

Some recent results on higher order elliptic equations

Marco Squassina

May 24 –27, 2002 – Wilmington USA

Lack of symmetry I: The subcritical case

Let $\varphi \in L^2(\Omega)$, $\Omega \subset \mathbb{R}^N$, $N \geq 3$. Existence of infinitely many solutions for

$$\begin{cases} -\Delta u = g(x, u) + \varphi & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where $g(x, -s) = -g(x, s)$ and there exist $q > 2$ and $R > 0$ such that

$$|g(x, s)| \leq a + b|s|^\sigma, \quad \sigma \in \left] 1, \frac{qN + 2(q - 1)}{qN - 2(q - 1)} \right[$$

$$|s| \geq R \implies 0 < qG(x, s) \leq g(x, s)s$$

has been obtained e.g. in the following papers:

A. BAHRI, H. BERESTYKI, A perturbation method in critical point theory and applications, *Trans. Amer. Math. Soc.* **267** (1981), 1–32.

A. BAHRI, P.L. LIONS, Morse index of some Min-Max critical points. I. Applications to multiplicity results, *Comm. Pure Appl. Math.* **45** (1992), 1205–1215.

P. H. RABINOWITZ, Critical points of perturbed symmetric functionals, *Trans. Amer. Math. Soc.* **272** (1982), 753–769.

M. STRUWE, Infinitely many critical points for functional which are not even and application to superlinear boundary value problems, *Manuscripta Math.* **32** (1980), 335–364.

A. CANDELA, A. SALVATORE, M. SQUASSINA, Multiple solutions for semilinear elliptic systems with nonhomogeneous boundary conditions, *Nonlinear Anal.* in press (2002).

Moreover, in the following papers an extension to more general classes of elliptic problems has been recently achieved.

S. PALEARI, M. SQUASSINA, Multiplicity results for perturbed symmetric quasilinear elliptic systems, *Differential Integral Equations* **14** (2001) 785–800.

M. SQUASSINA, C. TARSI, Multiple solutions for a class of quasilinear elliptic problems in \mathbb{R}^2 with exponential growth, *Manuscripta Math.* **106** (2001) 315–337.

Let now $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain with $N > 2K$, $K \geq 1$,

$$\phi_j \in H^{K-j-\frac{1}{2}}(\partial\Omega) \quad j = 0, \dots, K-1$$

and φ a function in $L^2(\Omega)$. Moreover let

$$2 < \sigma < K_*, \quad K_* = \frac{2N}{N-2K},$$

where K_* denotes the critical exponent for the embedding $H_0^K(\Omega) \hookrightarrow L^{K_*}(\Omega)$.

Consider the following polyharmonic elliptic problem

$$(-\Delta)^K u = |u|^{\sigma-2}u + \varphi \quad \text{in } \Omega \quad (P_\varphi^K)$$

with nonhomogeneous Dirichlet boundary conditions

$$\left(\frac{\partial}{\partial \nu} \right)^j u \Big|_{\partial\Omega} = \phi_j \quad j = 0, \dots, K-1, \quad (D_\phi)$$

where ν denotes the outer unit normal to $\partial\Omega$.

Theorem 1. *Let $N > 2K$. Assume that σ satisfies*

$$2 < \sigma < 2 \frac{N + K}{N}.$$

Then the problem

$$(-\Delta)^K u = |u|^{\sigma-2} u + \varphi \quad \text{in } \Omega$$

with boundary conditions

$$\left(\frac{\partial}{\partial \nu} \right)^j u \Big|_{\partial \Omega} = \phi_j \quad j = 0, \dots, K - 1$$

admits infinitely many solutions.

If $\phi_j = 0$ for every $j = 0, \dots, K - 1$, we have the following stronger result.

