation M has good properties that allow to *compare* representations of Q and representations of $End_Q(M)$.
- More precisely, it allows to compare the derived categories of representations of Q and $End_Q(M)$ - denoted by $\mathcal{D}^b(Q)$ and $\mathcal{D}^b(End_Q(M))$, respectively.
- The Auslander-Reiten quiver of the derived category of a quiver Q can be drawn by *repetition* of the Auslander-Reiten quiver of Q.

Tilting representations and Happel's theorem

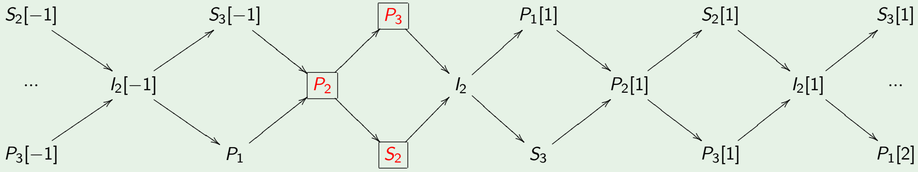
Example (Auslander-Reiten quiver of $\mathcal{D}^b(\mathbb{C}Q)$, $Q = 1 \longrightarrow 2 \longrightarrow 3$)



The Auslander-Reiten quiver of $\mathcal{D}^b(Q)$ is obtained *by repetition*, where the colours represent:
Auslander-Reiten quiver of Q in position -1
Auslander-Reiten quiver of Q in position 0
Auslander-Reiten quiver of Q in position 1
Auslander-Reiten quiver of Q in position 2

Tilting representations and Happel's theorem

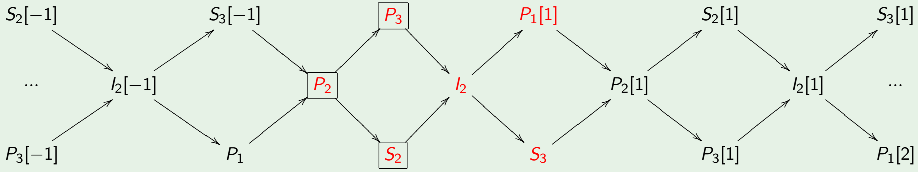
Example (Tilting representation $T = P_2 \oplus P_3 \oplus S_2$ over $Q = 1 \longrightarrow 2 \longrightarrow 3$)



Recall that $End_Q(T) \cong \mathbb{C}(Q')$, where $Q' = 1 \longleftarrow 2 \longrightarrow 3$,

Tilting representations and Happel's theorem

Example (Tilting representation $T = P_2 \oplus P_3 \oplus S_2$ over $Q = 1 \longrightarrow 2 \longrightarrow 3$)

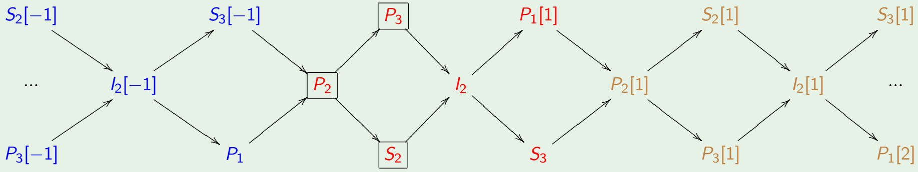


Recall that $End_Q(T) \cong \mathbb{C}(Q')$, where $Q' = 1 \longleftarrow 2 \longrightarrow 3$, and we know its Auslander-Reiten quiver.

If we draw its *repetition quiver*, then we get

Tilting representations and Happel's theorem

Example (Tilting representation $T = P_2 \oplus P_3 \oplus S_2$ over $Q = 1 \longrightarrow 2 \longrightarrow 3$)

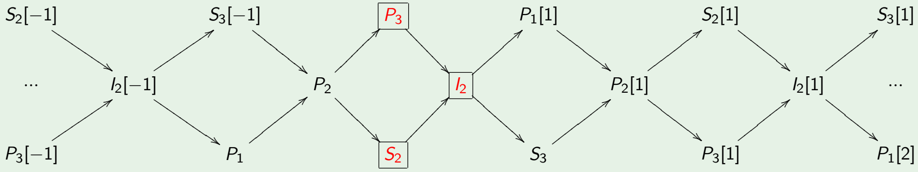


Recall that $End_Q(T) \cong \mathbb{C}(Q')$, where $Q' = 1 \longleftarrow 2 \longrightarrow 3$, and we know its Auslander-Reiten quiver.

If we draw its *repetition quiver*, then we get **the same quiver!**, i.e., the derived categories $D^b(Q)$ and $D^b(Q')$ are equivalent.

Tilting representations and Happel's theorem

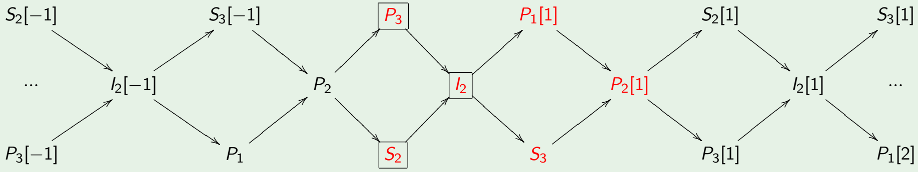
Example (Tilting representation $V = I_2 \oplus P_3 \oplus S_2$ over $Q = 1 \longrightarrow 2 \longrightarrow 3$)



