LOCALISATION IN ALGEBRA AND GEOMETRY

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CONTENTS

These lecture notes contain the material presented in a minicourse at the University of Verona in February/March 2012. Throughout the notes, some proofs will be sketchy, inviting the reader to complete them. Also some observations require checking and some exercises are proposed. The exercises signaled with *Background* are important to the consolidation of the material while the exercises signaled with *Needed* are integrant part of our exposition and may be used later in these notes. The exercises signaled with a * are of a more challenging nature that may go beyond the scope of these lectures and may require some bibliographical support.

All our rings are unital and we consider the zero ring to have identity equal to zero. $Mod(R)$ will denote the category of right R-modules and, unless otherwise stated, we will always be considering right modules.

1. COMMUTATIVE LOCALISATION

Throughout this section, unless otherwise stated, R will denote a commutative ring. Recall that a **multiplicative subset** of R is a set $S \subset R$ such that

(i) $1_R \in S$;
(ii) For all $x, y \in S$, we have $xy \in S$.

Trivial examples of such multiplicative sets are

- the set formed by the identity of the ring;
- the set of units (invertible elements) of the ring;
- the set of non-zero divisors of the ring;
- the set of non-negative powers of a non-zero element of the ring.

Many thanks are due to Lidia Angeleri Hügel for her kind invitation to Verona and hospitality during this visit. I am also thankful to Frederik Marks and Rishi Vyas for their comments on preliminary versions of these notes. Finally, thank you to all the participants in these lectures for making them such an enjoyable experience. This visit was supported by DFG's Schwerpunktprogramm Darstellungstheorie 1388.

Important examples of multiplicative sets are associated with prime ideals. We recall this notion.

Definition 1.1. For any ring R, an ideal $P \lhd R$ is said to be **prime** if for any two ideals $I \lhd R, J \lhd R$ such that $IJ \subset P$ we have $I \subset P$ or $J \subset P$; it is said to be **completely prime** if for any two elements $a, b \in R$ such that $ab \in P$ we have $a \in P$ or $b \in P$. The set of prime ideals of R is called the **spectrum of ring** R and we denote it by $Spec(R)$.

Example 1.2. In $\mathbb{Z}$, $p\mathbb{Z}$ is a completely prime ideal, for all prime $p \in \mathbb{Z}$ (check!).

Note that the complement of a completely prime ideal P in a ring R is a multiplicative set. If R is commutative, the complement of a prime ideal is also a multiplicative set since these notions are equivalent, as shown in the following exercise.

Exercise 1.3 (*Needed*)**.** Show that, for a (not necessarily commutative) ring R, a completely prime ideal is prime. Furthermore, prove that the converse holds if R is commutative but not in general.

A ring R is said to be a **domain** or **completely prime ring** if (0) is a completely prime ideal and a **prime ring** if (0) is a prime ideal. These notions coincide if R is commutative (see exercise).

Given a multiplicative set $S \subset R$, R commutative, consider the equivalence relation (check!) $\sim$ defined in $S \times R$ by: $(s,x) \sim (s',x')$ if and only if there is $t \in S$ such that $t(s'x - sx') = 0$. We will denote the equivalence class of a pair (s,x) by $\frac{x}{s}$ and the quotient set will be denoted by $S^{-1}R$.

Theorem 1.4. *Let R be a commutative ring and $S \subset R$ a multiplicative set. The operations*

$$+ : S^{-1}R \times S^{-1}R \longrightarrow S^{-1}R,\ (\frac{x}{s}, \frac{y}{t}) \mapsto \frac{tx + sy}{st}$$
$$\cdot\ : S^{-1}R \times S^{-1}R \longrightarrow S^{-1}R,\ (\frac{x}{s}, \frac{y}{t}) \mapsto \frac{xy}{st}$$

endow $S^{-1}R$ with a ring structure with unit $\frac{1}{1}$. Moreover, there is a natural ring homomorphism $Q_S : R \longrightarrow S^{-1}R$ such that $Q_S(r) = \frac{r}{1}$ and any ring homomorphism $\phi : R \longrightarrow B$, for some commutative ring B, such that the elements in $\phi(S)$ are invertible must factor through Q_S, i.e., there is $\bar{\phi} : S^{-1}R \longrightarrow B$ such that $\phi = \bar{\phi}Q_S$.

The ring $S^{-1}R$ is called the **localisation of** R **at** S. If $S = R \setminus P$ for a prime ideal P, then we denote $S^{-1}R$ by R_P.

Proof. We sketch the proof, leaving some details to the reader. Using the definition of the equivalence relation $\sim$ it is easy to check that the operations $+$ and $\cdot$ are well-defined and that they indeed endow $S^{-1}R$ with a ring structure. It is clear that Q_S is a ring homomorphism and all that we need to show is the universal property stated in the theorem. This can be done by defining $\bar{\phi}(\frac{x}{s}) = \phi(x)\phi(s)^{-1}$ and checking that indeed $\bar{\phi}$ factors through Q_S. □

Exercise 1.5 (*Background*)**.** Observe that if R is a domain, then Q_S is injective.

We can do a similar construction for R-modules. Given a module M over a commutative ring R and a multiplicative subset S, there is also an R-module $S^{-1}M$ constructed as the quotient of $S \times M$ by the equivalence relation (check!) $(s,x) \sim (s',x')$ if and only if there is $t \in S$ such that $t(s'x - sx') = 0$ and endowed with the natural operations. Note that $S^{-1}M$ is, first of all, a $S^{-1}R$-module and that its R-module structure comes from the map Q_S (check!).

Exercise 1.6 (*Background*)**.** For R commutative, M an R-module and $S \subset R$ a multiplicative set, state and prove an analogous theorem to 1.4 for localisation of modules, i.e., show that there is a homomorphism of R-modules $Q_{S,M} : M \longrightarrow S^{-1}M$ satisfying a suitable universal property.

Exercise 1.7 (*Background*)**.** Let R be a commutative ring. Show that the prime ideals of $S^{-1}R$ are in bijection with the prime ideals of R that do not intersect S. If P is a prime ideal, observe that the prime ideals of R_P are in bijection with the prime ideals of R contained in P and that this is a complementary behaviour to that of the prime ideals of R/P. **Hint:** $Q_S^{-1}(J) \lhd R$ for all $J \lhd S^{-1}R$.

Exercise 1.8 (*Needed*)**.** Let R be a domain and let f be a nonzero element. Let $S_f = \{f^n : n \in \mathbb{N}_0\}$. Check that S_f is a multiplicative set. Moreover prove that $R[X]/\langle 1 - fX \rangle \cong S_f^{-1}R$. In general, we will denote $S_f^{-1}R$ by R_f.

Recall that the **radical** of an ideal I is the set of elements $r \in R$ such that there is $n \in \mathbb{N}$ such that $r^n \in I$, and we denote it by $\sqrt{I}$. An ideal I is said to be radical if $\sqrt{I} = I$. As an interesting (and useful) application of localisation, we have the following lemma.

Lemma 1.9. *Let R be a commutative ring and I and ideal of R. Then, the radical of I is equal to the intersection of all prime ideals containing I, i.e., $\sqrt{I} = \cap_{I \subset P \in Spec(R)} P$.*

Proof. Clearly the prime ideals are, by definition, radical - and so is any intersection of prime ideals. Since I is contained in the intersection of all prime ideals containing I and this intersection is radical, the radical of I is also contained there. Conversely, suppose $f \in R$ such that $f \notin \sqrt{I}$. Consider localised ring R_f and the corresponding localised ideal I_f. By Zorn's lemma, the quotient R_f/I_f has a maximal ideal (just as any ring with identity - check!) which, by exercise 1.7, corresponds to an ideal of R_f contains I_f. By the same exercise, this corresponds to an ideal of R containing I and not intersecting the multiplicative set $\{f^n : n \in \mathbb{N}_0\}$. Moreover, since we started with a prime ideal in R_f/I_f, the corresponding ideal in R will also be prime. This finishes the proof since then $f \notin \cap_{I \subset P \in Spec(R)} P$. □

Remark 1.10. Note that an element is nilpotent if and only if it lies in $\sqrt{(0)}$ which, by this lemma, is equivalent to say that it lies in every prime ideal of the ring.

Exercise 1.11 (*Needed*)**.** Show that $S^{-1}M = 0$ if and only if for all $x \in M$, there is $s \in S$ such that $sx = 0$.

Exercise 1.12 (*)**.** S^{-1} can be regarded as an endofunctor of the category of R-modules by composing the forgetful functor with the natural functor from the category of R-modules to to the category of $S^{-1}R$-modules. Show that this endofunctor is exact, i.e., that it preserves short exact sequences.

2. Structure sheaves in affine algebraic geometry

In this section we will show how localisation of commutative rings arises naturally in algebraic geometry. The key notion that we will be dealing with requires some intuition. In geometry we usually deal with a topological space with some additional structure (for example, a Riemannian metric, a complex structure, a symplectic form, ...). Algebraically, we are interested on the rings of *regular functions* (for some appropriate notion of regularity) on the topological space. However, the geometry of the space is rarely entirely codified in the ring of regular functions on the whole of the topological space. Indeed we need to know regular functions on each open subset of the space, i.e., the geometry must be studied from a local point of view. A sheaf is a single object that allows us to store all that information.

Definition 2.1. Let X be a topological space. A **presheaf of rings** $\mathcal{F}$ is the association of a ring $\mathcal{F}(U)$, whose elements are called **sections**, to each open subset U of X together with a set of ring homomorphisms called **restriction functions**

$$\{\rho_{U,V} : \mathcal{F}(V) \longrightarrow \mathcal{F}(U) |\ U \subset V \text{ open subsets of X}\}$$

such that, for U,V and W open subsets of X, we have that if $U \subset V \subset W$, then $\rho_{U,W} = \rho_{U,V}\rho_{V,W}$. Furthermore, $\mathcal{F}$ is a **sheaf of rings** if it satisfies the following conditions

- If $(U_i)_{i\in I}$ is a family of open sets of X, $U = \bigcup_{i\in I} U_i$ and $s,t \in \mathcal{F}(U)$ are such that $\rho_{U_i,U}(s) = \rho_{U_i,U}(t)$, for all $i \in I$, then $s = t$.
- If $(U_i)_{i\in I}$ is a family of open sets of X, $U = \bigcup_{i\in I} U_i$ and $(s_i \in \mathcal{F}(U_i))_{i\in I}$ is a family of sections of $\mathcal{F}$ such that
$$\rho_{U_i\cap U_j,U_i}(s_i) = \rho_{U_i\cap U_j,U_j}(s_j), \forall i, j \in I,$$
then there is $s \in \mathcal{F}(U)$ such that $s_i = \rho_{U_i,U}(s)$. By the condition (1), s is then unique.

