Chapter 3

Mutations of quivers with potentials and derived equivalences

3.1 Introduction

For a quiver with potential, Derksen, Weyman and Zelevinsky defined in 2008 ([DWZ08]) a combinatorial transformation - mutations. Mukhopadhyay and Ray, on the other hand, tell us how to compute Seiberg dual quivers for some quivers with potential through a tilting procedure, thus obtaining derived equivalent algebras ([MR04]). In this chapter, we compare mutations with this approach to Seiberg duality, concluding that for a certain class of potentials and under certain conditions they coincide. Therefore mutations provide us with some derived equivalences.

A broad class of noncommutative algebras can be presented as a path algebra of a quiver with relations. We shall be studying the derived categories of some of these algebras, namely when their relations can be suitably encoded on a potential via cyclic derivatives as follows. Q_0 and Q_1 denotes the sets of vertices and arrows, respectively, of a quiver Q. $\mathbb{K}Q$ is the path algebra of the quiver Q over $\mathbb{K}$ and our convention is to write concatenation of paths as composition of functions. The following definitions are due to Derksen, Weyman and Zelevisnky ([DWZ08]).

Definition 3.1.1. *A* ***potential*** *on a quiver is an element of the vector space spanned by the cycles of the quiver (denote it by $\mathbb{K}Q_{cyc}$).*

Remark 3.1.2. *We will assume throughout this chapter, unless otherwise stated, that every cycle in any potential S is simple, i.e., it does not pass through the same vertex twice.*

Definition 3.1.3. *Let $A = \langle Q_1 \rangle$, i.e., the vector space spanned by all arrows. For each $\xi \in A^*$ (the dual of A), define a* ***cyclic derivative****:*

$$\partial/\partial\xi : \begin{array}{ccc} \mathbb{K}Q_{cyc} & \rightarrow & \mathbb{K}Q \\ a_1 \ldots a_n & \mapsto & \sum_{k=1}^{n} \xi(a_k) a_{k+1} \ldots a_n a_1 \ldots a_{k-1} \end{array}.$$

If $x \in Q_1$, we will denote by $\partial/\partial x$ the cyclic derivative correspondent to the element of A^* which is the dual of x in the dual basis of A. Potentials are regarded as a way to encode the relations of certain path algebras, when the relations are precisely given by the ideal generated by all the cyclic derivatives. Different potentials can, however, define the same set of relations. For example, the same cycle can be written with different starting vertices even though its cyclic derivatives do not depend on such choices. To identify these, the following equivalence relation is introduced.

Definition 3.1.4. *Two potentials are* ***cyclically equivalent*** *if $S - S'$ lies in the span of elements of the form $a_1 \ldots a_{n-1} a_n - a_2 \ldots a_n a_1$. A pair (Q, S) is said to* *be a **quiver with potential** if Q has no loops and no two cyclically equivalent paths appear on S.*

The following notion of (strong) right equivalence will be central in our discussion. However, one needs at this point to introduce the notion of complete path algebra. Recall that $\mathbb{K}Q$ can be seen as $\bigoplus_{i=0}^{\infty} A^i$.

Definition 3.1.5. *The **complete path algebra** is defined as $\widehat{\mathbb{K}Q} := \prod_{i=0}^{\infty} A^i$.*

Definition 3.1.6. *Two quivers with potentials (Q, S) and (Q', S') are said to be **right equivalent** if there is isomorphism ϕ between $\widehat{\mathbb{K}Q}$ and $\widehat{\mathbb{K}Q'}$ such that $\phi(S)$ is cyclically equivalent to S'. We shall say that they are **strongly right equivalent** if we can take ϕ to be an isomorphism between $\mathbb{K}Q$ and $\mathbb{K}Q'$ such that $\phi(S)$ is cyclically equivalent to S'.*

In particular it is clear that strong right equivalence implies right equivalence. We now introduce the algebras of our focus in this chapter.

Definition 3.1.7. *Given a quiver with potential (Q, S), define the **Jacobian algebra** of (Q, S) as $J(Q, S) = \mathbb{K}Q/\langle J(S)\rangle$, where $J(S) = (\partial S/\partial x)_{x\in Q_1}$. We call $\widehat{J(Q, S)} = \widehat{\mathbb{K}Q}/\langle\langle J(S)\rangle\rangle$ the **complete Jacobian algebra**, where $\langle\langle J(S)\rangle\rangle$ is the closure of the ideal generated by $J(S)$ in $\widehat{\mathbb{K}Q}$ in the m-adic topology, for m the maximal in $\widehat{\mathbb{K}Q}$ generated by all arrows.*

Remark 3.1.8. *Note that two strongly right equivalent quivers with potentials have isomorphic Jacobian algebras while two right equivalent ones have isomorphic complete Jacobian algebras ([DWZ08]).*

A very interesting class of examples arises naturally in toric geometry and homological mirror symmetry ([UY07]). These examples are constructed from bipartite graphs on the torus as we now explain. Let G be a bipartite graph embedded on a torus T, with the two sets of vertices being called W (white) and B (black). We can construct a quiver Q and a potential S as follows:

- The vertices of Q are the faces of G, i.e., the connected component of $T \setminus G$;
- There is an arrow between two vertices of Q if the corresponding faces of G share a common edge;
- The direction of the arrow a in Q is determined by the convention that the white vertex of the corresponding edge in G lies on the right side of a;
- The terms of the potential are the cycles that go around each vertex of G, assigning positive signs to those coming from white vertices and negative sign otherwise.

In some cases the quivers with potential obtained in this way are derived equivalent to toric varieties combinatorially related to the bipartite graphs ([UY07]). To get a quiver with potential we must ensure that no loops are allowed. For this we require the embedding of G to be such that each edge separates two distinct faces.

In the next section we will define mutation and Seiberg duality for a quiver with potential followed by some results on the links between them in section 3.3. Section 3.4 explores an example of algebro-geometric nature and we end this chapter by discussing the results of 3.3 in the 3-Calabi-Yau context.

3.2 Mutation and Seiberg Duality

For a quiver with potential (Q, S), $K^b(Q, S)$ and $D^b(Q, S)$ will denote, respectively, the bounded homotopy category and the bounded derived category of right modules over $J(Q,S)$. Given an arrow $\alpha \in Q_1$, let $t(\alpha)$ denote the **target** of α and $s(\alpha)$ the **source** of α (i.e., the arrival and departure vertices, respectively).

It is well known (see [Hap87]) that given a path algebra, reflection functors on vertices that are either sources (i.e., vertices with no incoming arrows) or sinks (i.e., vertices with no outgoing arrows) provide us with derived equivalences. Our aim is to identify some derived equivalent algebras and hence we shall consider a generalisation of these reflection functors, DWZ-mutations, for which we need first the following definition and theorem ([DWZ08]).

Definition 3.2.1. *A potential S (or a quiver with potential (Q,S)) is said to be **trivial** if it is homogeneous of degree 2, i.e., if it is constituted only by 2-cycles. A potential S (or a quiver with potential (Q,S)) is said to be **reduced** if it has no 2-cycles. For a quiver with potential (Q,S), if m is the ideal generated by the arrows in $\mathbb{K}Q$, we define m_{triv} as the ideal generated by arrows appearing in the two-cycles of the potential and $m_{red} = m/m_{triv}$.*

Note that, since we assume that all cycles in the potential are simple (i.e., no cycle in the potential passes through the same vertex twice), each 2-cycle of S is a summand of S.

