

t-structures, recollements and silting objects

Jorge Vitória

Institute for Algebra and Number Theory
University of Stuttgart

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Overview

- 1 t-structures and co-t-structures;
- 2 Recollements and glueing;
- 3 The piecewise hereditary case;
- 4 Silting objects;
- 5 Glueing of silting objects;
- 6 Glueing of tilting objects.

t-structures and co-t-structures

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with $A \in \mathcal{D}^{\leq 0}$ and $B \in \mathcal{D}^{\geq 0}[-1]$.

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Definition (Bondarko, Pauksztello'10)

A **co-t-structure** in \mathcal{D} is a pair of full subcategories $(\mathcal{D}_{\geq 0}, \mathcal{D}_{\leq 0})$ s.t.

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- **Co-heart:** $\mathcal{D}_{\geq 0} \cap \mathcal{D}_{\leq 0}$. It is additive but, in general, not abelian!
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satisfying:

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satisfying:

- 1 $(i^*, i_*, i^!)$, $(j_!, j^*, j_*)$ are adjoint triples;
- 2 i_* , j_* , $j_!$ are fully faithful;
- 3 $j^* i_* = 0$;
- 4 $\forall X \in \mathcal{D}$, there are triangles given by the (co)units of the adjunctions:

$$i_* i^! X \rightarrow X \rightarrow j_* j^* X \rightarrow i_* i^! X[1] \quad , \quad j_! j^* X \rightarrow X \rightarrow i_* i^* X \rightarrow j_! j^* X[1].$$

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where B is also a f.d. piecewise hereditary and $\text{End}_{\mathcal{D}^b(A)}(X)$ is a f.d. skew-field over \mathbb{K} .

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Theorem (Beilinson-Bernstein-Deligne'82)

Given \mathcal{R} and $(\mathcal{X}^{\leq 0}, \mathcal{X}^{\geq 0})$, $(\mathcal{Y}^{\leq 0}, \mathcal{Y}^{\geq 0})$ t -structures in \mathcal{X} and \mathcal{Y} , then

$$\mathcal{D}^{\leq 0} := \{Z \in \mathcal{D} : j^*Z \in \mathcal{X}^{\leq 0}, i^*Z \in \mathcal{Y}^{\leq 0}\}$$

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Theorem (Bondarko'10)

Given \mathcal{R} and $(\mathcal{X}_{\geq 0}, \mathcal{X}_{\leq 0})$, $(\mathcal{Y}_{\geq 0}, \mathcal{Y}_{\leq 0})$ co- t -structures in \mathcal{X} and \mathcal{Y} , then

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Lemma

Let $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ be a bounded t -structure with length heart in $\mathcal{D}^b(R)$.
Then $\mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0} \cong \text{mod}(S)$, where S is a f.d. directed algebra over \mathbb{K} .

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Theorem (Liu-V.'11)

Let $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ be a bounded t -structure with length heart in $\mathcal{D}^b(R)$.
Then there is a recollement of $\mathcal{D}^b(R)$ by derived module categories such that $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ is glued with respect to this recollement.

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- $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ is glued with respect to this recollement from t -structures $(j^* \mathcal{D}^{\leq 0}, j^* \mathcal{D}^{\geq 0})$ and $(i^* \mathcal{D}^{\leq 0}, i^! \mathcal{D}^{\geq 0})$

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Theorem (Keller-Vossieck'88)

Q Dynkin quiver. The bounded t -structures in $\mathcal{D}^b(\mathbb{K}Q)$ are in bijection with basic silting objects in $\mathcal{D}^b(\mathbb{K}Q)$.

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- **co- t -structure**: $(\mathcal{D}_{\geq 0}, \mathcal{D}_{\leq 0}) \rightsquigarrow$ **silting**: M s.t. $\text{add}(M) = \mathcal{D}_{\geq 0} \cap \mathcal{D}_{\leq 0}$.
- **t -structure**: $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0}) \rightsquigarrow$ **co- t -structure**: $(({}^{\perp}\mathcal{D}^{\leq 0})[1], \mathcal{D}^{\leq 0})$.

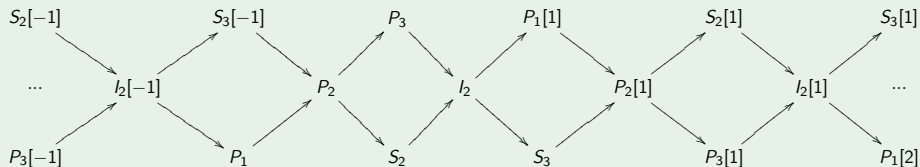
- **silting**: $M \rightsquigarrow$ **t -structure**:

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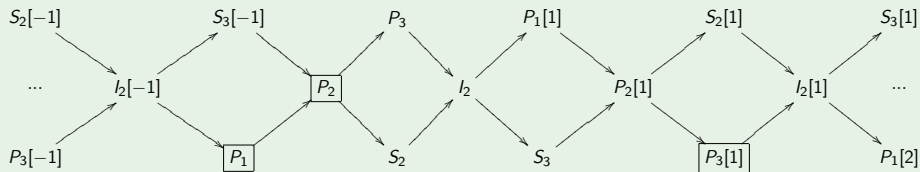
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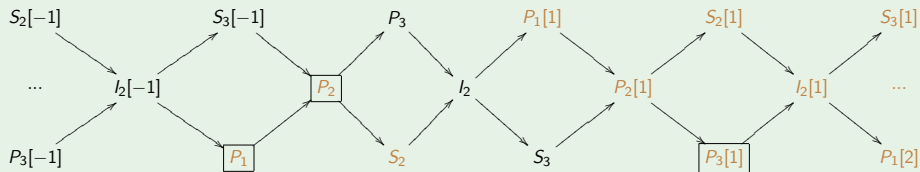
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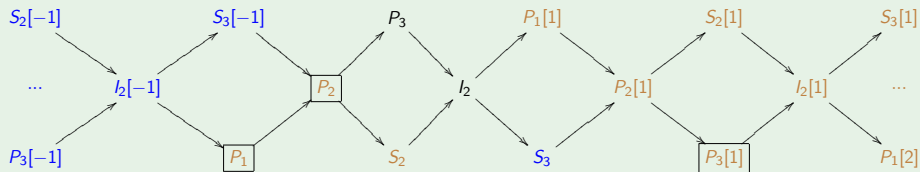


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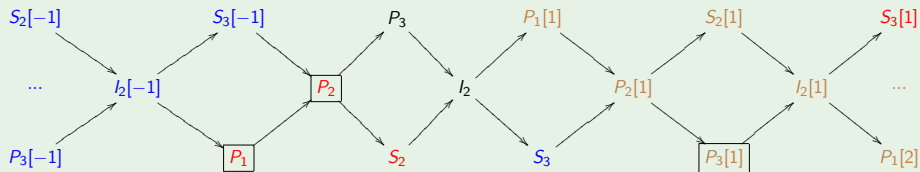
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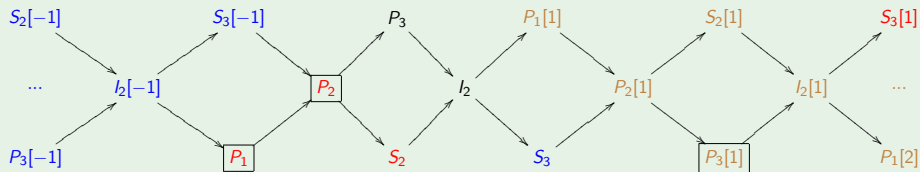
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Heart: $\mathcal{D}_M^{\leq 0} \cap \mathcal{D}_M^{\geq 0} \cong \text{End}_{\mathcal{D}^b(R)}(M) \cong \mathbb{K}A_2 \times \mathbb{K}$, $A_2 = 1 \longrightarrow 2$

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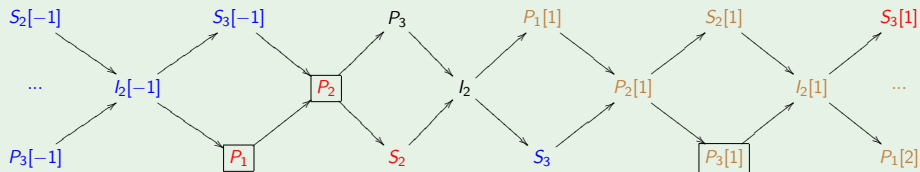
$\mathcal{D}_M^{\geq 1} = (\mathcal{D}_M^{\leq 0})^\perp$

Heart: $\mathcal{D}_M^{\leq 0} \cap \mathcal{D}_M^{\geq 0} \cong \text{End}_{\mathcal{D}^b(R)}(M) \cong \mathbb{K}A_2 \times \mathbb{K}$, $A_2 = 1 \longrightarrow 2$

In general:

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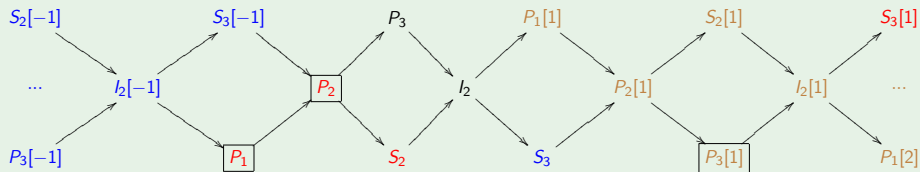
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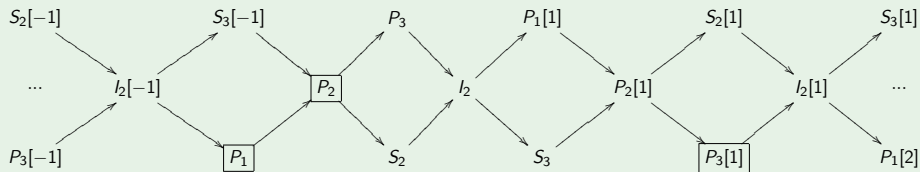
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In general:

- The heart of $(\mathcal{D}_M^{\leq 0}, \mathcal{D}_M^{\geq 0})$ is $\text{End}_{\mathcal{D}^b(R)}(M)$;
- M is tilting if and only if $M \in \mathcal{D}_M^{\leq 0} \cap \mathcal{D}_M^{\geq 0}$.