This is a rather complicated definition but intuitively one should keep in mind rings of functions over a manifold and restrictions between open sets. In this setting, condition (1) is essentially saying that a function defined on an open set is completely determined by its restrictions to an open cover of that set. Condition (2), on the other hand, says that a family of functions which are compatible on the double intersections of an open cover can be glued to the whole open set.

Exercise 2.2 (*Background*)**.** Let X be a topological space and let R be a ring. Show that C, obtained by associating to each open set U of X the ring $C(U) = \{s : U \longrightarrow R : s \text{ is constant}\}$ (which is isomorphic to R) and with restriction functions being the natural restrictions to smaller open sets (the same as the identity in R), is a presheaf of rings. Moreover show that it satisfies condition (1) but not condition (2) and, therefore, it is not a sheaf of rings.

Exercise 2.3 (*Background*)**.** Let X be a topological space and let R be a ring. Show that L, obtained by associating to each open set U of X the ring $L(U) = \{s : U \longrightarrow R : s \text{ is locally constant}\}$ (recall that a function is locally constant if for any point there is a neighbourhood where it is constant) and with restriction functions being the natural restrictions to smaller open sets is a sheaf of rings.

To look at functions at a point, we can define the **stalk** of a sheaf $\mathcal{F}$ at a point $x \in X$. The idea is to look at *smaller and smaller* open neighbourhoods of the point and *take a limit*. For a rigorous definition, one needs the notion of a direct limit. Here we will just present a construction of this limit, without proof that it is indeed a direct limit (i.e., it satisfies a certain universal property). For more details, we refer to our bibliography.

To construct this limit, let G be the disjoint union of $\mathcal{F}(U)$ for all open sets U containing the point $x \in X$. Clearly this ring G contains a lot of redundant information since it does not account for the information of which sections restrict to the same section in a smaller neighbourhood of the point. To enter this information we introduce an equivalence relation (check!) $\sim$ in G by saying that a section $s \in \mathcal{F}(U)$ is equivalent to a section $t \in \mathcal{F}(V)$ if and only if there is an open set $W \subset U \cap V$ (note that $x \in U \cap V \neq \emptyset$) such that $\rho_{W,U}(s) = \rho_{W,V}(t)$. We define the **stalk of** $\mathcal{F}$ **at** x as the quotient $G/\sim$ and we denote it by $\mathcal{F}_x$.

Exercise 2.4 (*Background*)**.** Compute the stalks for the presheaf C and for the sheaf L above defined.

Remark 2.5. There is a construction called **sheafification** that produces a sheaf of rings out of a presheaf of rings. The sheaf obtained will have as section over an open U functions from U to the disjoint union of the stalks at each point of U satisfying some properties. For more details, use our bibliography in algebraic geometry.

We are interested in some very specific types of sheafs, namely the ones that are intrinsically associated with affine schemes, a fundamental concept in algebraic geometry.

The usual approach to algebraic geometry starts with discussing algebraic sets, i.e., solution sets of polynomial equations. These are the motivation for algebraic varieties and algebraic varieties are then generalised to the language of schemes. We will introduce affine schemes directly and later we will see how they generalise affine algebraic varieties and solutions of polynomial equations.

For R a commutative ring, $Spec(R)$ is endowed with a natural topology: the **Zarisky topology**. The closed sets of this topology are

$$V(I) = \{P \in Spec(R) : I \subset P\}$$

for all $I \lhd R$.

Exercise 2.6 (*Background*)**.** Check that the Zarisky topology is indeed a topology. Namely, observe that, for a family of ideals $(I_\lambda)_{\lambda\in\Lambda}$ of R, $V(\sum_{\lambda\in\Lambda} I_\lambda) = \cap_{\lambda\in\Lambda} V(I_\lambda)$ and that, additionally, if Λ is finite, $V(\cap_{\lambda\in\Lambda} I_\lambda) = \cup_{\lambda\in\Lambda} V(I_\lambda)$.

Recall that a **noetherian ring** is a ring such that every ascending chain of ideals stabilises.

Exercise 2.7 (*Background*)**.** Show that if R is a commutative noetherian ring, then $Spec(R)$ is a compact topological space. More than that, every open subset of $Spec(R)$ is compact!

Remark 2.8. In algebraic geometry the term *compact* is usually replaced by *quasi-compact*, to emphasise the fact that the Zarisky topology is not Hausdorff (check!).

It is easy to observe that an ideal is maximal if and only if it is a closed point in $Spec(R)$ (check!). Also (0) is prime if and only if R is an integral domain, and its closure is the whole spectrum. This is called a **generic point**.

Exercise 2.9 (*Background*)**.** Describe the Zarisky topology in $Spec(\mathbb{Z})$. Which points are closed?

The open sets are, of course, the complements of the closed ones. We will deal with a basis for the topology (i.e., a set such that any open can be written as the union of basis elements). Given an element $f \in R$, let $U(f)$ denote the open set which is the complement of $V(fR)$. It is easy to see (check!) that $U(f)$ is the set of prime ideals that do not contain f.

Exercise 2.10 (*Needed*)**.** Show that $(U(f))_{f\in R}$ forms a basis for the Zarisky topology in $Spec(R)$. Moreover, show that we only need to consider the open sets $U(f)$ where f is not nilpotent, since for f nilpotent, $U(f) = \emptyset$ (check remark 1.10).

We will want to define a certain sheaf on $Spec(R)$ that will contain relevant algebro-geometric information (*regular functions on open sets*). It is, however, difficult to do so for all open sets and, therefore, we restrict ourselves to defining a sheaf on the basis introduced above. The following theorem is, therefore, important for our target.

Theorem 2.11. *Let X be a topological space and $(B_i)_{i\in I}$ a basis of open sets for X. Suppose $\mathcal{F}$ satisfies the sheaf axioms for the open sets of the basis. Then $\mathcal{F}$ extends as a sheaf of rings to X.*

Proof. We just give a vague idea of this proof. This is based on a sheafification process (see remark 2.5). The idea is to consider the stalks $\mathcal{F}_x$ for all $x \in X$ as the direct limit of section of $\mathcal{F}$ over the open sets of the basis containing the point $x \in X$. Then, with these stalks one can mimic the sheafification process of a presheaf and then check that indeed one gets a sheaf that extends $\mathcal{F}$, i.e., it agrees with $\mathcal{F}$ on the open sets of the basis. □

For each open set U in $Spec(R)$, let $\tilde{U}$ denote the union of the prime ideals in U. Then the complement of $\tilde{U}$ in R, denoted by S_U, is a multiplicative set (check!). We can define a presheaf $\tilde{O}_{Spec(R)}(U) := S_U^{-1}R$. We need the following lemma in order to define the restriction maps of our structure sheaf.

Lemma 2.12. *Let R be a commutative ring and $f \in R$ not nilpotent. Then, $s \in S_{U(f)}$ if and only if there is $t \in S_{U(f)}$ and $n \in \mathbb{N}_0$ such that $st = f^n$.*

Proof. Note that $s \in S_{U(f)}$ implies that $V(sR) \subset V(fR)$. Therefore $\cap_{fR \subset P \in Spec(R)} P \subset \cap_{sR \subset P \in Spec(R)} P$ and thus, by lemma 1.9, $\sqrt{fR} \subset \sqrt{sR}$. Therefore, we get that there is $n \in \mathbb{N}_0$ such that $f^n = st$ for some $t \in R$. Suppose that $t \notin S_{U(f)}$. Then there is a prime ideal P in $U(f)$ such that $t \in P$, implying that $f^n \in P$. Since prime ideals are radical, we get $f \in P$, a contradiction to the fact that $P \in U(f)$.

Conversely, suppose there is $t \in S_{U(f)}$ and $n \in \mathbb{N}_0$ such that $st = f^n$. Supposing that $s \notin S_{U(f)}$ leads to a contradiction just as the argument above, thus finishing the proof. □

Corollary 2.13. *Let R be a commutative ring and $f, g \in R$ non-nilpotent. If $U(g) \subset U(f)$, then there is a natural map $\psi_{g,f} : R_f \longrightarrow R_g$.*

Proof. If $U(g) \subset U(f)$, then $V(fR) \subset V(gR)$ and the proof of the previous lemma allows us to conclude that there is $t \in R$ and $n \in \mathbb{N}_0$ such that $ft = g^n$. This means that f is invertible in R_g and thus, the universal property of localisation gives us a natural map from R_f to R_g. □

We are now ready to introduce the definition of affine scheme and structure sheaf.

Definition 2.14. An **affine scheme** is a pair $(Spec(R), O_{Spec(R)})$, for R a commutative ring, where $Spec(R)$ is endowed with the Zarisky topology and $O_{Spec(R)}$ is a sheaf of rings, called the **structure sheaf of** $Spec(R)$, which is defined on the basis $(U(f))_{f \in R}$ by $O_{Spec(R)}(U(f)) = R_f$ with restriction maps $\rho_{U(g),U(f)} : R_f \longrightarrow R_g$ given by $\rho_{U(g),U(f)} = \psi_{g,f}$ or by 0 when either R_f or R_g is the zero ring, for $U(g) \subset U(f)$.

In fact, one can prove that the structure sheaf is the sheafification of the presheaf $\tilde{O}_{Spec(R)}$ defined above.

Exercise 2.15 (*Background*). Check that the presheaf $\tilde{O}_{Spec(R)}$ coincides with $O_{Spec(R)}$ on the open sets $U(f)$ for f non-nilpotent. What happens if f is nilpotent?

Let us compute the stalks of the structure sheaf. For this we use the idea in the proof of theorem 2.11, i.e., we compute the stalks as direct limits over the open sets of the basis $(U(f))_{f \in R}$.

Proposition 2.16. *Let R be a ring and P a prime ideal of R. Then the stalk of $O_{Spec(R)}$ at P, $O_{Spec(R),P}$ is isomorphic to R_P.*

Proof. Let $f \in R$ such that $P \in U(f)$ (i.e., $f \notin P$). Clearly, by the universal property of localisation, there is a canonical homomorphism of rings, call if ϕ_f from R_f to R_P. These maps are compatible with restriction maps to open sets $U(g) \subset U(f)$, i.e., $\phi_g \rho_{U(g),U(f)} = \phi_f$ (check! - use corollary

2.13). The universal property of the direct limit (omited in these notes) guarantees, therefore, a ring homomorphism, call it Φ, from the stalk $O_{Spec(R),P}$ to R_P. We need to prove that Φ is surjective and injective. Suppose $\Phi(s)=0$. Then, by definition of direct limit, there is an open neighbourhood of P, $U(f)$, where $s=\frac{b}{f^n}$, for some $b\in R$ and $n\in\mathbb{N}_0$, such that $\phi_f(\frac{b}{f^n})=0$. This means that there is $t\in R\setminus P$ such that $tb=0$. But this implies, by corollary 2.13, that $\rho_{U(f)\cap U(t),U(f)}(\frac{b}{f^n})=0$ and, thus, $\phi_f(s)=\phi_t(\rho_{U(f)\cap U(t),U(f)}(\frac{b}{f^n}))=0$.