Theorem 2. *Let $N > 2K$. Assume that σ satisfies*

$$2 < \sigma < 2 \frac{N - K}{N - 2K}.$$

Then the problem

$$(-\Delta)^K u = |u|^{\sigma-2} u + \varphi \quad \text{in } \Omega$$

with boundary conditions

$$\left(\frac{\partial}{\partial \nu} \right)^j u \Big|_{\partial \Omega} = 0 \quad j = 0, \dots, K - 1$$

has infinitely many solutions.

S. LANCELOTTI, A. MUSESTI, M. SQUASSINA, Infinitely many solutions for polyharmonic problems with broken symmetries, *preprint* (2001).

Notice that if $\partial\Omega$ is of class $C^{2K,\alpha}$, $\varphi \in C^{0,\alpha}(\overline{\Omega})$ and $\phi_j \in C^{2K-j}(\partial\Omega)$, then every solution of (P_φ^K) belongs to $C^{2K,\alpha}(\overline{\Omega})$, hence it is classical.

Conjecture 3. *Let $K \geq 1$ and $N > 2K$. Theorem 1 holds if*

$$2 < \sigma < \frac{2N}{N - K}$$

provided that Ω is of class C^{2K} , $\phi_j \in C^{2K-j}(\partial\Omega)$ for every j and $\varphi \in C(\overline{\Omega})$.

This fact has already been proved for $K = 1$ in:

P. BOLLE, N. GHOUSSOUB, H. TEHRANI, The multiplicity of solutions in nonhomogeneous boundary value problems, *Manuscripta Math.* **101** (2000), 325–350.

The case $K > 1$ involves some “particular” Pohozaev–type identity of higher order!

Problem 4. *Does Theorem 1 hold true with nonhomogeneous Navier boundary conditions $\Delta^j u|_{\partial\Omega} = \phi_j$ for $j = 0, \dots, K - 1$?*

MAIN TOOLS IN THE PROOF

- Variational methods, critical point theory ;
- Morse Theory (K. Tanaka) ;
- Bolle’s perturbation Theory ;
- Spectral estimates (see e.g. Laptev, Lieb, Li–Yau, Rozenblum, Weidl): Let

$$(-\Delta)^K u + V(x)u = \lambda u \quad \text{in } \mathbb{R}^N,$$

where $V \in L^{N/2K}(\mathbb{R}^N)$ and $(\mu_k) \subset \mathbb{R}$ denote the sequence of eigenvalues of the operator $(-\Delta)^K + V(x)$ repeated according to multiplicity. Then

$$\#\{k \in \mathbb{N} : \mu_k \leq 0\} \leq B_{N,K} \int_{\mathbb{R}^N} V^-(x)^{N/2K}. \quad (1)$$

for some $B_{N,K} > 0$, where $\#$ denotes the cardinality.

Lack of symmetry II: The critical case

Let $K \geq 1$ and let Ω be a bounded smooth domain in \mathbb{R}^N with $N > 2K$.

Consider the polyharmonic elliptic problem

$$\begin{cases} (-\Delta)^K u = |u|^{K_*-2}u + f & \text{in } \Omega \\ \left(\frac{\partial}{\partial \nu}\right)^j u \Big|_{\partial\Omega} = 0 \\ j = 0, \dots, K-1, \end{cases} \quad (P_{\Omega, f})$$

where $f \in H^{-K}(\Omega)$ and $K_* = \frac{2N}{N-2K}$ denotes the critical exponent for the Sobolev embedding $H_0^K(\Omega) \hookrightarrow L^{K_*}(\Omega)$.

Theorem 5. *There exists a $\kappa > 0$ such that, if $f \neq 0$ and $\|f\|_{H^{-K}} < \kappa$, then problem $(P_{\Omega, f})$ has at least 2 solutions.*

Theorem 6. *If $G \neq \{1\}$ acts freely on Ω then there exists a $\kappa > 0$ with the property that, for every $f \neq 0$ which is G -invariant and such that $\|f\|_{H^{-K}} < \kappa$, problem $(P_{\Omega, f})$ has at least 3 solutions one of which is G -invariant and one of*

which is not.