Silting objects

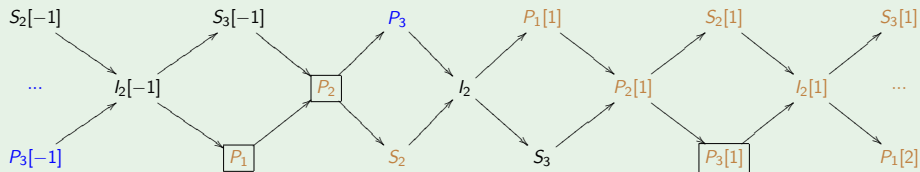
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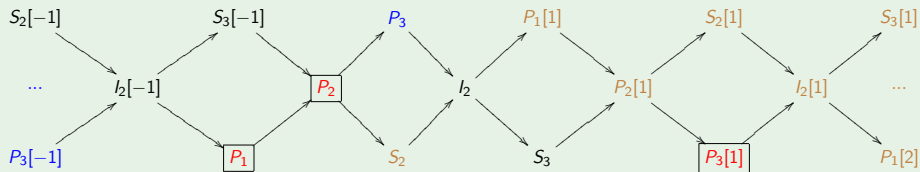
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Glueing silting

Theorem (Liu-V.-Yang'12)

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- \mathcal{R} recollement of triangulated category \mathcal{D} ;

$$(\mathcal{R}) : \quad \mathcal{Y} \begin{array}{c} \xleftarrow{i^*} \\ \xrightarrow{i_*} \\ \xleftarrow{i^!} \end{array} \mathcal{D} \begin{array}{c} \xleftarrow{j_!} \\ \xrightarrow{j^*} \\ \xleftarrow{j_*} \end{array} \mathcal{X} ;$$

- $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$ silting objects;
- $(\mathcal{X}_{\geq 0}, \mathcal{X}_{\leq 0})$ and $(\mathcal{Y}_{\geq 0}, \mathcal{Y}_{\leq 0})$ corresponding *co-t-structures* in \mathcal{X} and \mathcal{Y} .

Then

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Then the glued co-t-structure $(\mathcal{D}_{\geq 0}, \mathcal{D}_{\leq 0})$ corresponds to the silting $Z = i_* Y \oplus K_X$, where K_X is defined by

$$i_* \beta_{\geq 1} i^! j_! X \longrightarrow j_! X \longrightarrow K_X \longrightarrow (i_* \beta_{\geq 1} i^! j_! X)[1]$$

($\beta_{\geq 1}$ is a (non-functorial) choice of truncation for $(\mathcal{Y}_{\geq 0}, \mathcal{Y}_{\leq 0})$ in \mathcal{Y}).

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Idea of proof

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- [BBD'82] \rightsquigarrow functor $j_{!*}$ (7th/intermediate functor) \rightsquigarrow glueing simples in the hearts of **t-structures**;
- {Summands of glued silting} \leftrightarrow {"simples" of glued **coheart**};
- "Triangulated" description of 7th functor (using truncations);
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Remark

Glueing silting objects means glueing compatibly with the glueing of *co-t-structures*.

Glueing silting

Remark

\mathcal{R} recollement of $\mathcal{D}^b(R)$, R of *finite global dimension*;

- There is Serre functor in $\mathcal{D}^b(R)$;
- Recollements can be reflected (Jørgensen);

\rightsquigarrow *Glueing of silting can be done, in this setting, compatibly with the glueing of t-structures.*

This corresponds to glueing with respect to certain co-t-structures via a reflected recollement.

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Theorem (Liu-V.-Yang'12)

- \mathcal{R} recollement of $\mathcal{D}^b(R)$ by $\mathcal{D}^b(C)$ and $\mathcal{D}^b(B)$

$$(\mathcal{R}) : \quad \mathcal{D}^b(B) \begin{array}{c} \xleftarrow{i^*} \\ \xrightarrow{i_*} \\ \xleftarrow{i^!} \end{array} \mathcal{D}^b(R) \begin{array}{c} \xleftarrow{j_!} \\ \xrightarrow{j^*} \\ \xleftarrow{j_*} \end{array} \mathcal{D}^b(C) ;$$

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Then $Z = i_*Y \oplus K_X$ is tilting if and only if

- $\text{Hom}_{\mathcal{D}^b(B)}(Y, i^*j_*X[k]) = 0$ for all $k < -1$;
- $\text{Hom}_{\mathcal{D}^b(B)}(i^*j_*X, Y[k]) = 0$ for all $k < 0$;
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Then $Z = i_* Y \oplus K_X$ is tilting if and only if $\exists X'_{-1}, X'_0, X'_1, X'_2 \in \text{mod}(B)$:

- X'_2 projective;
- $i^* j_* X \cong X'_{-1}[1] \oplus X'_0 \oplus X'_1[-1] \oplus X'_2[-2]$;
- $\text{Hom}_B(X_m, X_n) = 0$ whenever $n - m \geq 2$.

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Thank you for your attention.