To check surjectivity, let $s\in R_P$, i.e., $s=\frac{a}{f}$. Then, clearly, $s=\phi_f(\frac{a}{f})$ and thus s will lie in the image of the direct limit. □

We now look to examples and try to develop some intuition about the affine schemes in the friendlier atmosphere of algebraic varieties. Let us start by looking at the affine scheme

$$\mathbb{A}^2_{\mathbb{C}}=(Spec(\mathbb{C}[X,Y]),O_{Spec(\mathbb{C}[X,Y])}).$$

It is well-known (check!) that the maximal ideals of $\mathbb{C}[X,Y]$ are of the form $\langle X-a,Y-b\rangle$, where $a,b\in\mathbb{C}$ (these are the closed points). Therefore the closed points in $\mathbb{A}^2$ is in bijection with the plane $\mathbb{C}^2$. It is a domain and, thus, the zero ideal (0) is a generic point. The Zarisky topology, however, is very different from the usual topology - see for example the next exercises.

Exercise 2.17 (*Background*)**.** Show that $\mathbb{Z}^2$ is dense in the Zarisky topology in $\mathbb{A}^2_{\mathbb{C}}$.

Exercise 2.18 (*Background*)**.** Show that any Zarisky-open set in $Spec(\mathbb{C}[X,Y])$ is not limited in the usual topology, i.e., it is not contained in a ball of finite radius (recall that the norm of a point (z,w) in $\mathbb{C}^2$ is given by $\sqrt{|z|^2+|w|^2}$).

Let us look at the affine scheme $W=(Spec(\mathbb{C}[X,Y]/\langle X\rangle),O_{Spec(\mathbb{C}[X,Y]/\langle X\rangle)})$. By exercise 1.7 we know that the maximal ideals of $\mathbb{C}[X,Y]/\langle X\rangle$ are the maximal ideals of $\mathbb{C}[X,Y]$ containing $\langle X\rangle$, i.e., ideals of the form $\langle X,Y-b\rangle$ for $b\in\mathbb{C}$ (the closed points). Note that the closed points are in bijection with the zeros of the polynomial X in $\mathbb{C}^2$. Therefore, W is a *complex line*, i.e., it is isomorphic as a scheme to $\mathbb{A}^1_{\mathbb{C}}$ (requires proof, which we will omit). The ideal $\langle X\rangle$ is the generic point. This is an observation that is justified by the following important result. For an ideal I of a polynomial ring $\mathbb{K}[X_1,...,X_n]$ over a field $\mathbb{K}$, we will denote by $Z(I)$ the set of elements in $\mathbb{K}^n$ such they are zeros of all polynomials in I.

Theorem 2.19 (Hilbert's Nullstellensatz)**.** *Let $\mathbb{K}$ an algebraically closed field, $n\in\mathbb{N}$, I an ideal of $R=\mathbb{K}[X_1,...,X_n]$ and let $\pi:R\longrightarrow R/I$ be the canonical projection. Then there is a bijection between $Z(I)$ and the closed points of $Spec(R/I)$, sending $(a_1,...,a_n)$ to the ideal $\langle\pi(X_1-a_1),...,\pi(X_n-a_n)\rangle$.*

Proof. Assume that the set of zeros of I is non-empty (this, although not obvious, holds over algebraically closed fields - we omit the proof of this fact). It is enough to prove the theorem for $I=0$ - the result then follows using exercise 1.7. It is clear that an ideal of the form $\langle X_1-a_1,...,X_n-a_n\rangle$ is maximal (check!). Conversely, if J is a maximal ideal, by our assumption it has a zero $(a_1,...,a_n)$ in $\mathbb{K}^n$. Let $J'=\langle X_1-a_1,...,X_n-a_n\rangle$. If J' is not contained in J, then $J'+J=R$ and therefore there are $f\in J$ and $g\in J'$ such that $f+g=1$. This is a contradiction since $f(a_1,...,a_n)=0=g(a_1,...,a_n)$. Therefore $J'\subset J$ and since J' is maximal we have $J'=J$. □

Let us now compute some stalks of the structure sheaf. In $\mathbb{A}^2$, let $P = \langle X \rangle$. By proposition 2.16, the stalk $O_{\mathbb{A}^2,P}$ is the localisation of $\mathbb{C}[X,Y]_P$, i.e.,

$$O_{\mathbb{A}^2,P} = \left\{ \frac{f}{g} : f,g \in \mathbb{C}[X,Y], g(0,a) \neq 0, \forall a \in \mathbb{C} \right\}.$$

Note that the elements of this ring are the fractions of polynomials whose denominator does not vanish on the zero locus of P in the complex plane! This is why we say that the structure sheaf is the sheaf of *regular* functions in algebraic geometry. Continuing with the other example above, the affine scheme W, let us compute the stalk at the closed point $Q = \langle Y-1+\langle X \rangle \rangle$. According to proposition 2.16, $O_{W,Q}$ is the localisation of $\mathbb{C}[X,Y]/\langle X \rangle$ at Q, i.e.,

$$O_{W,Q} = \left\{ \frac{f}{g} : f,g \in \mathbb{C}[X,Y]/\langle X \rangle, g(1+\langle X \rangle) \neq 0 \right\}.$$

Again, the same observation. The zero locus of Q is a point on the complex line W. The rational functions which are regular at that point are precisely those whose denominator does not vanish at that point.

Exercise 2.20 (*Background*)**.** Let R be a commutative ring and let $X = (Spec(R), O_{Spec(R)})$. Show that the stalk of the structure sheaf at a closed point is a local ring (i.e., it has a unique maximal ideal). Observe that, if R is an integral domain, the stalk of the structure sheaf at a generic point is the field of fractions of R.

3. Ore localisation

If R is a noncommutative ring and we want to work with *fractions* of the elements of R we run into an immediate problem: the fractions must be *1-sided*, i.e., it is diferent to talk about $s^{-1}x$ or xs^{-1}. And this is where the fun begins! For the rest of these notes we will work with right fractions, i.e.m fractions of the form xs^{-1}. We start by making sense of what a ring of fractions is in this context.

Definition 3.1. Let R be a ring and $S \subset R$ a multiplicative subset of elements of R A **right ring of fractions** of R with respect to S is a ring Q, with a homomorphism of rings $\phi_S : R \longrightarrow Q$ such that

- $\phi_S(s)$ is invertible in Q for all $s \in S$;
- every element of Q can be written as a product $\phi_S(x)\phi_S(s)^{-1}$, where $x \in R$ and $s \in S$;
- $Ker(\phi_S) = \{x \in R : \exists s \in S : xs = 0\}$

Note that if the elements of S are **regular** (i.e., elements which are not annihilated neither on the left nor on the right; they are neither left nor right zero divisors), then the last condition just says that the homomorphism is injective (compare with exercise 1.5).

Fixing a side is, however, far from enough as we run into another immediate problem: how to multiply to elements of our desired localised ring? How to calculate $xs^{-1}yt^{-1}$? A naive, commutative-inspired approach would involve changing the order of s^{-1} and y - but the ring is noncommutative! Also, what does it mean, the middle segment $s^{-1}y$, given that we have decided to work with right fractions? All of these problems hint that a multiplicative set will not, in general, suffice to allow a localisation process and indeed right rings of fractions do not always exist. To get a positive answer about existence we need the following key condition.

Definition 3.2. Let S be a multiplicative subset of a ring R. We say that S is a **right Ore set** in R if, for any $x \in R$ and $s \in S$, there are $y \in R$ and $t \in S$ such that $xt = sy$. Moreover, S is a **right denominator set** in R if it is a right Ore set and for all $x \in R$ and $s \in S$ such that $sx = 0$, then there is $t \in S$ such that $xt = 0$.

The Ore condition is popularly known as *the poor man's commutativity*. It is a commutative-inspired idea, that allows one to change the side of the denominator - aiming at solving the problem explained in the paragraph above.

Recall that, given a right R-module M, the (right) **annihilator** of an element $m \in M$, $Ann(m)$ is the set of elements $x \in R$ such that $mx = 0$. The annihilator of M, $Ann(M)$, is the intersection of all $Ann(m)$ with $m \in M$, i.e., the set of elements $x \in R$ such that $Mx = 0$. It is easy to see that $Ann(M)$ is always an ideal while $Ann(m)$, for $m \in M$ is, in general, only a right ideal. (check!).

Observe that if S contains only regular elements, then S is a right denominator set if and only if it is a right Ore set. Moreover, we will mostly be working with noetherian rings and, in that setting, we have the following nice result.

Proposition 3.3. *Let S be a right Ore set in a ring R. If R is noetherian, then S is a denominator set.*

Proof. Let $s \in S$ and $x \in R$ such that $sx = 0$. Since the R is noetherian and $Ann(s^n) \subset Ann(s^{n+1})$ for all $n \in \mathbb{N}$, we have that there is an integer $k \in \mathbb{N}$ such that $Ann(s^k) = Ann(s^{k+1})$. Since S is a right Ore set, there are $y \in R$ and $t \in S$ such that $xt = s^n y$. Now, $s^{n+1}y = sxt = 0$ since $sx = 0$ and thus $y \in Ann(s^{n+1}) = Ann(s^n)$, showing that $xt = s^n y = 0$. □

Theorem 3.4. *[Asano, Ore; Gabriel] Let R be a ring and S a multiplicative subset of elements of R. There exists a right ring of fractions of R with respect to S if and only if S is a right denominator set.*

Proof. We give some ideas of the proof and we leave some details in the form of exercise. Our strategy is to reduce to a setting in which S contains only regular elements, in which case localisation behaves better.

Exercise 3.5 (*Background*)**.** Show that if R has a right ring of fractions with respect to S, then S is a right denominator set.

The other direction is harder. Suppose S is a right denominator set. It is easy to check (do it!) that

$$t_S(R) := \{x \in R : \exists s \in S : xs = 0\}$$

is an ideal of R. Let ϕ denote the projection map to the corresponding quotient, $\tilde{R} = R/t_S(R)$.