Theorem 3.2.2 (Derksen, Weyman, Zelevinsky). *For a quiver with potential (Q,S), there exist a trivial quiver with potential (Q_{triv}, S_{triv}) (where the arrows of Q_{triv} generate m_{triv}) and a reduced quiver with potential (Q_{red}, S_{red}) (where the arrows in Q_{red} generate m_{red}) such that (Q,S) is right equivalent to $(Q_{triv} \oplus Q_{red}, S_{triv} + S_{red})$ (where the arrows in $Q_{triv} \oplus Q_{red}$ generate $m_{triv} \oplus m_{red}$).*

We can now describe the procedure of mutation of a quiver with potential (Q,S) on a vertex k (denote it by $\mu_k(Q,S)$).

1. Suppose k does not belong to any 2-cycle and that S does not have any cycle starting and finishing on k (if it does, substitute it by a cyclically equivalent potential that does not).

2. Change the quiver in the following way:

   - Reflect arrows starting or ending at k. Denote reflected arrows by $(.)^*$;
   - Create one new arrow for each path $\beta\alpha$ of length two, with $\alpha, \beta \in Q_1$ such that $t(\alpha) = s(\beta) = k$ and denote it by $[\beta\alpha]$.

   We denote the resulting quiver by $\tilde{Q}$.

3. Change the potential in the following way:

   - Substitute factors appearing in S of the form $\beta\alpha$ with middle vertex k by the new arrow $[\beta\alpha]$ and denote it by $[S]$;
   - Define $\tilde{S} := \Delta_k + [S]$ where $\Delta_k = \sum_{t(\alpha)=s(\beta)=k} [\beta\alpha]\alpha^*\beta^*$.

4. The mutation at k of (Q, S) is $\mu_k(Q, S) = (\bar{Q}, \bar{S}) := (\tilde{Q}_{red}, \tilde{S}_{red})$.

We proceed to define Seiberg duality ([MR04]). This is a tilting procedure and therefore it is an equivalence of derived categories. To check if a complex is tilting we will have to compute homomorphisms in the derived category between (finitely generated) projective modules. For this we will use remark 2.1.10.

From now on, we will assume that (Q, S) is a quiver with potential with n vertices such that every vertex is contained in some cycle. Let P_i be the projective right module over $J(Q, S)$ associated to the vertex i of Q, i.e., $P_i = e_i J(Q, S)$ where e_i is the stationary path on vertex i. For each vertex k, consider the following complex:

$$T^k = \bigoplus_{i=1}^{n} T_i^k$$

where

$$T_i^k = 0 \longrightarrow P_i \longrightarrow 0, \text{ if } i \neq k$$

(P_i is in degree zero of the complex) and

$$T_k^k = 0 \longrightarrow \bigoplus_{t(\alpha)=k} P_{s(\alpha)} \xrightarrow{(\alpha)} P_k \longrightarrow 0$$

($\bigoplus_{t(\alpha)=k} P_{s(\alpha)}$ is in degree zero of the complex), where (α) denotes the morphism defined in each component of the direct sum by

$$P_{s(\alpha)} \longrightarrow P_k : \ u \mapsto \alpha u.$$

Remark 3.2.3. *We observe that the projective modules $P_i = e_i J(Q,S)$ are indecomposable. This argument is due to Dong Yang and the result follows as a consequence of a lemma proved by Keller and Yang ([KY10]). In their paper, they observe that the projective modules $e_i\Gamma(Q,S)$ associated with the Ginzburg algebra $\Gamma(Q,S)$ - a differential graded algebra defined such that $H^0\Gamma(Q,S) = \widehat{J(Q,S)}$ - are indecomposable (indeed, they prove more: the perfect derived category, $per(\Gamma)$, is Krull-Schmidt). Hence, since*

$$\mathrm{Hom}_{D^b(\Gamma)}(e_i\Gamma, e_i\Gamma) = e_i H^0\Gamma e_i = \mathrm{Hom}_{H^0\Gamma}(e_iH^0\Gamma, e_iH^0\Gamma),$$

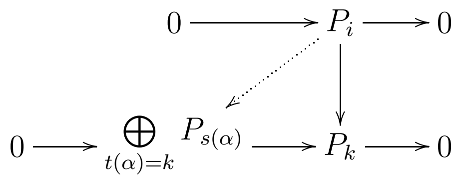
the endomorphism algebra of $\hat{P}_i = e_i\widehat{J(Q,S)} = P_i \otimes_{J(Q,S)} \widehat{J(Q,S)}$ is local and hence $\hat{P}_i$ is indecomposable. This implies that P_i is also indecomposable.

Lemma 3.2.4. *T^k is a tilting complex over the Jacobian algebra of (Q,S) if and only if Hom$_{K^b(P(J(Q,S)))}(T_k^k, T_s^k[-1]) = 0$, $\forall s \in Q_0$.*

Proof. We split the proof into two parts: homomorphism vanishing and generation.

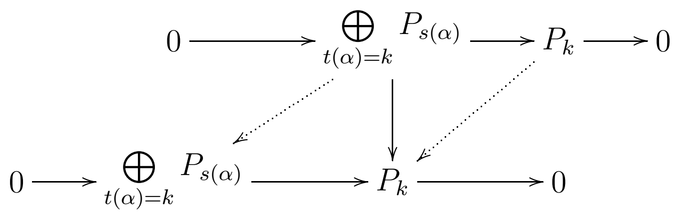
First we prove that Hom$_{K^b(P(J(Q,S)))}(T_r^k, T_s^k[i]) = 0$, for all $i \neq 0$ if $r \neq k$ and for all $i > 0$ if $r = k$.

It is clear that if $r, s \neq k$, then $\text{Hom}_{K^b(P(J(Q,S)))}(T_r^k, T_s^k[i]) = 0$, for all $i \neq 0$ (as this is some higher Ext-group between projectives). Now, suppose $s = k$ and $r \neq k$. We only have to check that $\text{Hom}_{K^b(P(J(Q,S)))}(T_r^k, T_k^k[1])$ is trivial. Note that, since a homomorphism between P_r to P_k is identified with an element of the path algebra with each term being a path from r to k, every such homomorphism factors through $\bigoplus_{t(\alpha)=k} P_{s(\alpha)}$.



This factorisation implies that such maps are homotopic to zero, thus zero in the homotopy category.

If $s = r = k$ then we also have such a homotopy just by taking identity maps.



Secondly we check that T^k generates $K^b(P(J(Q,S)))$ as a triangulated category.

It is enough to prove that the stalk complexes of indecomposable projective modules are generated by the direct summands of T^k.

