GLUEING SILTING OR HOW TWO HEARTS BECOME ONE

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ABSTRACT. The natural question of which module categories embed "nicely" in a fixed bounded derived category of a finite dimensional algebra can be approached by looking at endomorphism algebras of silting objects. In general, not much is known about these "silted" algebras. There are, however, methods to glue silting objects via recollements of derived categories. In this talk we discuss these methods and observe that, for piecewise hereditary algebras, they may help solving this problem. This is joint work with Qunhua Liu and Dong Yang ([5],[6]).

Let $\mathbb{K}$ be an algebraically closed field and let R be a finite dimensional $\mathbb{K}$-algebra of finite global dimension. We will discuss hearts of bounded t-structures in the bounded derived category of finitely generated right R-modules, $\mathcal{D}^b(R)$. In this talk we choose to hide the use of t-structures and of co-t-structures in order to convey the main ideas of our work in a simple and concise way.

1. MOTIVATION AND PRELIMINARY RESULTS

Definition 1.1. Let S be a finite dimensional $\mathbb{K}$-algebra. The abelian category *mod-S* is said to be a **heart** in $\mathcal{D}^b(R)$ if there is an exact functor $\Phi : mod\text{-}S \to \mathcal{D}^b(R)$, i.e., taking short exact sequences to triangles, such that

(1) Φ is fully faithful;
(2) $Hom_{\mathcal{D}^b(R)}(\Phi(mod\text{-}S), \Phi(mod\text{-}S)[k]) = 0$ for all $k < 0$;
(3) $tria(\Phi(mod\text{-}S)) = \mathcal{D}^b(R)$.

Question 1.2. For which S is *mod-S* a heart in $\mathcal{D}^b(R)$?

Example 1.3. The canonical embedding *mod-R* in $\mathcal{D}^b(R)$ makes *mod-R* into a heart of $\mathcal{D}^b(R)$, called the standard heart;

Example 1.4. Let T be a tilting complex in $\mathcal{D}^b(R)$. Rickard's Morita theory for derived categories shows that T induces an equivalence $\Psi : \mathcal{D}^b(End_{\mathcal{D}^b(R)}(T)) \to \mathcal{D}^b(R)$. The restriction of Ψ to $mod\text{-}End_{\mathcal{D}^b(R)}(T)$ makes $mod\text{-}End_{\mathcal{D}^b(R)}(T)$ into a heart of $\mathcal{D}^b(R)$.

The following concept will provide a way to tackle our main question.

Definition 1.5 (Keller-Vossieck'88, [3])**.** An object $M \in \mathcal{D}^b(R)$ is silting if:

- $tria(M) = \mathcal{D}^b(R)$;
- $Hom(M, M[k]) = 0$, for all $k > 0$.

Note that, in general, we need to require M to be compact as well. This assumption is omitted here since we assume that R has finite global dimension.

Theorem 1.6 (Keller-Nicolás'12 [2], Koenig-Yang'12 [4])**.** *mod-S is a heart in $\mathcal{D}^b(R)$ if and only if there is a silting object M in $\mathcal{D}^b(R)$ such that $S \cong End_{\mathcal{D}^b(R)}(M)$.*

This theorem is usually stated in a different way, exposing an underlying bijection between bounded t-structures with length heart, bounded co-t-structures and basic silting objects.

2. GLUEING

An idea that can be useful to characterise the endomorphism algebras of silting objects is that of glueing along a recollement.