Exercise 3.6 (*Background*)**.** Show that $\phi(S)$ is a right denominator set in $\tilde{R}$, and that, moreover, $\phi(S)$ contains only regular elements of $\tilde{R}$.

Now we are in the setting we want. Let us show that $\tilde{R}$ admits a right ring of fractions with respect to $\phi(S)$. Let us define an equivalence relation in $\tilde{R} \times \phi(S)$ as follows:

$$(x,s) \sim (y,t) \Leftrightarrow \exists z,u \in \tilde{R} : sz = tu, xz = yu.$$

Exercise 3.7 (*)**.** Check that $\sim$ is an equivalence relation. **Hint:** Show that, for all $a,b \in R$, if $sa = tb$ then $xa = yb$.

Given a pair $(x,s) \in \tilde{R} \times \phi(S)$, we will denote by xs^{-1} the equivalence class in the quotient $\tilde{R} \times \phi(S)/\sim$ which we will denote by Q. We define two operations in Q:

$$+ : Q \times Q \longrightarrow Q,\ (xs^{-1}, yt^{-1}) \mapsto (xz + yu)(tu)^{-1}$$

where $(z,u) \in \tilde{R} \times \phi(S)$ such that $sz = tu$, and

$$\cdot : Q \times Q \longrightarrow Q,\ (xs^{-1}, yt^{-1}) \mapsto xz(tu)^{-1}$$

where $(z,u) \in \tilde{R} \times \phi(S)$ such that $yu = sz$.

Exercise 3.8 (*Background*)**.** Check that the operations are well-defined and that they endow Q with a ring structure. **Hint:** Use the same hint of the previous exercise.

With these operations it is clear that Q is a right ring of fractions of $\tilde{R}$ with respect to $\phi(S)$, in which $\tilde{R}$ is a subring (check!).

Exercise 3.9 (*Background*)**.** Check that, indeed, Q is also a right ring of fractions for R with respect to S, where the map from R to Q is the composition of ϕ with the inclusion of $\tilde{R}$ in Q.

□

The right ring of fractions of R with respect to S will commonly be denoted by RS^{-1}. The following proposition states that, indeed, our construction gives us a localisation in the universal sense.

Proposition 3.10. *Let $\phi : R \longrightarrow Q$ be a right ring of fractions for R with respect to a right denominator set S. For any ring homomorphism $\psi : R \longrightarrow T$ such that $\psi(s)$ is invertible for all $s \in S$, there is a unique factorisation of ψ by ϕ, i.e., there is a unique ring homomorphism $\theta : Q \longrightarrow T$ such that $\psi = \theta\phi$.*

Proof. If it exists, θ is uniquely determined since it must be defined by $\theta(\phi(x)\phi(s)^{-1}) = \psi(x)\psi(s)^{-1}$ for all $x \in R, s \in S$.

To prove existence, we again reduce to the setting in which S contains only regular elements. Let $r \in Ker(\phi)$.

Exercise 3.11 (*Background*)**.** Show that $\psi(r) = 0$ and that, therefore, there is $\tilde{\theta} : \phi(R) \longrightarrow T$ such that $\psi = \tilde{\theta}\phi$.

As in the previous theorem, Q is a right ring of fractions of $\phi(R)$ with respect to $\phi(S)$ and $\phi(S)$ contains only regular elements in $\phi(R)$. We leave the remainder of the proof as an exercise.

Exercise 3.12 (*)**.** Show that $\tilde{\theta}$ extends to a ring homomorphism θ as wanted.

□

Similar results can be obtained, as in the commutative case, for the localisation of modules.

Definition 3.13. Let R be a ring and $S \subset R$ a right denominator subset of R and M a right R-module. Then a **module of fractions** of M with respect to S is a right RS^{-1}-module MS^{-1} (and, thus, an R-module, via the localisation homomorphism $\phi : R \longrightarrow RS^{-1}$) together with an R-module homomorphism $\psi : M \longrightarrow MS^{-1}$ such that

- every element of MS^{-1} can be written as a product $\psi(m)s^{-1}$, where $m \in M$ and $s \in S$;
- $Ker(\psi) = \{m \in M : \exists s \in S : ms = 0\}$

The proof of the following theorem mimics the one of theorem 3.4.

Theorem 3.14. *Let R be a ring, S a right denominator set in R and M a right R-module. Then M admits a right module of fractions.*

Also, the right module of fractions satisfies the expected universal property.

Proposition 3.15. *Let R be a ring, S a right denominator subset of R, M a right R-module. For any K a right RS^{-1}-module and $f : M \longrightarrow K$ an R-module homomorphism, f must factor (uniquely) by the canonical map $\psi : M \longrightarrow MS^{-1}$.*

The following exercises consolidates the material of this section so far and provides an explicit construction of a right module of fractions.

Exercise 3.16 (**Background**)**.** [Goodearl, Warfield] Let S be a right denominator set in a ring R, M a right R-module. Let $T = End_R(M)$, view M as a (T,R)-bimodule and consider the ring

$$U = \begin{pmatrix} T & M \\ 0 & R \end{pmatrix}.$$

Show that the set of diagonal matrices

$$V = \begin{pmatrix} 1 & 0 \\ 0 & S \end{pmatrix}$$

is a right denominator set in U and that

$$K = \left\{ \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & s \end{pmatrix}^{-1} : m \in M, s \in S \right\}$$

is a right ideal of UV^{-1}. Describe K as a right RS^{-1}-module and show that K together with the map $M \longrightarrow K$ defined by

$$m \mapsto \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^{-1}$$

is a module of fractions for M with respect to S.

The period that followed the discovery of theorem 3.4, whose first version is due to Ore in 1931, until the 50's is described by some as a *golden age of classical ring theory*. One could change this to say that it was the Goldie's age of classical ring theory. The following results, which we present without proof due to its lengthy requirements, were proved by Alfred Goldie (often referred to as *The Lord of the Rings*) in the 50's. They have lead to new approaches to the subject, namely the study of orders in noncommutative rings.

Recall that a submodule N of a module M is said to be **essential** if N intersects nontrivially every nonzero submodule of M.

Exercise 3.17 (*Background*)**.** Show that every ideal of a prime ring is essential as a right ideal.

To avoid defining additional terminology, we present the theorems in weaker forms.

Theorem 3.18 (Goldie)**.** *Let R be a prime noetherian ring. Then every essential right ideal of R contains a regular element.*

At a first look this seems a rather unexciting result. But it does have powerful consequences.

Corollary 3.19 (*Goldie*, 1958)**.** *Let R be a prime noetherian ring. Then the set of regular elements S of R is a denominator set and the right ring of fractions RS^{-1} is a simple artinian ring.*

Note that by Artin-Wedderburn's theory, we know precisely who are the simple artinian rings: they are matrix rings over division rings. This is an amazing result, showing the power of Goldie's theorem.

We finish this section with a few comments on prime ideals.

(1) In the commutative setting we always have a notion of R_P, a localisation at a prime P. This is done by inverting the elements in the complement of P. Note that these elements are regular in the quotient R/P.

(2) In the noncommutative setting, if we want to localise at a prime P, we need to make sure that some subset of R associated with it is a right denominator set. The reasonable analogue to the commutative case is not the complement of P (which is not, in general, a multiplicative system! - see definition of prime ideal...) but the set $\mathcal{C}(P)$ formed by the elements $r \in R$ such that $r+P$ is regular in R/P (check that $\mathcal{C}(P)$ is, indeed, a multiplicative set). Note that $\mathcal{C}(P) = R \setminus P$ precisely when P is completely prime (check!)

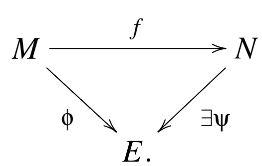
(3) $\mathcal{C}(P)$ is not always a denominator set. For example, let $R = \mathbb{K}\{X,Y\}$ be the free algebra in two variables over a field $\mathbb{K}$ and let P be the prime ideal generated by X. Of course P is completely prime and, therefore, $\mathcal{C}(P) = R \setminus P$. Of course the right Ore condition is not satisfied by $\mathcal{C}(P)$: there would have to be some element t of $\mathcal{C}(P)$ and $r \in R$ such that $Xt = Yr$, which is clearly impossible in the free algebra.

(4) Still, $Spec(R)$ with the Zarisky topology defined as before is a topological space. So, one idea to get noncommutative affine schemes would be to replace localisation at prime ideals by something more general...

4. Injective modules

In this section we start discussing geometric alternatives to the prime spectrum in the noncommutative setting. As we will see, a good such alternative is the **injective spectrum**, i.e., the set formed by all indecomposable injective modules. Recall that a module is **indecomposable** if it is not the direct sum of two submodules. Let us start by studying some properties of injective modules, starting by their definition.

Definition 4.1. A module E over a ring R is said to be **injective** if, for any pair of R-modules M,N and any R-monomorphism $f : M \longrightarrow N$, whenever there is a map ϕ from M to E, there is an extension of it to the whole of N, i.e., there is a map ψ such that the diagram below commutes



To check whether a module is injective or not can be, from the definition, quite difficult. Fortunately there exists an easier criterion, proved by Baer, which can be very useful as we will see. It essentially says that we do not need to check that maps to the injective right module are extendable for all injective homomorphism, but rather only for inclusions of right ideals in the ring.

Theorem 4.2 (Baer's criterion)**.** *Let R be a ring and M a right R-module. Then M is injective if and only if, for every right ideal I of R and every $f \in Hom_R(I,M)$, f extends to a map $\tilde{f} \in Hom_R(R,M)$.*

Proof. We leave it as a (nontrivial) background exercise. **Hint**: For an inclusion of right R-modules $B \leq C$ and a map $f : B \longrightarrow M$, consider pairs (B_i, f_i) where $B \leq B_i \leq C$ and f_i is a homomorphism extending f and make a suitable use of Zorn's lemma. □

The following is a well-known fact about injective modules, the proof of which we omit for brevity.

Theorem 4.3. *For any ring R and any R-module M, there is an injective module $E(M)$ such that $M \leq E(M)$ is an essential submodule, i.e., M intersects all other nonzero submodules. Moreover, $E(M)$ is uniquely determined up to isomorphism.*

For a module M, the module $E(M)$ is called the **injective hull** or the **injective envelope** of M. It is the *smallest* injective module containing M.

Exercise 4.4 (*Needed*)**.** Show that if $M \leq N$ then $E(M) \leq E(N)$.

Exercise 4.5 (*Background*)**.** Check that $\mathbb{Q}$ is the injective envelope of $\mathbb{Z}$ in the category of $\mathbb{Z}$-modules.