Consider the direct summands of T^k and take the cone of the map from

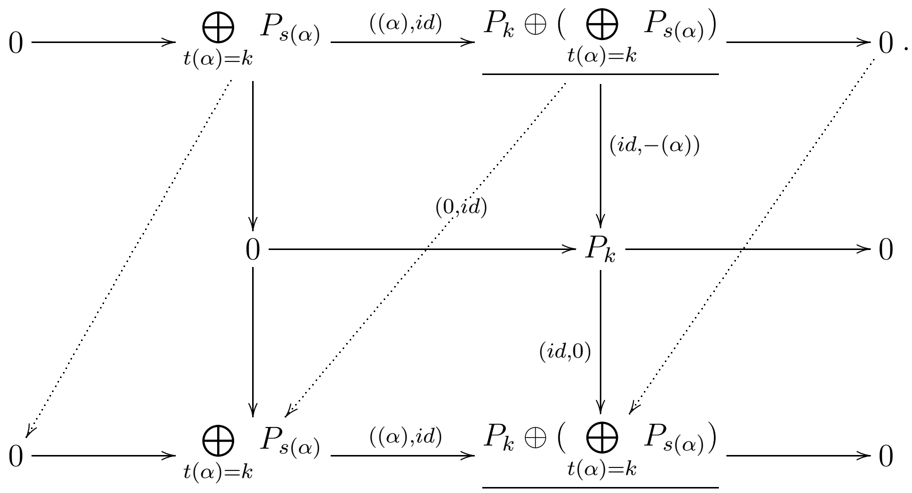
T_k^k to $\bigoplus\limits_{t(\alpha)=k} T^k_{s(\alpha)}$ defined by:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \bigoplus\limits_{t(\alpha)=k} P_{s(\alpha)} & \longrightarrow & P_k & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle id} & & & & \\ 0 & \longrightarrow & \bigoplus\limits_{t(\alpha)=k} P_{s(\alpha)} & \longrightarrow & 0 & & \end{array}$$

That cone is just the following complex (the underlined term is in degree zero):

$$0 \longrightarrow \bigoplus_{t(\alpha)=k} P_{s(\alpha)} \xrightarrow{((\alpha),id)} \underline{P_k \oplus (\bigoplus_{t(\alpha)=k} P_{s(\alpha)})} \longrightarrow 0 \qquad (3.2.1)$$

Consider the map from the complex (3.2.1) to the stalk complex of P_k in degree zero defined by identity in the first component and $-(\alpha)$ in the second component. Consider also the map from this same stalk complex to (3.2.1) defined by the inclusion of P_k. We observe that the composition of these maps is homotopic to the identity map, hence proving that these complexes are isomorphic in the derived category. In fact, that follows from the following diagram:



Similarly we can see the same phenomenon for the reverse composition and therefore (3.2.1) is isomorphic to the stalk complex P_k in degree zero.

Therefore, the complex is tilting if and only if the remaining conditions, i.e., $\mathrm{Hom}_{K^b(P(J(Q,S)))}(T_k^k, T_s^k[-1]) = 0$, for all $s \in Q_0$, are verified. $\square$

Definition 3.2.5. *Given a quiver with potential (Q, S), define $\delta(Q, S)$ as the set of vertices for which the complex above is tilting over $J(Q, S)$, i.e.,*

$$\delta(Q,S) = \{k \in Q_0 : Hom_{K^b(P(J(Q,S)))}(T^k_k, T^k_s[-1]) = 0, \ \forall s\}.$$

*If $\delta(Q, S) \neq \emptyset$, then we say that (Q, S) is **locally dualisable** in $\delta(Q, S)$. Furthermore, if $\delta(Q, S) = Q_0$ then we say that (Q, S) is **globally dualisable**.*

Remark 3.2.6. *Note that to check whether the complex is tilting we just need to check that, for any $s \neq k$, there is no element $f \neq 0$ in the path algebra such that*

$$\begin{array}{ccc} \bigoplus_{t(\alpha)=k} P_{s(\alpha)} & \xrightarrow{(\alpha)} & P_k \\ \downarrow & & \downarrow f \\ 0 & \longrightarrow & P_s \end{array}$$

commutes. The existence of such an f implies that the ideal of relations must contain the set $\{f\alpha : t(\alpha) = k\}$. This allows us, given a potential S for Q, to determine $\delta(Q, S)$.

Moreover, observe that if such f exists, then $fJ(Q, S) \cong S_k$, where S_k is the simple module at the vertex k. This means that $soc(P_s) \neq 0$. So, if Hom$(S_k, P_s) = 0$ for all $s \neq k$ then T^k is tilting.

From now on we will drop the superscript on T whenever the vertex with respect to which we are considering the tilting complex is fixed.

Definition 3.2.7. *The **Seiberg dual algebra** of a quiver Q with potential S (or of its Jacobian algebra) at the vertex $k \in \delta(Q, S)$ is End$_{D^b(Q,S)}(T^k)$, the endomorphism algebra of T^k.*

Rickard's theorem then asserts that Seiberg dual algebras have derived equivalent categories of modules. For an illustrative example see section 3.4.

3.3 Seiberg duality for good potentials

Let us consider the following class of potentials:

Definition 3.3.1. *A potential on a quiver Q is a* ***good potential*** *if its cycles are simple (i.e., do not pass through the same vertex twice), each arrow of Q appears at least twice and no subpath of length two appears repeated in any two distinct cycles of the potential.*

Note that, in particular, a quiver with a good potential has the property that every arrow is contained in at least two distinct cycles.

Proposition 3.3.2. *A quiver with good potential is globally dualisable.*

Proof. This is immediate from the definition of good potential. In fact, the generators for the ideal of relations are of the form $\partial S/\partial a = \sum_{i=1}^{d} \lambda_i v_i$, where $\lambda_i \in \mathbb{K}$. Hence, $d \geq 2$ and the v_i's are paths starting with different arrows thanks to the requirement that no subpath of length two should be repeated in two distinct terms of the potential. Thus, the ideal cannot contain any element of the form $u\alpha$ where u is not a relation and $\alpha \in Q_1$. Therefore $\delta(Q, S) = Q_0$. □

Remark 3.3.3. *Let G be a bipartite graph embedded on a torus (such that each edge separates two distinct faces) and (Q,S) the quiver with potential associated to it as explained in the introduction of this chapter. Under very mild assumptions on G, S is a good potential. In fact, it is always true that each arrow appears exactly twice in S since there are no loops in G and thus each edge of G has two vertices (thus, dually, each arrow appears in two terms of the potential). To guarantee that no subpath of length two appears repeated we just have to ensure that no face of G is limited by only two edges.*

Let (Q, S) be a quiver with good potential. We want to give a presentation of its Seiberg dual algebra at a fixed vertex k. We will see that this algebra is in fact the Jacobian algebra of a quiver with potential, which we shall call the **Seiberg dual quiver**.

First we compute the quiver. It has the same number of vertices as the initial quiver (since the indecomposable projectives of $\text{End}_{D^b(Q,S)}(T)$ correspond to the direct summands of T) and, for each irreducible homomorphism between the T_i's, we draw an arrow between the correspondent vertices. As we shall see later, those irreducible homomorphisms are of three types (the terminology below, used for simplicity of language, is inspired by Mukhopadhyay and Ray's work, [MR04]). Also theorem 3.3.7 shows that our repeated choice of notation below is adequate since the procedure to get the of the Seiberg dual quiver is the same as mutation of the initial quiver.

- Arrows of the form a, where a is also an arrow in Q, will be called **internal arrows**. These arrows correspond to morphisms between T_i^k and T_j^k (which are stalk complexes of projective modules over $J(Q, S)$), for $i \neq k$, that do not factor through the stalk complex of P_k;
- Arrows of the form α^* will be called **dual arrows**. These arrows correspond to morphims either from or to T_k^k;
- Arrows of the form $[\beta\alpha]$ will be called **mesonic arrows**. These arrows correspond to morphisms between T_i^k and T_j^k (which are stalk complexes of projective modules over $J(Q, S)$), for $i \neq k$, that factor through the stalk complex of P_k.