Theorem 2.1 (Liu-V.-Yang'12 [6]). *Given a recollement of derived modules categories*

$$\mathcal{D}^b(B) \begin{array}{c} \xleftarrow{i^*} \\ \xrightarrow{i_*} \\ \xleftarrow[i^!]{} \end{array} \mathcal{D}^b(R) \begin{array}{c} \xleftarrow{j_!} \\ \xrightarrow{j^*} \\ \xleftarrow[j_*]{} \end{array} \mathcal{D}^b(C) \; .$$

and silting objects $X \in \mathcal{D}^b(C)$ and $Y \in \mathcal{D}^b(B)$, then there is a silting object $Z = j_!X \oplus K_Y \in \mathcal{D}^b(R)$ such that:

(1) *K_Y is uniquely determined (and explicitly computable) up to summands and it depends on both X and Y;*
(2) *There is a cohomological functor $H^0_Z : \mathcal{D}^b(R) \to mod\text{-}End_{\mathcal{D}^b(R)}(M)$ and a recollement of abelian categories making the below diagram commute:*

$$\begin{array}{ccccc} \mathcal{D}^b(B) & \begin{array}{c} \longleftarrow \\ \longrightarrow \\ \longleftarrow \end{array} & \mathcal{D}^b(R) & \begin{array}{c} \longleftarrow \\ \longrightarrow \\ \longleftarrow \end{array} & \mathcal{D}^b(C) \\ \uparrow & & \downarrow{\scriptstyle H^0_Z} & & \uparrow \\ mod\text{-}End_{\mathcal{D}^b(B)}(Y) & \begin{array}{c} \longleftarrow \\ \longrightarrow \\ \longleftarrow \end{array} & mod\text{-}End_{\mathcal{D}^b(R)}(Z) & \begin{array}{c} \longleftarrow \\ \longrightarrow \\ \longleftarrow \end{array} & mod\text{-}End_{\mathcal{D}^b(C)}(X) \end{array} \; .$$

where the arrow pointing upwards are the functors given by the fact that $mod\text{-}End_{\mathcal{D}^b(B)}(Y)$ and $mod\text{-}End_{\mathcal{D}^b(C)}(X)$ are hearts in $\mathcal{D}^b(B)$ and $\mathcal{D}^b(C)$ respectively.

This theorem also appears, in a different form, in [1]. See [6] for details and for a more precise statement.

Also, the explicit nature of K_Y allows to draw conclusions on the glueing of tilting objects (or of derived equivalences). See [6] for details.

3. RESTRICTING

To be able to answer the question posed in section 1 with glueing techniques we will need to find out which silting objects are glued from simples ones. This can be answered in the case of piecewise hereditary algebras.

Theorem 3.1 (Liu-V.'12 [5]). *Suppose that R is piecewise hereditary. Let $mod\text{-}S$ be a heart in $\mathcal{D}^b(R)$ and Z a silting object in $\mathcal{D}^b(R)$ such that $S \cong End_{\mathcal{D}^b(R)}(Z)$. Then there is a recollement*

$$\mathcal{D}^b(B) \begin{array}{c} \xleftarrow{i^*} \\ \xrightarrow{i_*} \\ \xleftarrow[i^!]{} \end{array} \mathcal{D}^b(R) \begin{array}{c} \xleftarrow{j_!} \\ \xrightarrow{j^*} \\ \xleftarrow[j_*]{} \end{array} \mathcal{D}^b(C) \; .$$

with B and C finite dimensional piecewise hereditary $\mathbb{K}$-algebras and $X \in \mathcal{D}^b(C)$, $Y \in \mathcal{D}^b(B)$ silting objects such that $Z = j_!X \oplus K_Y$, thus inducing a recollement of abelian categories

$$mod\text{-}End_{\mathcal{D}^b(B)}(Y) \begin{array}{c} \longleftarrow \\ \longrightarrow \\ \longleftarrow \end{array} mod\text{-}End_{\mathcal{D}^b(R)}(Z) \begin{array}{c} \longleftarrow \\ \longrightarrow \\ \longleftarrow \end{array} mod\text{-}End_{\mathcal{D}^b(C)}(X)$$

The recollement in the theorem is obtained by taking C as the endomorphism ring of a simple projective module in $mod\text{-}End_{\mathcal{D}^b(R)}(Z)$, which exists since one can show that R being piecewise hereditary implies that $End_{\mathcal{D}^b(R)}(Z)$ is a directed algebra.

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