For the remainder of this section, the rings under consideration will be noetherian, unless otherwise stated. Indeed we can characterise the property of being noetherian in terms of injective modules.

Note that it is always true that products, summands and finite sums of injective modules are still injective (check!).

Theorem 4.6 (Papp, Bass)**.** *Let R be a ring. Then R is right noetherian if and only if the direct sum of injective right R-modules is still injective.*

Proof. We prove one direction and leave the other as an exercise. Suppose R is right noetherian and let $E = \oplus_{i \in I} E_i$ be a direct sum of injective right R-modules E_i. We shall use Baer's criterion to show E is injective. Let J be a right ideal of R and $f : I \longrightarrow E$ a homomorphism. Since R is right noetherian, J is finitely generated and we let $x_1, ..., x_n$ be a system of generators of J. Each $f(x_k)$ lies in a finite direct sum of E_i's and, therefore, so does the image of J by f. Let E^* be a finite direct sum of E_i's containing the image of f. Since E^* is a finite sum, it is injective and therefore f extends as a homomorphism of right modules $R \longrightarrow E^* \leq E$, finishing the proof.

For the converse we leave a **hint**: Consider an ascending chain $(J_i)_{i \in I}$ of right ideals of R, define a map of R-modules from the union of the chain to the product of the injective hulls $E(R/I_i)$ and show that the image of this map lies in the direct sum. □

Over noetherian rings, injective modules have a particularly nice structure.

Theorem 4.7 (Bass, Matlis, Papp)**.** *If R is a noetherian ring then any injective right R-module is a direct sum of indecomposable injective R-modules.*

Proof. Let E be a nonzero injective right R-module and let $\mathcal{N}$ be a maximal family of independent (i.e., any two elements of the family intersect trivially) nonzero finitely generated submodules of E. The direct sum of these objects is essential (check!). Moreover, the family of injective envelopes of the elements in $\mathcal{N}$ is independent as well (check!) and thus the direct sum of the injective envelopes is an essential submodule of E. Since R is noetherian, the direct sum of injective modules is injective and, therefore, E equals the direct sum of the injective envelopes of the elements in $\mathcal{N}$ (note that injective homomorphisms with an injective domain always split). Now, each module

$N \in \mathcal{N}$ is noetherian and thus any maximal family of independent nonzero submodules of N is finite. Note that each submodule of this family is indecomposable (otherwise it would not be a maximal family) and thus the corresponding injective envelope is an indecomposable submodule of $E(N)$ (check!). This shows that E decomposes as a direct sum of indecomposable injective modules, as wanted. $\square$

We say that a module is **uniform** if all nonzero submodules intersect nontrivially, i.e., all nonzero submodules are essential. It is clear that a uniform module is indecomposable.

Exercise 4.8 (*Needed*)**.** Let R be a ring. Show that every noetherian right R-module contains a uniform submodule.

Exercise 4.9 (*Nedded*)**.** Let R be a ring and E an injective right R-module. Show that U is uniform if and only if $E(U)$ is indecomposable.

Definition 4.10. Let R be a ring. The set of indecomposable injective right R-modules, up to isomorphism, is called the **right injective spectrum of** R and will be denoted by $Inj(R)$.

Exercise 4.11 (*Needed*)**.** Show that, if R is commutative and M is an R-module, then $Ann(m) = Ann(mR)$ for all $m \in M$.

Lemma 4.12. *Let R be a right noetherian ring and U a uniform right R-module. Then there is a unique prime ideal P of R such that $P = Ann(V)$ for some $0 \neq V \leq U$ and $Ann(W) \subset P$ for any $W \leq U$.*

Proof. Let $P = Ann(V)$, for some $V \leq U$, be an ideal which is maximal among the annihilators of submodules of U. First we prove that P is prime. Indeed, suppose $IJ \subset P$ for some ideals $I, J \subset R$. Then $VIJ = 0$ which implies that $P \subset J \subset Ann(VI)$ and therefore, by maximality of P, $P = J$ and P is prime. If W is a nonzero submodule of U then $V \cap W \neq 0$ since U is uniform and, by maximality of P, $P = Ann(V \cap W)$. It is clear from this proof that P is uniquely determined. $\square$

The ideal P in the previous lemma is called the **assassinator ideal of** U and we denote it by $Ass(U)$.

Lemma 4.13. *Let R be a noetherian ring, P a prime ideal and U a uniform right R-submodule of R/P. Then $Ass(U) = Ass(E(U)) = P$.*

Proof. Let $\tilde{U}$ be the right ideal of R containing P such that $\tilde{U}/P = U$. Then it is clear that $\tilde{U}x = 0$ implies $(R\tilde{U}R)(RxR) = 0$ and thus that $x \in P$, proving that $Ann(\tilde{U}) = Ann(U) = P$. The same argument works for any nonzero submodule of U, showing that $Ass(U) = P$. By definition of the assassinator ideal, $P \subset Ass(E(U))$ and let $V \leq E(U)$ be such that $Ann(V) = Ass(E(U))$. Since $E(U)$ is uniform, $V \cap U \neq 0$ and since $V \cap U \leq V$, $Ann(V \cap U) = Ass(E(U))$. But $V \cap U \leq U$ and thus $Ann(V \cap U) = P$, finishing the proof. $\square$

Exercise 4.14 (*Needed*)**.** Show that for P a prime ideal of a commutative ring R, R/P is uniform and thus $E(R/P)$ is indecomposable (and uniform).

For R a noetherian ring, consider the following correspondences:

$$Spec(R) \longrightarrow Inj(R)$$

$$\Phi : P \longmapsto E(U)$$

where U is a uniform right R-submodule of R/P, and

$$Spec(R) \longleftarrow Inj(R)$$
$$Ass(E) \longleftarrow\!\shortmid E : \Psi.$$

Remark 4.15. Note that, in view of lemma 4.13, the correspondence Φ is always injective with the assumption that R is noetherian.

If R is commutative the following result is easy to prove.

Theorem 4.16 (Matlis)**.** *If R is a commutative Noetherian ring, then there is a bijection between prime ideals of R and indecomposable injective modules over R.*

Proof. Let us first show that $\Phi\Psi$ is the identity. Let E be an indecomposable injective module over R and let $P = Ass(E)$ and $V \leq E$ such that $P = Ann(V)$. Observe that, for all $x \in V$, $Ann(xR) = P$ (since it must contain P, but P is maximal among annihilators of submodules of E). Thus we have that $xR \cong R/P$ as R-modules (check!). It is clear that the injective hull of xR is contained in E and therefore it must equal E, thus proving that $E \cong E(R/P)$.

To prove that $\Psi\Phi$ is the identity we just use lemma 4.13 on the uniform module R/P.

□

Exercise 4.17. Compute all indecomposable injective modules of $Mod(\mathbb{Z})$.

Note that in the proof of the theorem, we used commutativity for the fact that R/P is uniform (and $E(R/P)$ is indecomposable). So there is hope that we can generalise this phenomenon to some noncommutative rings. For this effect, we will work with some special class of noetherian rings that, in a sense, are not very far from commutative. This will allow us to get a nice result concerning the prime spectrum.

Definition 4.18. A ring R is called **right fully bounded** if every essential right ideal of R/P contains a nonzero two-sided ideal, for all prime ideals P of R.

This is a seemingly odd definition, but it covers a large class of examples. An important one comes from the following proposition.

Proposition 4.19. *Let R be a ring which is finitely generated as a module over its centre, which we assume to be noetherian. Then R is fully bounded noetherian (**FBN**).*

Proof. It is easy to see that R is also noetherian (check!). Let P be a prime ideal of R and Z the centre of R. Then R/P is finitely generated over $S+P/P$ (check!) and therefore we can assume that R is prime and we only need to prove right boundedness. Suppose R is prime and that $P = (0)$ and let E be an essential right ideal of R. By Goldie's theorem, there is a regular element $c \in E$ and we consider the chain $Z \subset Z + cZ \subset Z + c^2Z \subset \ldots$ which must stop since R is noetherian. Therefore, there is $n \in \mathbb{N}$ (and we choose it to be minimal) such that $c^n \in Z + cZ + \ldots + c^{n-1}Z$, i.e., $c^n = z_0 + cz_1 + c^2z_2 + \ldots. + c^{n-1}z_{n-1}$. Since c is regular, $z_0 \neq 0$ (check!) and thus $z_0 \in cR \cap Z$. Therefore z_0R is a two-sided ideal contained in E. □

Example 4.20. Any finite dimensional algebra over a field is fully bounded noetherian!

Theorem 4.21. *Let R be an FBN ring. Then the correspondence $\Psi : Inj(R) \longrightarrow Spec(R)$, $E \mapsto Ass(E)$ is a bijection.*

Proof. We present just a sketch of the proof. Let $P = Ass(E)$ and let U be a uniform right R-submodule of R/P (which exists by exercise 4.8). We will show $E(U) = E$ and, for this, it is enough to show that U is isomorphic to a submodule of E (since E is indecomposable). Let $V \leq E$ be such that $P = Ann(V)$ and consider a finitely generated submodule $M \leq V$. Clearly M is a uniform R-module with $Ann(M) = P$. We will use the following fact about M without proof (relying heavily on the fact that R/P is a prime noetherian ring and that M is an R/P-module with zero annihilator).

Fact: M is not annihilated by any element in $C(P)$ and, as a consequence, every uniform right ideal of R/P is isomorphic to a submodule of M.

Using this fact, we see that U is isomorphic to a submodule of M, thus proving that P determines E. To finish the proof we use lemma 4.13 and the existence of uniform submodules for noetherian modules. □

Exercise 4.22 (*Background*)**.** Make this bijection explicit for the path algebra over a field $\mathbb{K}$ of the linearly ordered quiver A_n.

Remark 4.23. The converse of the above theorem is also true, i.e., for a noetherian ring R, if the correspondences are bijections, then R is fully right bounded. This means FBN rings are *as far as we can go* in a noncommutative context while still keeping a *large* prime spectrum...

5. Torsion theories and localisation

In this section we introduce the language of torsion theories in the category of R-modules, for a ring R, and explore how this notion relates with some properties of localisation.

The notion of torsion theory is motivated by the following example.

Example 5.1. In the category of abelian groups (or $\mathbb{Z}$-modules), there is a classical notion of torsion. An element g of a group G is torsion if there is $n \in \mathbb{Z}$ such that $ng = 0$. It is easy to see (check!) that the torsion elements form a subgroup of G, call it $\tau(G)$, and that $G/\tau(G)$ does not have any nonzero torsion elements, i.e., it is torsion-free.