Similarly to the mutation process, we will do Seiberg duality in two main steps:

obtain a quiver $\tilde{Q}$ that may contain more arrows than the irreducible homomorphisms and then, looking at relations, eliminate the appropriate arrows that do not correspond to irreducible ones (those will be the arrows lying in the 2-cycles of the potential). It turns out that relations on the Seiberg dual quiver can also be encoded in a potential (see proposition 3.3.9) and it will be determined as follows:

1. Determine $\tilde{S} := [S] + \sum_{t(\alpha)=s(\beta)=k} [\beta\alpha]\alpha^*\beta^*$ (eventually containing some arrows representing non-irreducible homomorphisms);

2. For every arrow a in a two cycle ab, take the relation $\partial\tilde{S}/\partial a = 0$ and substitute b in $\tilde{S}$ using this equality (and thus eliminate b from the quiver, since b is not irreducible as it can be written as a composition of arrows). Call $\bar{S}$ to the potential thus obtained.

Remark 3.3.4. *Again, for simplicity of language, arrows appearing in two cycles will be called* ***massive arrows*** *and the process described on item 2 of the algorithm above will be called* ***integration over massive arrows****.*

Definition 3.3.5. *If one massive arrow a appears in two or more different 2-cycles of $\tilde{S}$, that is, if we get an expression of the form:*

$$\tilde{S} = \sum_{i=1}^{d} \lambda_i a b_i + \sum_{j=1}^{l} a u_j + W$$

where $\lambda_i \in \mathbb{K}$, each b_i is an arrow, $d \geq 2$, each u_i is a path of length greater or equal than 2 and a does not appear in W, then we say that the b_i's are ***related arrows*** *(by a).*

Given a quiver Q with good potential S, suppose that no related arrows occur in $\tilde{S}$. Then $\tilde{S}$ can be written as follows:

$$\tilde{S} = \sum_{i=1}^{N} (\lambda_i a_i b_i + \sum_{j} \sigma_{i,j} a_i u_{i,j} + b_i v_i) + W \qquad (3.3.1)$$

where $\sigma_{i,j}$, $\lambda_i \in \mathbb{K}$, each $a_i b_i$ is a 2-cycle (i.e., a_i and b_i are massive arrows), each b_i is mesonic (thus the coefficient of $b_i v_i$ is 1), u_{ij} does not contain any arrow b_k, v_i is a composition of dual arrows and W does not have any term involving massive arrows. Since there are no related arrows we have $a_i \neq a_j$ and since each b_i is mesonic (and S, being good, does not have repeated subpaths of length two) $b_i \neq b_j$, for all $i \neq j$.

Theorem 3.3.6. *Let Q be a quiver with a good potential S. If k is a vertex such that no related arrows occur in the mutation, there is a strong right equivalence ϕ from $(\tilde{Q}, \tilde{S})$ to $(\tilde{Q}, S' + \bar{S})$, where S' is trivial and $\bar{S}$ is obtained by Seiberg duality.*

Proof. Since there are no related arrows, let us assume that $\tilde{S}$ is of the form (3.3.1). Take the family of homomorphisms given by

$$\begin{array}{rcl} \phi_i : \mathbb{K}\tilde{Q} & \to & \mathbb{K}\tilde{Q} \\ a_i & \mapsto & a_i - \frac{1}{\lambda_i} v_i \\ b_i & \mapsto & b_i - \frac{1}{\lambda_i} \sum_j \sigma_{i,j} u_{i,j} \\ z & \mapsto & z \quad \text{if } z \neq a_i, b_i,\ z \in Q_1 \end{array}$$

where i ranges from 1 to N, the number of 2-cycles in $\tilde{S}$. Note that ϕ_i is in fact an automorphism for all $1 \leq i \leq N$: injectivity is clear and all arrows lie in the image since

$$\phi_i(a_i + \frac{1}{\lambda_i} v_i) = a_i \quad \text{and} \quad \phi_i(b_i + \frac{1}{\lambda_i} \sum_j \sigma_{i,j} u_{i,j}) = b_i$$

and since they generate the algebra, surjectivity holds.

Let ϕ be the composition of all ϕ_i's. Then we may compute $\phi(\tilde{S})$ thus getting

$$\phi(\tilde{S}) = \sum_{i=1}^{N} (\lambda_i a_i b_i - \frac{1}{\lambda_i} \sum_j \sigma_{i,j} u_{i,j} v_i) + W$$

whose reduced part is exactly

$$\sum_{i=1}^{N}(-\frac{1}{\lambda_i}\sum_j \sigma_{i,j}u_{i,j}v_i) + W.$$

Now, if we integrate over massive arrows in 3.3.1, taking in account that

$$\partial\tilde{S}/\partial a_i = \lambda_i b_i + \sum_j \sigma_{i,j}u_{i,j} \qquad \partial\tilde{S}/\partial b_i = \lambda_i a_i + v_i$$

and using the relations $\partial\tilde{S}/\partial a_i = 0$ and $\partial\tilde{S}/\partial b_i = 0$ in $\tilde{S}$ we get

$$\sum_{i=1}^{N}(-\frac{1}{\lambda_i}\sum_j \sigma_{i,j}u_{i,j}v_i) + W$$

which is the same as $\phi(\tilde{S})_{red}$. Thus ϕ is such a strong right equivalence. □

The following theorem establishes the desired comparison between the mutated quiver and the Seiberg dual quiver.

Theorem 3.3.7. *Let Q be a quiver with a good potential S such that no related arrows occur in the mutation at a vertex k. Then the jacobian algebra of the quiver with potential obtained by mutation at k is isomorphic to with the Seiberg dual algebra of (Q, S) at k.*

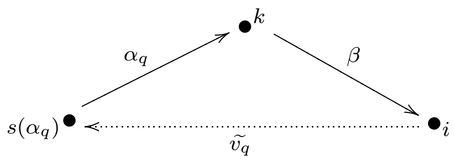
Proof. We start by looking at the shape of the quiver.

1. First we prove that Seiberg duality at k inverts incoming arrows to k. The complex T_k has in degree zero one copy of P_j for every arrow from j to k. Therefore, for each such arrow we get one projection map from the direct sum to P_j and thus an irreducible homomorphism from T_k to T_j, which translates into an arrow from k to j in the dual quiver. For each arrow α_j from j to k, denote the correspondent homomorphism from T_k to T_j by α_j^*. There are no more irreducible homomorphims from T_k to T_j: any other homomorphism factors through some factor of the direct sum first.

2. Now we prove that Seiberg duality at k inverts outgoing arrows from k. This requires the commutativity of a diagram like the following:

$$\begin{array}{ccccccc} 0 & \longrightarrow & P_i & \longrightarrow & 0 & & \\ & & \downarrow f & & \downarrow & & \\ 0 & \longrightarrow & \bigoplus\limits_{t(\alpha)=k} P_{s(\alpha)} & \xrightarrow{(\alpha)} & P_k & \longrightarrow & 0 \end{array} .$$

The diagram commutes if and only if $(\alpha)f = 0$ and so we have to check the relations in the quiver to obtain such a condition. Fix an arrow β from k to i and take the (cyclic) derivative of the potential with respect to β. Since S is a good potential, $\partial S/\partial\beta = \sum_{q=1}^{d} \lambda_q v_q$ where the v_q's are paths from i to k (since βv_q is a cycle for all q) and $d \geq 2$. To give a homomorphism from P_i to $\bigoplus_{t(\alpha)=k} P_{s(\alpha)}$ we just need to give a homomorphism from P_i to each $P_{s(\alpha)}$. Let α_q be the arrow such that $t(\alpha_q) = k$ and it is on the path v_q. Observe that $v_q = \alpha_q \widetilde{v_q}$, where $\widetilde{v_q}$ is a path from i to $s(\alpha_q)$ as in the picture.