Recall that $End_Q(T) \cong \mathbb{C}(Q'')$, where $Q'' = 1 \longrightarrow 2 \longleftarrow 3$,

Tilting representations and Happel's theorem

Example (Tilting representation $T = P_2 \oplus P_3 \oplus S_2$ over $Q = 1 \longrightarrow 2 \longrightarrow 3$)

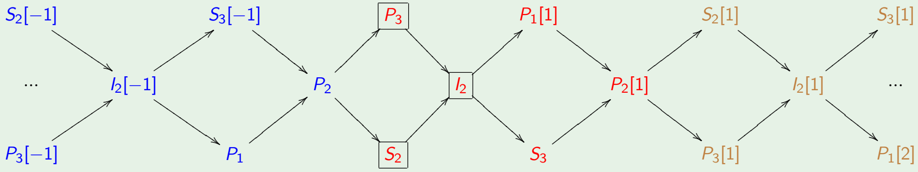


Recall that $End_Q(T) \cong \mathbb{C}(Q'')$, where $Q'' = 1 \longrightarrow 2 \longleftarrow 3$, and we know its Auslander-Reiten quiver.

If we draw its *repetition quiver*, then we get

Tilting representations and Happel's theorem

Example (Tilting representation $T = P_2 \oplus P_3 \oplus S_2$ over $Q = 1 \longrightarrow 2 \longrightarrow 3$)



Recall that $End_Q(T) \cong \mathbb{C}(Q'')$, where $Q'' = 1 \longrightarrow 2 \longleftarrow 3$, and we know its Auslander-Reiten quiver.

If we draw its *repetition quiver*, then we get **the same quiver!**, i.e., the derived categories $D^b(Q)$ and $D^b(Q'')$ are equivalent.

Tilting representations and Happel's theorem

Definition (Tilting representation of a quiver)

A finite dimensional representation T of a quiver Q is said to be tilting if

1. $Ext^1_Q(T, T) = 0$, i.e., every short exact sequence of representations of the form $0 \to T \to M \to T \to 0$ splits.
2. The number of indecomposable summands of T equals the number of vertices in Q.

Theorem (Happel, 1989)

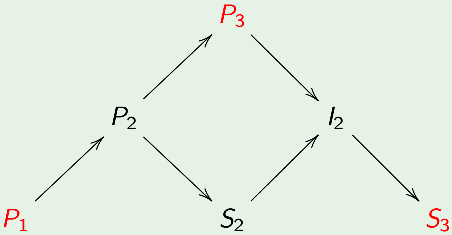
Let T be a tilting representation of a quiver Q. Then $D^b(Q)$ is equivalent to $D^b(End_Q(T))$.

Note that $End_Q(T)$ is not always of the form $\mathbb{C}Q'$ for some quiver Q'.

Tilting representations and Happel's theorem

Example (The tilting representation $W = P_1 \oplus P_3 \oplus S_3$)

$W = P_1 \oplus P_3 \oplus S_3$ over $Q = 1 \longrightarrow 2 \longrightarrow 3$

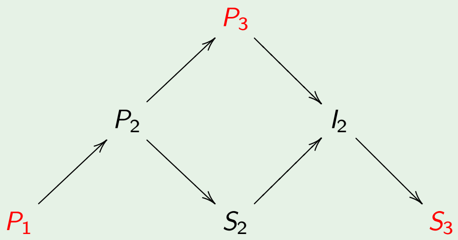


To understand $End_Q(T)$, identify P_1, P_3 and S_3 with the vertices of a quiver but remember that the composition $P_1 \to P_3 \to S_3$ is a morphism between the representations $P_1 = (1\ 0\ 0)$ and $S_3 = (0\ 0\ 1)$,

Tilting representations and Happel's theorem

Example (The tilting representation $W = P_1 \oplus P_3 \oplus S_3$)

$W = P_1 \oplus P_3 \oplus S_3$ over $Q = 1 \longrightarrow 2 \longrightarrow 3$



To understand $End_Q(T)$, identify P_1, P_3 and S_3 with the vertices of a quiver but remember that the composition $P_1 \to P_3 \to S_3$ is a morphism between the representations $P_1 = (1\ 0\ 0)$ and $S_3 = (0\ 0\ 1)$, i.e., it is the zero morphism. Thus,

$End_Q(W) \cong \mathbb{C}(\ 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3\ )/ < \beta\alpha >$,
where $< \beta\alpha >$ is the ideal generated by the path $\beta\alpha$.

Tilting representations and Happel's theorem

Example (The tilting representation $W = P_1 \oplus P_3 \oplus S_3$)

Still, Happel's theorem applies, and $D^b(Q) \cong D^b(End_Q(W))$, with

$$End_Q(W) \cong \mathbb{C}(1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3)/ < \beta\alpha > .$$

Representations of $End_Q(W)$ are representations $((M_i)_{i \in Q_0}, (f_\gamma)_{\gamma \in Q_1})$ of Q satisfying the relation $f_\beta f_\alpha = 0$.

References

- Angeleri Hügel, L., Happel, D., Krause, H., *Handbook of tilting theory*, London Mathematical Society, 2007;
- Assem, I., Simson, D., Skowronski, A., *Elements of the Representation Theory of Associative Algebras, 1: Techniques of Representation Theory*, London Mathematical Society, 2006;
- Auslander, M., Reiten, I., Smalø, S., *Representation Theory of Artin Algebras*, Cambridge University Press, 1997;
- Happel, D., *Triangulated categories in the representation theory of finite dimensional algebras*, London Mathematical Society, 1988.