More generally, if R is a commutative ring and M is a R-module, an element $m \in M$ is classically said to be torsion if it is annihilated by some element of R, i.e., there is $r \in R$ such that $rm = 0$. Similarly, the set of torsion elements is a submodule of M and the respective quotient has no nonzero torsion elements (check!).

We now define torsion theory.

Definition 5.2. Let R be a ring and $Mod(R)$ its category of right modules. A pair of full subcategories $(\mathcal{T}, \mathcal{F})$ of $Mod(R)$ is said to be a **torsion theory** if:

(1) $\mathrm{Hom}(T, F) = 0$, for all $T \in \mathcal{T}$ and $F \in \mathcal{F}$,
(2) For all $M \in Mod(R)$ there is an exact sequence
$$0 \longrightarrow \tau(M) \longrightarrow M \longrightarrow M/\tau(M) \longrightarrow 0$$
where $\tau(M) \in \mathcal{T}$ and $M/\tau(M) \in \mathcal{F}$. We call $\mathcal{T}$ the **torsion class** and $\mathcal{F}$ the **torsion-free class**. A torsion theory is **hereditary** if the torsion class is closed under submodules.

The association $M \mapsto \tau(M)$ with respect to a fixed torsion pair $(\mathcal{T}, \mathcal{F})$ is functorial (check!). This functor associates to an object M of $Mod(R)$ the maximal submodule of M that lies in $\mathcal{T}$.

We can characterise the subcategories $\mathcal{T}$ which are torsion classes in $Mod(R)$.

Proposition 5.3. *A full subcategory $\mathcal{T}$ of $Mod(R)$ is a torsion class if and only if $\mathcal{T}$ is closed under homomorphic images, direct sums and extensions.*

Proof. If $\mathcal{T}$ is a torsion class, it clearly satisfies the desired properties (check!). Suppose now that $\mathcal{T}$ is closed under extensions, images and direct sums. Define $\mathcal{F}$ as the set of modules F in $Mod(R)$ such that, for any $T \in \mathcal{T}$, $\mathrm{Hom}(T,F) = 0$. For $M \in Mod(R)$, let $\tau(M)$ be the sum of all submodules of M lying in $\mathcal{T}$. Let f be a map from an object $T \in \mathcal{T}$ to $M/\tau(M)$. We will show that this map is zero and, thus, $M/\tau(M) \in \mathcal{F}$. It is easy to see (but check!), there is $N \leq M$, such that

$$0 \longrightarrow \tau(M) \longrightarrow N \longrightarrow \mathrm{im}(f) \longrightarrow 0$$

is an exact sequence. Since $\mathcal{T}$ is closed under quotients, $\mathrm{im}(f)$ lies in $\mathcal{T}$ and since it is closed under extensions, so does N. But, by definition of $\tau(M)$, $N \subset \tau(M)$ and, therefore, $\tau(M) \cong N$. Hence, $im(f) = 0$ thus finishing the proof. $\square$

Exercise 5.4. [*Needed*] Find and prove a dual statement, characterising torsion-free classes in $Mod(R)$.

Definition 5.5. A torsion theory (or its torsion class) in $Mod(R)$ is said to be **cogenerated by an injective object** E if the torsion objects are precisely those M satisfying $\mathrm{Hom}_{Gr(R)}(M,E) = 0$. We denote this torsion class by $\mathcal{T}_E$

Exercise 5.6. [*Needed*] Show that a torsion theory cogenerated by an injective object is hereditary. Also, observe that a torsion theory is hereditary if and only if the torsion-free class is closed under taking injective envelopes.

Indeed, the phonomenon of the previous exercise has a converse.

Proposition 5.7. *Let R be a ring. A torsion theory in $Mod(R)$ is hereditary if and only if it is cogenerated by an injective module.*

Proof. Suppose $(\mathcal{T},\mathcal{F})$ is a torsion theory in $Mod(R)$ and let $E = \prod E(R/I)$, where the product runs over all right ideals $I \lhd R$ such that $R/I \in \mathcal{F}$. Clearly, E is a torsion-free module (check exercises 5.4 and 5.6) and $\mathcal{T} \subset \mathcal{T}_E$ (check!). Now if $M \notin \mathcal{T}$ then there is a cyclic submodule (i.e., generated by one element) $C \leq M$ and a nonzero homomorphism $\alpha : C \longrightarrow F$ to a torsion-free module F (check!). Then the image of α is cyclic and torsion-free, thus there is a nonzero homomorphism $C \longrightarrow E$ (check!), which can be extended to M since E is injective. Therefore, $\mathcal{T} = \mathcal{T}_E$. $\square$

For a right ideal J of a ring R we will use the notation $J \lhd_r R$ and, given $r \in R$ we define a right ideal $r^{-1}J$ as follows

$$r^{-1}J := \{a \in R : ra \in J\}.$$

As we will see, hereditary torsion theories in $Mod(R)$ are intimately related with certain *topological structures* on the ring R. The following key concept will be used throughout the rest of these lecture notes.

Definition 5.8. A **right Gabriel filter** in a ring R is a set of right ideals $\mathbb{L}$ satisfying the following properties:

(1) If $I \in \mathbb{L}$ and $I \subset J$, then $J \in \mathbb{L}$;
(2) If $I,J \in \mathbb{L}$, then $I \cap J \in \mathbb{L}$;
(3) If $I \in \mathbb{L}$ and $x \in R$, then $x^{-1}I \in \mathbb{L}$;

(4) If $I \triangleleft_r R$ and there is $J \in \mathbb{L}$ such that $j^{-1}I \in \mathbb{L}$ for all $j \in J$, then $I \in \mathbb{L}$.

Exercise 5.9 (Background). Suppose $\mathbb{L}$ is a right Gabriel filter in a ring R. Show that if $I \in \mathbb{L}$ and $J \in \mathbb{L}$ then $IJ \in \mathbb{L}$.

Theorem 5.10. *There is a bijection between hereditary torsion theories in $Mod(R)$ and right Gabriel filters in R.*

Proof. We give explicit bijections and we leave to the reader to check all the axioms.

Given a right Gabriel filter $\mathbb{L}$ we define a hereditary torsion class

$$\mathcal{T}_{\mathbb{L}} := \{M \in Mod(R) : Ann(x) \in \mathbb{L}, \forall x \in M\},$$

and given a hereditary torsion class we define a right Gabriel filter by

$$\mathbb{L}_{\mathcal{T}} := \{I \triangleleft_r R : R/I \in \mathcal{T}\}.$$

We leave as a (nontrivial) exercise to check that these correspondences are well-defined and that, indeed, they are inverse to each other. □

Let R be a, not necessarily commutative, noetherian ring. Recall that, for a prime ideal P we define $C(P)$ to be the set of regular elements modulo P, i.e., the set of elements x of R such that $x+P$ is neither left nor right zero divisor in R/P. If R is commutative then $C(P) = R \setminus P$.

The following lemma proves a useful criterion for modules to be torsion with respect to the torsion theory cogenerated by an injective object.

Lemma 5.11. *Given right modules M and N over a ring R, the following conditions are equivalent:*

(1) $Hom_R(M, E(N)) = 0$;
(2) $\forall m \in M$, $\forall n \in N \setminus 0$, $\exists r \in R$: $mr = 0 \wedge nr \neq 0$.

Proof. Suppose $Hom_R(M, E(N)) \neq 0$. Let α be one of its nonzero elements. Choose $u \in M$ such that $\alpha(u) \neq 0$. Now, N is an essential submodule of $E(N)$, and thus there is $x \in R$ such that $0 \neq \alpha(u)x = \alpha(ux) \in N$. If we choose $m = ux$ and $n = \alpha(ux)$, then clearly, given $r \in R$, if $mr = 0$ then $nr = 0$.

Suppose now that (2) is false, i.e., there are $m \in M$ and $n \in N \setminus \{0\}$ such that for all $r \in R$, if $mr = 0$ then $nr = 0$. Then, there is a well defined nonzero homomorphism

$$mR \longrightarrow N,\ mr \mapsto nr.$$

Since $E(N)$ is an injective right R-module, we can find a nonzero homomorphism from M to $E(N)$. □

As a corollary we have, for commutative rings, a clear connection between zero localisation and torsion theories.

Corollary 5.12. *Let R be a commutative ring, P a prime ideal in R and $S = R \setminus P$. Given M an R-module then $S^{-1}M = 0$ if and only if $Hom_R(M, E(R/P)) = 0$, i.e., $M_P = 0$ if and only if M is torsion with respect to the torsion theory cogenerated by $E(R/P)$.*

Proof. This follows from the fact $S^{-1}M = 0$ is equivalent, by definition of commutative localisation, to condition (2) of the above lemma where $N = R/P$. □

Our goal for this section is to prove a similar result for noncommutative noetherian rings. We face, however, an immediate problem: the localisation at a prime ideal may not be defined, as we have seen on section 3. Our approach is to make sense of what zero localisation means without actually localising the module. This can be done by using Gabriel filters and torsion theories.

Thus, for a prime ideal P of a commutative ring R, we have that

$$\mathcal{T}_{E(R/P)} := \{M \in Mod(R) : Hom_R(M, E(R/P)) = 0\} = \{M \in Mod(R) : M_P = 0\}$$

is a hereditary torsion class. By theorem 5.10, the associated right Gabriel filter can be written in the following two ways

$$\mathbb{L}_{\mathcal{T}_{E(R/P)}} = \{I \lhd_r R : Hom_R(R/I, E(R/P)) = 0\} = \{I \lhd_r R : (M/I)_P = 0\}.$$

What exactly does it mean $(R/I)_P = 0$? It means that for all $x + I \in R/I$, there is $s \in R \setminus P$ such that $xs \in I$ or, equivalently, that for all $x \in R$, $x^{-1}I \cap R \setminus P \neq \emptyset$. This is, in fact, a quite general phenomenon.

Exercise 5.13 (*Needed*)**.** Let R be a ring and S a multiplicative set in R. Show that

$$\mathbb{L}_S := \left\{I \lhd_r R : x^{-1}I \cap S \neq \emptyset, \forall x \in R\right\}$$

is a right Gabriel filter.

If $S = C(P)$ for some homogeneous prime ideal P, then we denote the filter by $\mathbb{L}_P$. We say that a module M is **torsion with respect to a multiplicative set** S if M belongs to the torsion class associated with $\mathbb{L}_S$.

Note that, by the previous section, given a ring R and a prime ideal P, $E(R/P)$ is a direct sum of copies of a single indecomposable injective R-module E, namely the injective hull of a uniform submodule of R/P. It is clear, therefore, that the torsion theory cogenerated by $E(R/P)$ is the same as the one cogenerated by E (check!).