Set a homomorphism from P_i to $P_{s(\alpha)}$, for α such that $t(\alpha) = k$, as follows:

- zero if $\alpha \neq \alpha_q$ for some q;
- $\lambda_q \widetilde{v_q}$ if $\alpha = \alpha_q$ for some q;

and set β^* to be the homomorphism induced by this set of homomorphisms to the direct sum and therefore to the complex T_k. Clearly this map makes the diagram above commute. Now we need to prove that it is irreducible. If not, then it factors through other T_r via an element $u \in e_r J(Q,S) e_i$. This would imply that $\widetilde{v_q} = w_q u$ for some $w_q \in e_{s(\alpha_q)} J(Q,S) e_r$, for all $1 \leq q \leq d$, which cannot happen since the potential is good. Hence β^* is irreducible. By construction, these homomorphisms are the only irreducible ones from T_i to T_k.

3. For each path of length two $\beta\alpha$ such that $t(\alpha) = s(\beta) = k$ we clearly get a homomorphism from $T_{s(\alpha)}$ to $T_{t(\beta)}$. Denote this homomorphism by $[\beta\alpha]$. We show that it is irreducible if and only it is not contained in a two cycle of the potential $\tilde{S}$. Suppose a is an arrow such that $[\beta\alpha]a$ is a 2-cycle of $\tilde{S}$. Then $\partial\tilde{S}/\partial a$ gives an explicit factorisation of the mesonic arrow. On the other hand, if it is not contained in a 2-cycle of $\tilde{S}$ then it is irreducible since it could only factor through the stalk complex of P_k which does not correspond to an indecomposable projective module over $\text{End}_{K^b(Q,S)}(T)$.

4. Finally, if none of the previous cases apply, then the homomorphisms between T_j and T_i that can be irreducible are just arrows from j to i. Again, they are in fact irreducible if and only if they are not contained in a 2-cycle of $\tilde{S}$ and a similar argument to the one above applies to this case.

Let $\tilde{Q}$ be the quiver obtained by taking all homomorphisms above considered (α^* for every arrow α with target k, β^* for every arrow β with source k, $[\beta\alpha]$ for every path $\beta\alpha$ with middle vertex k, and a for every arrow a not starting or ending at k), even if they are not irreducible. Determining $\tilde{Q}$ is clearly the same procedure either via mutations or via Seiberg duality. Now, by 3.3.6, we see that the reduced part of $(\tilde{Q}, \tilde{S})$ can be found by eliminating the 2-cycles of $\tilde{Q}$ appearing in $\tilde{S}$ and taking the potential obtained through integration over those massive arrows (thus eliminating the non-irreducible morphisms). Thus the result follows. □

Corollary 3.3.8. *If Q is a quiver with a good potential S and if k is a vertex such that no related arrows arise in the mutation procedure, then mutation at k*

produces a derived equivalence between the Jacobian algebras of (Q, S) and of $\mu_k(Q, S)$.

Proof. From the previous theorem we have $J(\mu_k(Q, S)) \cong \text{End}_{D^b(Q,S)}(T^k)$. Then, given that T^k is a tilting complex over $J(Q, S)$ (by lemma 3.3.2), Rickard's theorem 2.1.9 gives the desired derived equivalence. □

To finish this section, we shall prove that the algorithm previously described actually computes the Seiberg dual potential of a quiver with potential (Q, S) at a fixed vertex k.

Proposition 3.3.9. *The algorithm described in the beginning of this section computes a potential for the Seiberg dual quiver such that its Jacobian algebra is $\mathrm{End}_{D^b(Q,S)}(T^k)$, for a quiver with a good potential (Q, S).*

Proof. Let the homomorphisms represented by dual arrows of outgoing arrows be as it is described in the proof of theorem 3.3.7 and keep the notation therein. Denote by $\tau_{\beta\alpha}$ the coefficient of $[\beta\alpha]$ in $[S]$. We will first prove that the relations induced by the potential $\tilde{S}$ obtained through the algorithm above are satisfied in $\text{End}_{D^b(Q,S)}(T)$. Case by case, we analyse relations coming from differentiating:

- with respect to β^* (dual of an outgoing arrow):

$$\partial\tilde{S}/\partial\beta^* = \sum_{t(\alpha)=k} [\beta\alpha]\alpha^* = \beta(\alpha) = 0,$$

  since it is homotopic to zero in the category of complexes;

- with respect to α^* (dual of an incoming arrow):

$$\partial\tilde{S}/\partial\alpha^* = \sum_{s(\beta)=k} \beta^*[\beta\alpha] = (\sum_{s(\beta)=k} \beta^*\beta)\alpha.$$

Let us check that $\sum_{s(\beta)=k} \beta^*\beta = 0$. For this we compute each component of this vector by looking at the occurrences of a fixed γ incoming to k in S. We have in $[S]$ some sub expression of the form

$$\sum_{q=1}^{d} \tau_{\beta_i\gamma}[\beta_i\gamma]\widetilde{v_i}$$

for some β_i's with source k, where each $\tilde{v}_i$ completes the corresponding cycle and $\tau_{\beta_i\alpha} \neq 0$. Then we have the corresponding entry of $\sum_{s(\beta)=k} \beta^*\beta$ given by

$$\sum_{q=1}^{d} \tau_{\beta_i\alpha}\widetilde{v_i}\beta_i$$

which is zero since it equals $\partial S/\partial\gamma$;

- with respect to a, an internal arrow:

$$\partial\tilde{S}/\partial a = \partial[S]/\partial a = 0,$$

  since this is essentially the same as $\partial S/\partial a$ (with some extra square brackets);

- and, finally, with respect to $[\beta_i\alpha_j]$ (mesonic arrow):

$$\partial\tilde{S}/\partial[\beta_i\alpha_j] = \alpha_j^*\beta_i^* = 0$$

  this follows from the definition of α_j^* and β_i^* as homomorphisms (see proof of theorem 3.3.7).

Integration over massive arrows does not change the relations induced by the potential since the expressions obtained by differentiating with respect to a massive arrow are zero in the Jacobian algebra, according to the proof above.

The last thing we need to check is that this potential $\tilde{S}$ gives generators for the ideal of relations. Let r be a nonzero relation in the new quiver such that none of its factors are relations (i.e., if $r = uv$ then neither u nor v lie in the ideal of relations). We prove that this relation has already been contemplated. We split the proof into several cases.

- r **does not pass by** k. Observe that if r does not involve morphisms to or from T_k^k then it can be expressed as linear combinations of elements of the path algebra of Q from j to i, for some vertices i and j (where we identify such a path with the corresponding endomorphism of T^k). Therefore there are some internal arrows $a_1, ..., a_n$ such that a linear combination of $\partial S/\partial a_1$,...,$\partial S/\partial a_n$ equals r up to square brackets.

- r **passes by** k **and** $t(r), s(r) \neq k$. If r involves morphisms both to T_k^k and from T_k^k, then each of its terms involve both duals of arrows β outgoing from k (β^* is the natural map from T_i^k to T_k^k defined by the relation $\partial S/\partial \beta$ - see proof of 3.3.7) and duals of arrows α incoming to k (α^* is just a projection map - see proof of theorem 3.3.7). Then r can be also be identified with some linear combination of paths in Q, as the factor involving dual arrows can be read as the projection of a component of β^*, which is an element of $\mathbb{K}Q$. Since r is a zero morphism and none of its factors are relations, it can be identified with a linear combination of terms of the form $\partial S/\partial a_i$ for some internal arrows a_i from $t(r)$ to $s(r)$. Now, each a_i is also an arrow in $\tilde{Q}$ and therefore r is a linear combination of $\partial \tilde{S}/\partial a$.