For example, if Ω is symmetric with respect to the origin (i.e. $x \in \Omega$ if $-x \in \Omega$) and $0 \notin \Omega$ then, for every even function $f \neq 0$ with $\|f\|_{H^{-K}}$ small enough, problem (P_Ω, f) has at least 3 solutions, one of which is even and one of which is not.

Denote by S_K the best Sobolev constant for the embedding $H_0^K(\Omega) \hookrightarrow L^{K^*}(\Omega)$,

$$S_K = \inf \left\{ \|u\|_{K,2}^2 : u \in H_0^K(\Omega), \int_\Omega |u|^{K^*} = 1 \right\}.$$

We write $\#G_j$ for the cardinality of G_j . Our main result is the following.

Theorem 7. *Let $\{1\} = G_1, \dots, G_m$ be closed subgroups of $\mathcal{O}(N)$ acting freely on Ω such that $\#G_1 < \dots < \#G_m$ and $G_{m-1} \subset G_m$. Then at least one of the following assertions holds:*

(a) *$m > 1$ and for $f = 0$ problem $(P_{\Omega,0})$ has a nontrivial solution u such that*

$$\|u\|_{K,2}^2 \leq (\#G_{m-1})(S_K)^{N/2K};$$

(b) *$m \geq 1$ and there exists a $\kappa > 0$ with the property that, if f is G_i -invariant for every $i = 1, \dots, m$, $f \neq 0$ and $\|f\|_{H^{-\kappa}} < \kappa$, then the problem*

$$\begin{cases} (-\Delta)^K u = |u|^{K^*-2}u + f & \text{in } \Omega \\ \left(\frac{\partial}{\partial \nu}\right)^j u \Big|_{\partial\Omega} = 0 \\ j = 0, \dots, K-1 \end{cases}$$

has at least $m + 1$ solutions u_0, u_1, \dots, u_m such that u_i is G_i -invariant but not G_{i+1} -invariant for $i = 1, \dots, m-1$, and u_m is G_m -invariant.

M. CLAPP, M. SQUASSINA, Nonhomogeneous polyharmonic elliptic problems at critical growth with symmetric data, *preprint* (2001).

In the case $K = 1$ stronger results hold, see

M. CLAPP, O. KAVIAN, B. RUF, Multiple solutions of nonhomogenous elliptic equations with critical nonlinearity on symmetric domains *Communications in Contemporary Mathematics*, to appear (2002).

This is due to stronger nonexistence result than in the case $K > 1!!!$

Problem 8. *Does Theorem 1 hold true with nonhomogeneous boundary conditions*

$$\left(\frac{\partial}{\partial \nu}\right)^j u \Big|_{\partial\Omega} = \phi_j \text{ for } j = 0, \dots, K - 1?$$

MAIN TOOLS IN THE PROOF

- Variational methods, G – invariant critical point theory ;
- Splitting of the Nehari manifold into two components ;
- Proof of fine estimates due to BN (Ann. Pisa, 1989) in the polyharmonic setting.

Perturbation of the domain: The biharmonic problem

Let Ω be a bounded smooth domain in \mathbb{R}^N . To solve the problem

$$\begin{cases} -\Delta u = |u|^{2^*-2} u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

either perturb the critical growth nonlinearity by adding subcritical terms or change the geometry (topology) of Ω . We focus on the second approach which has been studied e.g. in the following papers:

A. BAHRI, J.M. CORON, On a nonlinear elliptic equation involving the Sobolev exponent: the effect of the topology of the domain, *Comm. Pure Appl. Math.* **41** (1988), 253–294.

E.N. DANCER, A note on an equation with critical exponent, *Bull. London Math. Soc.* **20** (1988), 600–602.

W. DING, Positive solutions of $\Delta u + u^{(N+2)/(N-2)} = 0$ on contractible domains,

J. Partial Differential Equations **2** (1989), 83–88.

D. PASSASEO, The effect of the domain shape on the existence of positive solutions of the equation $\Delta u + u^{2^*-1} = 0$, *Top. Meth. Nonlin. Anal.* **3** (1994), 27–54.