Proposition 5.14. *Let R be a noetherian ring and M a right R-module. Then $J \in \mathbb{L}_{E(M)}$ if and only if $m(r^{-1}J) \neq 0$ for all $m \in M \setminus \{0\}, r \in R$.*

Proof. Note that $J \in \mathbb{L}_{E(M)}$ if and only if, for every cyclic submodule C of R/J, $Hom_R(C, E(M)) = 0$ (check!). Now, it is easy to see (check!) that C is isomorphic to $R/x^{-1}J$ for some $x \in R$ (where $x + I$ is a generator for C). Now, of course, $Hom_R(R/x^{-1}J, M) = 0$ if and only if, for all $m \in M$, $mx^{-1}J \neq 0$ (check!). ☐

The noncommutative version of corollary 5.12 is the following.

Theorem 5.15. *[Lambek-Michler, 1971] Let P be a prime ideal of a noetherian ring R and let M be a right R-module. Then M is torsion with respect to $C(P)$ if and only if M is torsion with respect to the torsion theory associated to $E(R/P)$.*

Proof. We will prove that the Gabriel filters of both torsion theories coincide (and this is enough by theorem 5.10). Suppose also that $J \in \mathbb{L}_{E(R/P)}$, i.e., $J \lhd_r R : Hom_R(R/J, E(R/P)) = 0$. By lemma 5.11, for any choice of $x \in R$ and $0 \neq u \in R/P$, there is a choice of $r \in R$ such that $xr \in J$ and $ur \neq 0$ (in particular, $r \notin P$). Thus, we conclude that for all $x \in R$, $x^{-1}J$ is not contained in P, meaning that $(x^{-1}J + P)/P \lhd_r R/P$ is nonzero. We will prove that $(x^{-1}J + P)/P$ is an essential right ideal in the prime noetherian ring R/P and thus, by Goldie's theorem, we have that $(x^{-1}J + P)/P$ has a regular element concluding that, for all $x \in R$, $x^{-1}J \cap C(P) \neq \emptyset$ - meaning that $J \in \mathbb{L}_P$.

First we observe that $x^{-1}J+P \in \mathbb{L}_{E(R/P)}$. This follows from the axioms of Gabriel filter, since $x^{-1}J \in \mathbb{L}_{E(R/P)}$ and $x^{-1}J \subset x^{-1}J+P$. Therefore, $Hom_R(R/(x^{-1}J+P), E(R/P)) = 0$. If $(x^{-1}J+P)/P$ is not essential as a right ideal of R/P, then there is a non-trivial right ideal I/P of R/P disjoint from $(x^{-1}J+P)/P$. This implies that there must be an element of R/P annihilating $(x^{-1}J+P)/P$ on the left (check!).By proposition , this means that $(R/P)/((x^{-1}J+P)/P) \cong R/(x^{-1}J+P)$ is not torsion with respect to $E(R/P)$, which is a contradiction.

Conversely, suppose $J \in \mathbb{L}_P$ and let $a,b \in R$, $b \notin P$. By hypothesis, $a^{-1}J \cap C(P) \neq \emptyset$. Let z be one of its elements. Then, clearly, $az \in J$ and $bz \notin P$. Again, by lemma 5.11, the result follows. □

6. GABRIEL LOCALISATION

In this section we will defined a process of localisation that depends only on the existence of a right Gabriel filter on the ring R. Moreover, we will show that this strictly generalises Ore localisation. Note that a right Gabriel filter on R is a *downwards directed family* of right ideals with respect to the inclusion.

Definition 6.1. Let R be a ring with a right Gabriel filter $\mathbb{L}$ and M a right R-module. We define the *pre-localisation* of M with respect to $\mathbb{L}$ to be

$$M_{(\mathbb{L})} := \varinjlim_{J \in \mathbb{L}} Hom_R(J,M).$$

Taking into account our description of inductive limits in the second section, an element of $M_{\mathbb{L}}$ is represented by some R-homomorphism $\xi : J \longrightarrow M$, $J \in \mathbb{L}$. Another R-homomorphism $\eta : K \longrightarrow M$, $K \in \mathbb{L}$, represents the same element as ξ if and only if there is $D \subset J \cap K$ such that $D \in \mathbb{L}$ and $\xi_{|D} = \eta_{|D}$.

Lemma 6.2. *If $I,J \in \mathbb{L}$ and $\alpha : I \longrightarrow R$, then $\alpha^{-1}(J) \in \mathbb{L}$.*

Proof. This is a consequence of the definition of Gabriel filter. Indeed, for all $x \in I$,

$$x^{-1}\alpha^{-1}(J) = \{r \in R : \alpha(xr) \in J\} = \alpha(x)^{-1}J$$

which lies in $\mathbb{L}$ and thus so does $\alpha^{-1}J$. □

Using this we can define a ring structure in $R_{(\mathbb{L})}$ and a right $R_{(\mathbb{L})}$-module structure in $M_{(\mathbb{L})}$ as follows. Define a pairing

$$\mu_M : M_{(\mathbb{L})} \times R_{(\mathbb{L})} \longrightarrow M_{(\mathbb{L})}$$

such that, for representatives $\xi : I \longrightarrow M$ and $\eta : J \longrightarrow R$ in $M_{(\mathbb{L})}$ and $R_{(\mathbb{L})}$ respectively, the image is defined as the composition

$$\eta^{-1}(I) \xrightarrow{\eta} I \xrightarrow{\xi} M \ .$$

Exercise 6.3 (*Needed*)**.** Check that μ_M is well-defined and bi-additive. Additionally, check that μ_R endows $R_{(\mathbb{L})}$ with a ring structure. Finally, show that μ_M indeed gives $M_{(\mathbb{L})}$ a right $R_{(\mathbb{L})}$-module structure.

Now note that there is a canonical homomorphism of abelian groups

$$\phi_M : M \cong Hom_R(R,M) \longrightarrow \varinjlim_{J \in \mathbb{L}} Hom_R(J,M) = M_{(\mathbb{L})}$$

defined by restriction, and, in fact, ϕ_R is a homomorphism of rings $R \longrightarrow R_{(\mathbb{L})}$. This endows $M_{(\mathbb{L})}$ with a right R-module structure (by pulling back the action along ϕ_R). Moreover, ϕ_M becomes a homomorphism of right R-modules.

Lemma 6.4. *Let $\mathcal{T}_{\mathbb{L}}$ be the torsion class associated with $\mathbb{L}$ and let $\tau_{\mathbb{L}}$ the functor that to each right R-module R associates the largest submodule contained in $\mathcal{T}_{\mathbb{L}}$. Then $ker(\phi_M) = \tau_{\mathbb{L}}(M)$.*

Proof. If $\phi_M(m) = 0$, then there is a right ideal $J \in \mathbb{L}$ such that the map $J \longrightarrow M, x \mapsto mx$ is zero. Therefore $mJ = 0$, which implies that $J \subset Ann(m)$ and thus $Ann(m) \in \mathbb{L}$, proving that $m \in \tau_{\mathbb{L}}(M)$. The converse is obtained by running the same argument in the opposite direction. □

Exercise 6.5. [*Needed*] Show that $M \in \mathcal{T}_{\mathbb{L}}$ if and only if $M_{(\mathbb{L})} = 0$. Compare this with corollary 5.12.

Lemma 6.6. *The R-module $coker(\phi_M)$ lies in $\mathcal{T}_{\mathbb{L}}$.*

Proof. First observe that, for an element $x \in M_{(\mathbb{L})}$ represented by a map $\xi : J \longrightarrow M$, $J \in \mathbb{L}$, we have the following commutative diagram

$$\begin{array}{ccc} J & \longrightarrow & R \\ {\scriptstyle \xi}\downarrow & & \downarrow{\scriptstyle \beta} \\ M & \xrightarrow[\phi_M]{} & M_{(\mathbb{L})} \end{array}$$

where $\beta(r) = xr$. Indeed, if $r \in J$, xr is represented by the composition $R = r^{-1}J \longrightarrow J \longrightarrow M$ given by $y \mapsto \xi(ry) = \xi(r)y$. Therefore $xr = \phi_M\xi(r)$ (i.e., xr is the restriction of the multiplication by $\xi(r)$ to J). As a consequence, we see that for all $x \in M_{(\mathbb{L})}$, there is an ideal $J \in \mathbb{L}$ such that $xJ \in im(\phi_M)$. Thus, for all element $x + im(\phi_M)$ in the cokernel of ϕ_M, $Ann(x + im(\phi_M)) \in \mathbb{L}$ and hence $coker(\phi_M) \in \mathcal{T}_{\mathbb{L}}$. □

Definition 6.7. Let R be a ring, M a right R-module and $\mathbb{L}$ a right Gabriel filter on R. The **Gabriel localisation of M with respect to $\mathbb{L}$** is the right R-module $M_{\mathbb{L}} := (M_{(\mathbb{L})})_{(\mathbb{L})}$.

Lemma 6.8. *The localisation $M_{\mathbb{L}}$ can be described as $\varinjlim_{J \in \mathbb{L}} Hom_R(J, M/\tau_{\mathbb{L}}(M))$.*

Proof. Consider the homomorphism induced by ϕ_M from $M/\tau_{\mathbb{L}}(M)$ to $M_{(\mathbb{L})}$. Regarding $(\_)_{(\mathbb{L})}$ as a functor from $Mod(R)$ to $Mod(R)$, it is left exact (following from the left exactness of Hom and the exactness of direct limits). Applying it to the short exact sequence

$$0 \longrightarrow M/\tau_{\mathbb{L}}(M) \longrightarrow M_{(\mathbb{L})} \longrightarrow coker(\phi_M) \longrightarrow 0,$$

by lemma 6.6 and exercise 6.5, we get $(coker(\phi_M))_{(\mathbb{L})} = 0$ and thus $(M/\tau_{\mathbb{L}}(M))_{(\mathbb{L})} \cong (M_{(\mathbb{L})})_{(\mathbb{L})} = M_{\mathbb{L}}$, concluding the proof. □

Remark 6.9. It is easy to see, as before, that $R_{\mathbb{L}}$ is endowed with a ring structure and that $M_{\mathbb{L}}$ is a right $R_{\mathbb{L}}$-module (and an R-module as well, by pulling back the action along a canonical ring homomorphism $\psi_R : R \longrightarrow R_{(\mathbb{L})} \longrightarrow R_{\mathbb{L}}$. More concretely, the operation $M_{\mathbb{L}} \times R_{\mathbb{L}} \longrightarrow M_{\mathbb{L}}$, for a representative $\xi : I \longrightarrow M/\tau_{\mathbb{L}}(M)$ of an element in $M_{\mathbb{L}}$ and a representative $\alpha : J \longrightarrow R/\tau_{\mathbb{L}}(R)$ of an element in $R_{\mathbb{L}}$, can be described by the following representative of an element in $M_{\mathbb{L}}$:

$$\alpha^{-1}(I/\tau_{\mathbb{L}}(I)) \longrightarrow I/\tau_{\mathbb{L}}(I) \longrightarrow M/\tau_{\mathbb{L}}(M).$$

In fact this holds because $\tau_{\mathbb{L}}$ is left exact and ξ induces a map from $I/\tau_{\mathbb{L}}(I)$ to $M/\tau_{\mathbb{L}}(M)$. Note also that $ker(\psi_M) = \tau_{\mathbb{L}}(M)$ and $coker(\psi_M) \in \mathcal{T}_{\mathbb{L}}$ (check!).