- $t(r) = k$ **and** $s(r) = l \neq k$. Suppose $r = \sum_i \beta_i^* r_i$ where each r_i is an element of $\mathbb{K}\tilde{Q}$. Then, as a map from T_l^k to T_k^k, it is identified with n elements of $J(Q,S)$, $u_{s(\alpha)}$, starting at l and ending at some $s(\alpha)$ (where $t(\alpha) = k$), n being the number of terms in the direct sum $\bigoplus_{\alpha : t(\alpha)=k} P_{s(\alpha)}$ (see proof of theorem 3.3.7). Each $\beta_i^* r_i$ appears in at least two components of the direct sum, by construction of β_i^*, since S is good. In each such component, $\beta_i^* r_i$ provides a summand of $u_{s(\alpha)}$ (and $u_{s(\alpha)}$ yields a zero morphism from P_l to $P_{s(\alpha)}$). Also we identify that summand with an element of $\mathbb{K}Q$ by definition of β_i^*. In order to be zero, $u_{s(\alpha)}$ must have as a factor some relation in $J(Q,S)$ and thus the summand mentioned above contains as a factor some terms of this factor. Furthermore, this factor of the summand must contain the terms coming from β_i^* otherwise r would not be irreducible (factoring through the projective corresponding to the target of this factor).Now, in order to be able to read a relation involving the terms from the morphisms β_i^*, r_i must pass through the vertex k (and the relation is the factor of that component which starts at k). This follows from the definition of β_i^* as a morphism and from the fact that the potential S is good, not allowing repetition of subpaths of length two. Therefore all terms in each component $P_{s(\alpha)}$ begin with a common nontrivial path from $l \neq k$ to k. This path must be the same in every component since β_i^* appears at least in two components and such pairs of components do not coincide for any given two indices i, j (this also follows from S being good). Since r is irreducible (in the sense above), this path can only be an arrow to k (or a scalar multiple of it), otherwise r would factor through some T_m^k. This is because arrows to k in Q are no longer arrows of $\tilde{Q}$.Denote this arrow by γ. Hence, $r_i = r_i'\gamma$ for some paths r_i' starting at k with different arrows (otherwise, again, r would split) and, as an element of $J(Q,S)$, $\sum_i \beta_i^* r_i' = 0$, i.e., $\sum_i \beta_i^* r_i'$ is a n-tuple of relations in $J(Q,S)$ from k to all the vertices of the form $s(\alpha)$ with $t(\alpha) = k$. Therefore, by construction of β_i^*, $r_i' = \beta_i$ for all i (again because S is good) and the sum needs to run over all arrows β_i starting at k. Thus $r = \sum_{s(\beta)=k} \beta^*[\beta\gamma]$ which is precisely $\partial\tilde{S}/\partial\gamma^*$.

- $s(r) = k$ **and** $t(r) = l \neq k$. If r starts at k, the first arrow appearing in each term is a dual of an arrow α incoming to k (again, α^* is just a projection map). This is a situation different in nature to the previous ones: we are looking at a map from the direct sum $\bigoplus_{\alpha:t(\alpha)=k} P_{s(\alpha)}$ to some projective P_l and hence we are not able to identify r with a relation of $J(Q, S)$. Therefore the fact that r is zero must come from the fact that the map is homotopic to zero in the category of complexes over $J(Q, S)$. That is equivalent to the existence of a linear combination of paths in Q, call it u, from k to l such that $u(\alpha) = r$ in $J(Q, S)$. Since r is minimal, u must irreducible - and the space of irreducible maps from P_k to P_l has the arrows between k and l as a basis. Therefore r must be a linear combination of terms of the form $\sum_{t(\alpha)=k}[\beta\alpha]\alpha^*$ (which is precisely $\partial\tilde{S}/\partial\beta^*$) for each β a summand of u.

To complete the proof we need to show that no such relation r can both start and end at k. Suppose we have such a map r from T_k^k to T_k^k, i.e.,

$$\begin{array}{ccccccc} 0 & \longrightarrow & \bigoplus_{\alpha:t(\alpha)=k} P_{s(\alpha)} & \xrightarrow{(\alpha)} & P_k & \longrightarrow & 0 \\ & & \downarrow r_0 & & \downarrow r_1 & & \\ 0 & \longrightarrow & \bigoplus_{\alpha:t(\alpha)=k} P_{s(\alpha)} & \xrightarrow{(\alpha)} & P_k & \longrightarrow & 0 \end{array}$$

It can not be identified with an element of $J(Q, S)$ so there is a homotopy to zero h such that $h(\alpha) = r_0$. On the other hand, this is also a homotopy to zero for r_0 as a map from T_k^k to $\bigoplus_{\alpha:t(\alpha)=k} T_{s(\alpha)}^k$ and r_0 lies in the ideal of relations of $J(\tilde{Q}, \tilde{S})$ and it is covered by the cases above. Since r_1 is determined from r_0 by the commutation of the diagram as a linear combination of cycles in $J(Q, S)$, this means that r is generated by the relations contemplated above.

This completes the proof.

$\square$

3.4 An example

We shall exemplify mutation on a quiver with potential arising in derived algebraic geometry. Given a Del Pezzo surface, we can study its derived category of coherent sheaves using the existence of a strong exceptional sequence.

Theorem 3.4.1 (Kuleshov, Orlov, Hille, Perling, [KO95], [HP08]). *If X is a Del Pezzo Surface, we have strong exceptional sequences of sheaves given by:*

- $\{O, O(1), O(2)\}$ *if* $X = \mathbb{P}^2$
- $\{O, O(1,0), O(0,1), O(1,1)\}$ *if* $S = \mathbb{P}^1 \times \mathbb{P}^1$
- $\{O, O(E_1), ..., O(E_r), O(1), O(2)\}$ *if* X *is* dP_r *with* $r \leq 8$*, where each* E_i *is an exceptional curve of the blow up and* dP_r *is the Del Pezzo obtained by blowing up* $1 \leq r \leq 8$ *points in* $\mathbb{P}^2$.

As mentioned in the introduction, the direct sum of a strong exceptional sequence over a projective variety X is a tilting sheaf, yielding a derived equivalence between $coh(X)$ and $mod(\mathbb{K}Q/I)$ for some quiver Q and some ideal of relations I. These are determined by looking at the irreducible homomorphisms between the sheaves in the sequence and taking relations between those homomorphisms.

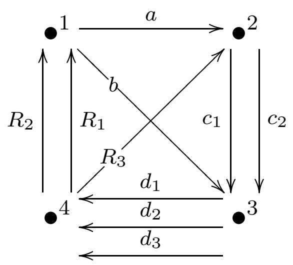
We shall focus on X as in example 2.1.7, i.e., a blow-up of $\mathbb{P}^2$ at one point. To get a derived equivalence to a Jacobian algebra of a quiver with potential, we ought to consider not X itself but $Y = \omega_X$ - the total space of the canonical bundle of X - instead. This is a local Calabi-Yau three-fold. If we let $\pi : Y \to X$ be the natural projection, we get that $\tilde{B} = End_Y(\oplus_i \pi^* E_i)$ is derived equivalent to $coh(Y)$, whenever the exceptional sequence $(E_i)_i$ is geometric over X ([Bri05]). Geometric in this context means that its associated helix satisfies some extra

Ext-vanishing conditions ([Bri05]), but we will not explore this. We proceed to characterise the algebra $\tilde{B}$ and that is enough for our purposes.