We consider the following biharmonic critical growth problem

$$\Delta^2 u = |u|^{2^*-2} u \quad \text{in } \Omega$$

either with Navier boundary conditions

$$u = \Delta u = 0 \quad \text{on } \partial\Omega$$

or with Dirichlet boundary conditions

$$u = \nabla u = 0 \quad \text{on } \partial\Omega$$

Here $\Omega \subset \mathbb{R}^N$ ($N \geq 5$) is an open bounded smooth domain and $2_* = \frac{2N}{N-4}$ is the critical Sobolev exponent for the embedding $H^2(\mathbb{R}^N) \hookrightarrow L^{2_*}(\mathbb{R}^N)$.

Let us set

$$\text{cap}^*(D) = \inf \left\{ \int_{\mathbb{R}^n} |\Delta u|^2 ; u \in \mathcal{D}^{2,2}(\mathbb{R}^n), u = 1 \text{ in } D \right\} (\geq \text{cap}(D)).$$

Theorem 9. *Let $\tilde{\Omega}$ be a smooth bounded domain of \mathbb{R}^N ($N \geq 5$) and let H be a closed subset contained in $\tilde{\Omega}$. Then, there exists $\varepsilon > 0$ such that if $\Omega \subset \tilde{\Omega}$ is a smooth domain with $\text{cap}^*(\tilde{\Omega} \setminus \Omega) < \varepsilon$ and such that H cannot be deformed in $\tilde{\Omega}$ into a subset of Ω , then there exists a nontrivial solution of*

$$\begin{cases} \Delta^2 u = |u|^{2^*-2}u & \text{in } \Omega \\ u = \nabla u = 0 & \text{on } \partial\Omega. \end{cases}$$

Theorem 10. *Let $\tilde{\Omega}$ be a smooth bounded domain of \mathbb{R}^N ($N \geq 5$) and let H be a closed subset contained in $\tilde{\Omega}$. Then, there exists $\varepsilon > 0$ such that if $\Omega \subset \tilde{\Omega}$ is a smooth domain with $\text{cap}^*(\tilde{\Omega} \setminus \Omega) < \varepsilon$ and such that H cannot be deformed in $\tilde{\Omega}$ into a subset of Ω , then there exists a positive solution of*

$$\begin{cases} \Delta^2 u = u^{(N+4)/(N-4)} & \text{in } \Omega \\ u = \Delta u = 0 & \text{on } \partial\Omega. \end{cases}$$

F. GAZZOLA, H.–CH. GRUNAU, M. SQUASSINA, Existence and non–existence results for critical growth biharmonic elliptic equations, *preprint* (2002).

Bahri–Coron type results have been recently proved by T. Bartsch, T. Weth and M. Willem for the polyharmonic operator with Dirichlet b.c. and by F. Ebobisse and M.O. Ahmedou for the biharmonic operator with Navier b.c.

MAIN TOOLS IN THE PROOF

- Variational methods, critical point theory.

Constrained critical points of the functional

$$f(u) = \int_{\Omega} |\Delta u|^2 \quad \text{on} \quad \mathcal{V}(\Omega) = \{u \in H_0^2(\Omega) : \|u\|_{2_*} = 1\};$$

- Nodal solutions “double” the energy (in the space \mathbb{R}^N , in the half space \mathbb{R}_+^N): no test with u^\pm , decomposition in dual cones;
- *PS* condition holds at level $c \in (S, 2^{4/N} S)$;
- Construct a *PS* sequence at level $c \in (S, 2^{4/N} S)$: here we need the “topological” constraint;
- With Navier boundary conditions we have no “uniform” extension operator. The global compactness result (Struwe, Math. Z. 1984) does not extend in a straightforward way to Navier boundary condition, some work needs to be done.