Definition 6.10. A right R-module M is said to be $\mathbb{L}$**-closed** if the canonical homomorphisms of right R-modules $M \cong Hom_R(R,M) \longrightarrow Hom(J,M)$ are isomorphisms for all $J \in \mathbb{L}$.

Remark 6.11.
1. Note that, if M is torsion-free with respect to $\mathbb{L}$, then ϕ_M is injective. Therefore, a torsion-free module is $\mathbb{L}$-closed if and only if the maps $M \cong Hom_R(R,M) \longrightarrow Hom(J,M)$ are surjective for all $J \in \mathbb{L}$.
2. If M is $\mathbb{L}$-closed, then the map $\psi_M : M \longrightarrow M_{\mathbb{L}}$ is an isomorphism (check!).

Proposition 6.12. *For any right R-module M, $M_{\mathbb{L}}$ is $\mathbb{L}$-closed. Moreover, the category of $\mathbb{L}$-closed right R-modules is equivalent to the subcategory of right $R_{\mathbb{L}}$-modules of the form $M_{\mathbb{L}}$.*

Proof. We will show first that $M_{\mathbb{L}}$ is torsion-free with respect to $\mathcal{T}_{\mathbb{L}}$. We have just shown that $M_{\mathbb{L}} = (M/\tau_{\mathbb{L}}(M))_{(\mathbb{L})}$ and thus we only need to show that given a torsion-free module N, $N_{(\mathbb{L})}$ is torsion-free as well. Indeed, let $x \in N_{(\mathbb{L})}$ (represented by $\xi : J \longrightarrow N$) and $I \in \mathbb{L}$ such that $xI = 0$. As in the proof of lemma 6.6, there is a commutative diagram

(6.1)
$$\begin{array}{ccc} J & \longrightarrow & R \\ \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \beta} \\ N & \xrightarrow[\phi_N]{} & N_{(\mathbb{L})} \end{array}$$

where $\beta(r) = xr$. This shows that $\phi_N \xi_{|I\cap J} = 0$. Since N is torsion-free, ϕ_N is injective and, thus, $\xi_{|I\cap J} = 0$, proving that $x = 0$ (since $I \cap J \in \mathbb{L}$).

Now it suffices to see that the maps $M_{\mathbb{L}} \cong Hom_R(R, M_{\mathbb{L}}) \longrightarrow Hom(J, M_{\mathbb{L}})$ are surjective for all $J \in \mathbb{L}$. Let $f : I \longrightarrow M_{\mathcal{F}}$ and consider the pullback of $\phi_{M/\tau_{\mathbb{L}}(M)} : M/\tau_{\mathbb{L}}(M) \longrightarrow M_{\mathbb{L}}$ (which is injective since $M/\tau_{\mathbb{L}}(M)$ is torsion-free with respect to $\mathbb{L}$) along f, thus giving raise the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J & \longrightarrow & I & \longrightarrow & I/J & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \cong} & & \\ 0 & \longrightarrow & M/\tau_{\mathbb{L}}(M) & \xrightarrow[\phi_{M/\tau_{\mathbb{L}}(M)}]{} & M_{\mathbb{L}} & \longrightarrow & coker(\phi_{M/\tau_{\mathbb{L}}(M)}) & \longrightarrow & 0 \end{array}$$

with exact rows. Since $coker(\phi_{M/\tau_{\mathbb{L}}(M)})$ is torsion with respect to $\mathbb{L}$, then so is I/J and, therefore, $J \in \mathbb{L}$. Using again the argument represented by the commutative diagram (6.1) (replacing N with $M/\tau_{\mathbb{L}}(M)$ and ξ by g), $\phi_{M/\tau_{\mathbb{L}}(M)} g : J \longrightarrow M_{\mathbb{L}}$ can be extended to a map $h : R \longrightarrow M_{\mathbb{L}}$. We will prove that h extends f, thus finishing the proof. Indeed, $h_{|I} - f : I \longrightarrow M_{\mathbb{L}}$ factors through I/J (since $h_{|J} = f_{|J} = \phi_{M/\tau_{\mathbb{L}}(M)} g$) which is torsion. Since $M_{\mathbb{L}}$ is torsion-free, $h_{|I} - f = 0$ and thus h extends f, as wanted.

The final statement follows from the first statement and from the previous remark, item (2). □

Theorem 6.13. *Let R be a ring, $\mathbb{L}$ a right Gabriel filter in R and M a right R-module. If N is an $\mathbb{L}$-closed right R-module and $f : M \longrightarrow N$ a homomorphism then f factors through $M_{\mathbb{L}}$, i.e., there is a unique R-homomorphism $g : M_{\mathbb{L}} \longrightarrow N$ such that $f = g\psi_M$.*

Proof. First note that, since N is $\mathbb{L}$-closed, it is torsion-free and therefore $f(\tau_{\mathbb{L}}(M)) = 0$ and f factors through $\tilde{f} : M/\tau_{\mathbb{L}}(M) \longrightarrow N$. Indeed, given a representative $\alpha : J \longrightarrow M/\tau_{\mathbb{L}}(M)$, $J \in \mathbb{L}$, of an element of $M_{\mathbb{L}}$, define an element of $N_{\mathbb{L}} \cong N$ (since N is $\mathbb{L}$-closed) by the representative $\tilde{f}\alpha$.

Exercise 6.14 (*). Show that this map $M_{\mathbb{L}} \longrightarrow N$ is well-defined and that it is unique, proving the theorem.

□

Theorem 6.15. *Let R be a ring and S a multiplicative set of regular elements satisfying the right Ore condition. Let M be a right R-module. Then $M_{\mathbb{L}_S} \cong MS^{-1}$.*

Proof. First observe that if S contains only regular elements and satisfies the right Ore condition, then S is a right denominator set and thus MS^{-1} exists. Observe that every right ideal in $\mathbb{L}_S$ contains a cyclic submodule of the form sR, for some $s \in S$ (since every right ideal intersects S nontrivially). Therefore,

$$M_{\mathbb{L}_S} = \varinjlim_{s \in S} Hom_R(sR, M/\tau_{\mathbb{L}_S}(M))$$

where $\tau_{\mathbb{L}_S}(M) = \{m \in M : \exists s \in S : ms = 0\}$ (check!). Hence, a pair $(m,s) \in M \times S$ determines an element of $M_{\mathbb{L}_S}$, say $x^{(m,s)}$ represented by a map $\eta^{(m,s)} : sR \longrightarrow M/\tau_{\mathbb{L}_s}(M), s \mapsto \bar{m}$ (where $\bar{m} = m + \tau_{\mathbb{L}_S}(M)$). Now, $x^{(m,s)} = x^{(k,t)}$, for two elements $(m,s),(k,t)$ of $M \times S$ if and only if there is a right ideal wR with $w \in S$ such that $wR \subset sR \cap tR$ and $\eta^{(m,s)}_{|wR} = \eta^{(k,t)}_{|wR}$. Equivalently, there are $a,b \in R$ such that $sa = tb = w$ and $ms - kt \in \tau_{\mathbb{L}_S}(M)$, i.e., there is $u \in S$ such that $(ms-kt)u = 0$. But this is precisely the relation defining MS^{-1} (check theorem 3.14 and the proof of theorem 3.4). □

7. Noncommutative affine geometry?

We end these lectures with a small discussion on how to approach noncommutative affine geometry. As motivated by section 2, we need a topological space and a structure sheaf. Section 3 shows us that using classic localisation might not be a very good idea since $C(P)$ is not always a denominator set, as in the commutative case. However, the tools introduced in chapters 5 and 6 tell us that one can hope for a noncommutative affine geometry using Gabriel localisation. As observed in the end of section 3, $Spec(R)$ is a topological space (with the Zarisky topology) and the techniques developed in sections 5 and 6 give us hope of forming a structure sheaf. Indeed, we have the following results by Van Ostayern and Verschoren.

Theorem 7.1 (Van Ostayern, Verschoren, 1981). *Let R be a noncommutative prime noetherian ring and $Spec(R)$ the set of prime ideals with the Zarisky topology. Let $U(I)$ denote the open set associated with a two-sided ideal I of R and define a presheaf on $Spec(R)$ by setting $O(U(I)) := R_{\mathbb{L}_I}$ where $\mathbb{L}_I$ is the Gabriel filter*

$$\mathbb{L}_I := \{J \triangleleft_r R : \exists n \in \mathbb{N} : I^n \subset J\}.$$

Then O is a sheaf in $Spec(R)$.

This is a very interesting result that has allowed the development of some ideas in noncommutative affine geometry. But this is not yet an approach without problems...

1. The first problem with this approach is of a practical nature: many rings do not have *enough* prime ideals.
2. The second problem has to do with, if not $Spec(R)$, what is the natural topological space underlying a noncommutative affine scheme?
3. To solve the second problem, we need to think about the third and vice-versa: what are the basic properties of a *successful* theory of noncommutative affine schemes? What should be the properties of this category?

A possible solution to the first two problems is to look at the injective spectrum, following the philosophy of section 4 of these notes. Indeed, for a noetherian ring R, there is an injection from $Spec(R)$ to $Inj(R)$, so this could solve the problem of having *few* prime ideals for some rings. Other approaches (famously Rosenberg's book) suggest other spectra to be considered.

Other fundamental idea to deal with some sort of noncommutative algebraic geometry is to look at categories of coherent sheaves rather than the schemes themselves. This categorical approach has also proved to be very useful, giving rise to a branch of noncommutative algebra called non-commutative projective geometry.

The material in these notes is essentially contained in the following references. This is a short list, mostly of survey/textbook material rather than original work. For references to original work, we refer to the references therein.

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