The algebra $\tilde{B}$ can be seen as the path algebra of a quiver with relations. It can be obtained from the correspondent quiver of a geometric exceptional sequence $(E_i)_i$ adding one arrow for each generator of the ideal of relations in the opposite direction of the composition of arrows in that relation ([Seg08]). This will be a quiver with potential, where the potential is the sum of the cycles obtained through the composition of each new arrow with the correspondent relations. This process is also described in [ABS08]. In fact it is easy to observe in our concrete example that the quiver with potential obtained via this construction is the same whether we consider the exceptional sequence of example 2.1.7 or other sequences frequently found in the literature ([Kin97],[Per09])

Example 3.4.2. *The algebra $\tilde{B}$ associated to X, $\mathbb{P}^2$ blown-up at one point, with exceptional sequence $\{O, O(E_1), O(1), O(2)\}$ is:*

$Q =$

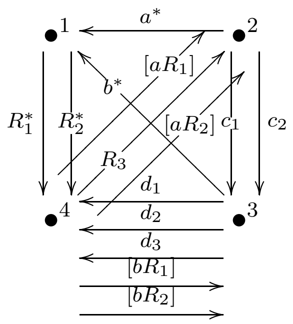


with potential:

$$S = R_3(d_3c_1 - d_1c_2) + R_1(d_1b - d_2c_1a) + R_2(d_2c_2a - d_3b)$$

Note that this is a good potential. For any fixed vertex k, it is easy to check that no related arrows occur in the mutation. Hence, the results in this section yield that mutations give derived equivalent path algebras. Since the one above is derived equivalent to $coh(Y)$ ([Bri05], [BS09]), so will be $J(\mu_k(Q,S))$. Let us present $\mu_1(Q,S)$.

$\tilde{Q} =$



We take a cyclically equivalent potential since there are terms on it starting and ending at 1. Then we substitute paths of length two passing through 1 by new arrows and add Δ_1.

$$\tilde{S} = R_3 d_3 c_1 - R_3 d_1 c_2 - d_2 c_1 [aR_1] + d_1 [bR_1] - d_3 [bR_2] + d_2 c_2 [aR_2]$$

$$+ [aR_1] R_1^* a^* + [aR_2] R_2^* a^* + [bR_1] R_1^* b^* + [bR_2] R_2^* b^*$$

Clearly this potential is not reduced. Following the proof of theorem 3.3.6, let us condsider the following (strong) right equivalence:

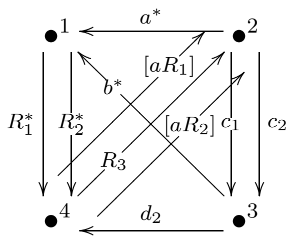
$$\begin{array}{rcl} \phi : \mathbb{K}\tilde{Q} & \to & \mathbb{K}\tilde{Q} \\ d_1 & \mapsto & d_1 - R_1^* b^* \\ d_3 & \mapsto & -d_3 + R_2^* b^* \\ {[bR_1]} & \mapsto & [bR_1] + c_2 R_3 \\ {[bR_2]} & \mapsto & [bR_2] + c_1 R_3 \\ u & \mapsto & u \ \text{ if } u \neq d_1, d_3, [bR_1], [bR_2], \ \ u \in Q_1 \end{array} .$$

If we compute $\phi(\tilde{S})$, it is of the form $S' + \bar{S}$ and thus we can take the reduced part. More simply, we can integrate over massive arrows by taking the relations:

$$[bR_1] = c_2 R_3, \ [bR_2] = c_1 R_3.$$

In any case, as proved in theorem 3.3.6, we get the same result which is:

$\bar{Q} =$



with potential

$$\bar{S} = c_2R_3R_1^*b^* + c_1R_3R_2^*b^* + d_2c_2[aR_2] - d_2c_1[aR_1] + [aR_1]R_1^*a^* + [aR_2]R_2^*a^*.$$

3.5 3-Calabi-Yau algebras

Suggested by the example of section 3.4, we investigate ideas of the previous sections in the 3-Calabi-Yau context. Here, the restrictions on the Jacobian algebras will be of homological rather than of combinatorial nature. 3-Calabi-Yau (3-CY for short) algebras are, in general, quotients of smooth algebras by ideals of relations coming from potentials ([Gin06]). In fact very recent results of Van den Bergh ([VdB10]) show that complete 3-CY algebras come from quivers with potential. We use the following definition.

Definition 3.5.1. *A $\mathbb{K}$-algebra R is said to be n-Calabi-Yau ($n \geq 1$) if:*

1. *R is homologically smooth, i.e., as an $R^{op} \otimes R$-module it has a finite resolution by finitely generated projective modules;*

2. *RHom$_{R^{op}\otimes R}(R, R^{op} \otimes R) \cong R[-d]$ in $D^b(R^{op} \otimes R)$.*

This definition is due to Ginzburg ([Gin06]). The following lemma is crucial to our approach. In fact it is common to find in the literature definitions of Calabi-Yau algebra based on the duality of the lemma.

Lemma 3.5.2 (Keller, [Kel08]). *Let R be an n-Calabi-Yau algebra. Suppose $X, Y \in D^b(Mod(R))$ such that $X \in D^b(fd(R))$, i.e., X is a complex of finite dimensional modules over $\mathbb{K}$. Then we have a canonical isomorphism*

$$\mathrm{Hom}_{D^b(Mod(R))}(X, Y)^* \cong \mathrm{Hom}_{D^b(Mod(R))}(Y, X[n]), \tag{3.5.1}$$

where $*$ denotes $\mathbb{K}$-duality.

Remark 3.5.3. *For an n-Calabi-Yau algebra R, it is clear that $D^b(fd(R))$ is Hom-finite, i.e., the Hom-spaces are finite dimensional over $\mathbb{K}$. Indeed, the duality in lemma 3.5.2 applied twice (which is possible when both X and Y are elements of $D^b(fd(R))$) shows that*

$$\mathrm{Hom}_{D^b(Mod(R))}(X, Y)^{**} = \mathrm{Hom}_{D^b(Mod(R))}(X[n], Y[n]) = \mathrm{Hom}_{D^b(Mod(R))}(X, Y).$$

The results obtained by Keller and Yang on the relations between mutations and derived equivalences ([KY10]) are far more general than the remarks we present here. There, it is proven that mutations hold derived equivalences between the dg-algebras obtained through Ginzburg's construction ([Gin06]) over the complete Jacobian algebra. 3-CY complete Jacobian algebras are such that the associated Ginzburg dg-algebras have their cohomology concentrated in degree zero (and equal to the original algebra). Our approach, however, will be as before, not working on the complete setting nor making use of Ginzburg's differential graded construction. Also Iyama and Reiten have obtained similar results for mutations of quivers without potentials ([IR08]).

Let R be a 3-Calabi-Yau algebra such that there is (Q, S) quiver with potential satisfying $J(Q, S) = R$. Let every vertex of Q be contained in some cycle (this seems to be a reasonable assumption as we can see from the graded 3-Calabi-Yau case - [Boc08]) and let Q be without loops or two cycles.

Fix a vertex k in Q. We want to prove that T^k is tilting for any vertex k of Q. It is enough to prove that Hom$(S_k, P_s) = 0$ for all $s \neq k$ (see 3.2.6). Indeed, as a consequence of 3.5.2 we have the following result

Corollary 3.5.4. *If R is n-CY algebra, then Hom$(S_k, P_s) = 0$ for all $s \neq k$ and hence T^k is tilting for any vertex k of Q.*

Proof. Lemma 3.5.2 shows that

$$Hom_{D^b(mod(R))}(S_k, P_s)^* = Hom_{D^b(mod(R))}(P_s, S_k[n]) = Ext^n(P_s, S_k) = 0$$

and thus the result follows. □

Remark 3.5.5. *If we take as definition of a Calabi-Yau algebra the existence of a duality (3.5.1) in $D^b(fd(R))$, then it is still possible to prove corollary 3.5.4 through a result proved by Iyama and Reiten ([IR08]). Indeed, they prove that for such algebras the duality can be extended to work also when one of the variables is a complex in $K^b(P(R))$. Even though their results are primarily concerned with finite dimensional algebras, the result is true in this generality as well.*

We are now able to prove similar results to the ones obtained in previous sections.

Theorem 3.5.6. *If $J(Q, S)$ is a 3-Calabi-Yau algebra, then End$_{K^b(J(Q,S))}(T^k) \cong J(\tilde{Q}, \tilde{S})$ where $(\tilde{Q}, \tilde{S})$ are obtained in the process of mutation at k before reduction. Furthermore, $\widehat{\mathrm{End}_{K^b(J(Q,S))}}(T^k) \cong J(\widehat{\mu_k(Q, S)})$, where $\mu_k(Q, S)$ is the reduced part of $(\tilde{Q}, \tilde{S})$.*

Proof. Let us fix a vertex k and drop the superscript on T^k for simplicity. First we take the indecomposable projective modules T_i of End$_{K^b(J(Q,S))}(T)$ and determine 'candidates' to irreducible homomorphisms between them. This gives us a quiver (call it G). We'll start by proving that $G = \tilde{Q}$

1. *Inversion of incoming arrow*: The argument on item 1 of proof 3.3.7 works here;

2. *Inversion of outgoing arrows*: This requires the commutativity of a diagram of the form:

$$\begin{array}{ccccccc} 0 & \longrightarrow & P_i & \longrightarrow & 0 & & \\ & & \downarrow{\scriptstyle f} & & \downarrow & & \\ 0 & \longrightarrow & \bigoplus_{t(\alpha)=k} P_{s(\alpha)} & \xrightarrow{(\alpha)} & P_k & \longrightarrow & 0 \end{array}$$

i.e. the existence of an f such that $(\alpha)f = 0$. Thus we have to look for relations on the quiver that may allow us to obtain such f. Fix an arrow β from k to i and differentiate the potential with respect to β getting $\partial S/\partial\beta = \sum_{t=1}^{d} \lambda_t v_t$ where the v_t's are paths from i to k (since βv_t is a cycle for all t). To give a homomorphism from P_i to $\bigoplus_{t(\alpha)=k} P_{s(\alpha)}$ we just need to give a homomorphism from P_i to each $P_{s(\alpha)}$, by the universal property of the direct sum. Define $\alpha^{-1}\gamma$ for any path γ to be zero if γ does not end with the arrow α and to be u if $\gamma = \alpha u$ for some $u \in \mathbb{K}Q$. Then we can define the following maps:

$$\begin{array}{rcl} \beta_\alpha^* : P_i & \rightarrow & P_{s(\alpha)} \\ \gamma & \mapsto & \alpha^{-1}\frac{\partial S}{\partial \beta}\gamma \end{array}$$

and set β^* to be the homomorphism induced by this set of homomorphisms in the direct sum and therefore to the complex T_k. Clearly this map makes the diagram above commute, as

$$(\alpha)\beta^* = \sum_{t(\alpha)=k} \alpha\beta_\alpha^* = \sum_{t(\alpha)=k} \alpha(\alpha^{-1}\frac{\partial S}{\partial \beta}) = \frac{\partial S}{\partial \beta}$$

which is zero in the Jacobian algebra. Now we need to prove that this is irreducible.

Suppose this homomorphism is not irreducible, factoring through T_l for some $l \in Q_0$. Then we have the following diagram:

$$
\begin{array}{ccc}
P_i & \longrightarrow & 0 \\
{\scriptstyle h}\downarrow & & \downarrow \\
P_l & \longrightarrow & 0 \\
{\scriptstyle g}\downarrow & & \downarrow \\
\bigoplus_{t(\alpha)=k} P_{s(\alpha)} & \xrightarrow{(\alpha)} & P_k
\end{array}
$$

where $\beta^* = gh$ and each square commutes. The commutativity of the bottom diagram requires the existence of such relation in $J(Q,S)$. If we denote this relation by θ, then

$$\frac{\partial S}{\partial \beta} = (\alpha)\beta^* = (\alpha)gh = \theta h.$$

Now, let R be a minimal set of generators of the ideal of $\mathbb{K}Q$ generated by all the cyclic derivatives of the potential S. We recall that the dimension of Ext$^1(S_j, S_l)$ (respectively Ext$^2(S_j, S_l)$), for $j, l \in Q_0$, measure the number of arrows from l to j (respectively the number of elements in R from l to j). This can be understood by computing a projective resolution for S_j. Then, since $J(Q,S)$ is 3-CY, we have:

$$|\{r \in R : t(r) = k,\ s(r) = i\}| = \dim \operatorname{Ext}^2(S_k, S_i) =_{3-CY}$$

$$= \dim \operatorname{Ext}^1(S_i, S_k) = |\{a \in Q_1 : t(a) = i,\ s(a) = k\}|\,,$$

However this yields a contradiction since, by the equation above, the relation induced by β is not in R (θ is, and θ is not induced by β as $l \neq i$). Thus β^* is irreducible.

3. *Gluing arrows* The argument on item 3 of proof 3.3.7 works here.

4. Finally, if none of the previous cases apply, then homomorphisms between T_j and T_i are just arrows from j to i. Again, these homomorphisms are irreducible if and only if they are not contained in a 3-cycle of the potential going through k and a similar argument to the one above applies to this case.

Let G then be the quiver obtained by taking all the homomorphisms considered in the cases above, even if they are not irreducible. We just proved that this quiver is the same as $\tilde{Q}$. Using proposition 3.3.9 we have that $\mathrm{End}_{K^b(J(Q,S))}(T) \cong J(G, \tilde{S}) = J(\tilde{Q}, \tilde{S})$. Now, since $(\tilde{Q}, \tilde{S})$ is right equivalent to $\mu_k(Q, S)$, we have an isomorphism of complete path algebras as stated. □

Remark 3.5.7. *Note that we need to consider completions because, in general, the removal of 2-cycles in the mutation procedure is not guaranteed. Derksen, Weyman and Zelevinsky have produced examples of such phenomenon ([DWZ08]). Indeed, we can only produce a strong right equivalence using the techniques of section 3.3 when the mutated quiver has no 2-cycles. Therefore, we have a derived equivalence between $J(Q, S)$ and $J(\tilde{Q}, \tilde{S})$ but we cannot guarantee the existence of a strong right equivalence between $(\tilde{Q}, \tilde{S})$ and $\mu_k(Q, S)$.*