Sulla localizzazione dei punti di concentrazione nei problemi ellittici singolarmente perturbati
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Roma1, 27-06-05.

Sunto: Si discutono alcune condizioni necessarie per la concentrazione di soluzioni, attorno ad un dato punto, di alcune classi di problemi ellittici singolarmente perturbati. Le condizioni coinvolgono i sottodifferenziali delle funzioni di minima energia.

1. *A Few Observations*

Consider the singularly perturbed equation

$$-\varepsilon^2\Delta u + u = f(x,u) \quad \text{in } \mathbb{R}^n, \qquad u > 0 \quad \text{in } \mathbb{R}^n, \tag{P_ε}$$

and, for every fixed $z \in \mathbb{R}^n$, the limiting autonomous equation

$$-\Delta u + u = f(z,u) \quad \text{in } \mathbb{R}^n, \qquad u > 0 \quad \text{in } \mathbb{R}^n. \tag{P_0}$$

There are (classical) choices of $f(x,u)$ such that the *ground energy function* Σ [1]

$$\Sigma(z) = \inf_{\phi\in\mathcal{N}_z} I_z(\phi), \qquad I_z(u) = \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u|^2 + u^2 - \int_{\mathbb{R}^n} F(z,u)$$

is C^1-smooth (with an explicit formula) and around its nondegenerate critical points the solutions u_ε of P_ε exhibit a ***spike-like profile*** as ε goes to zero:

$$\boxed{\forall\{u_{\varepsilon_h}\} \textit{ concentrating at } z \quad \Rightarrow \quad \nabla\Sigma(z) = 0}$$

$$\boxed{\nabla\Sigma(z) = 0 \textit{ and } z \textit{ is nondegenerate} \quad \Rightarrow \quad \exists\{u_{\varepsilon_h}\} \textit{ concentrating at } z.}$$

This is the case, for instance, for

$$f(x,u) = K(x)u^q, \qquad 1 < q < \tfrac{n+2}{n-2}, \quad n \geq 3,$$

where $K(x)$ is a suitable C^1 function [2] [3]. It turns out that the C^1 (and higher) smoothness of Σ is related to the crucial fact that, for every fixed $z \in \mathbb{R}^n$, the limiting equation P_0 admits a ***unique solution***, up to translations. However, unfortunately, the uniqueness feature for P_0 is a delicate matter and it is currently available only under rather restrictive assumptions on the function f.

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[1] ▶ X. Wang, B. Zeng, *On concentration of positive bound stated of nonlinear Schrödinger equations with competing potential functions*, SIAM J. Math. Anal. **28** (1997), 633–655.

[2] ▶ A. Ambrosetti, M. Badiale, S. Cingolani, *Semiclassical states of nonlinear Schrödinger equations*, Arch. Ration. Mech. Anal. **140** (1997), 285–300.

[3] ▶ A. Ambrosetti, A. Malchiodi, S. Secchi, *Multiplicity results for some nonlinear Schrödinger equations with potentials*, Arch. Ration. Mech. Anal. **159** (2001), 253–271.

What it is know, in general, is that:

- Σ is a *locally Lipschitz continuous* function;
- Σ admits *representation formulas for the left and right derivatives*.

Motivated by these facts, recently, some conditions for locating the concentration points for P_ε in presence of a ***more general nonlinearity*** f, not necessarily a power, have been investigated (in a few slides, the references).

The underlying philosophy is that when the limit problem P_0 ***lacks of uniqueness*** up to translations, then the ground energy function Σ could loose its additional regularity properties.

Nevertheless, in this (possibly nonsmooth) framework, it turns out that a necessary condition for the solutions u_ε to concentrate (in a suitable sense) around a given point z is that it is critical for Σ in the sense of the ***Clarke subdifferential*** ∂_C, that is $0 \in \partial_C \Sigma(z)$, or in a even weaker sense.

2. *Outline of the Recipe (Roughly Speaking)*

We are interested in semilinear elliptic equations/systems. On the other hand, we try to outline the procedure in a more general setting.

The program to be developed is the following:

I. Let $\varepsilon > 0$. Consider a *nonautonomous singularly pertubed PDE* in the form

$$\boxed{-\varepsilon^2 \operatorname{div}\{D_\xi \mathcal{L}(x,u,Du)\} + \varepsilon^2 D_s \mathcal{L}(x,u,Du) = g(x,u) \quad \text{in } \mathbb{R}^n} \qquad (\boldsymbol{P_\varepsilon})$$

namely the Euler equation associated with the functional $I_\varepsilon : H^1(\mathbb{R}^n) \to \mathbb{R}$ (for $\mathcal{L}$ sufficiently smooth and having a quadratic growth in Du) given by

$$I_\varepsilon(u) = \varepsilon^2 \int_{\mathbb{R}^n} \mathcal{L}(x,u,Du)dx - \int_{\mathbb{R}^n} G(x,u)dx.$$

We shall *assume* that the solutions to $\boldsymbol{P_\varepsilon}$ enjoy some nice *qualitative properties* such as *symmetry, exponential decay, existence of ground state solutions, etc.* Consider also the 'formally limiting' autonomous equation, for $z \in \mathbb{R}^n$ fixed,

$$\boxed{-\operatorname{div}\{D_\xi \mathcal{L}(z,u,Du)\} + D_s \mathcal{L}(z,u,Du) = g(z,u) \qquad \text{in } \mathbb{R}^n} \qquad (\boldsymbol{P_z})$$

namely the Euler equation associated with $I_z : H^1(\mathbb{R}^n) \to \mathbb{R}$,

$$I_z(u) = \int_{\mathbb{R}^n} \mathcal{L}(z,u,Du)dx - \int_{\mathbb{R}^n} G(z,u)dx.$$

II. Introduce the *ground energy function* $\Sigma : \mathbb{R}^n \to \mathbb{R}$ of (S_z)

$$\boxed{\Sigma(z) = \inf_{\phi \in \mathcal{N}_z} I_z(\phi),}$$

where $\mathcal{N}_z$ is the *Nehari manifold* of I_z, that is

$$\mathcal{N}_z = \big\{\phi \in H^1(\mathbb{R}^n) \setminus \{0\} : \; I_z'(\phi)[\phi] = 0\big\}.$$

III. Prove some regularity property for Σ, more precisely:

▶ $\Sigma \in C(\mathbb{R}^n)$ and, for every $z \in \mathbb{R}^n$, if

$$b_1(z) = \inf_{\phi \in H^1(\mathbb{R}^n)\setminus\{0\}} \sup_{t \geq 0} I_z(t\phi),$$

$$b_2(z) = \inf\big\{I_z(\phi) : \phi \in H^1(\mathbb{R}^n) \setminus \{0\} \text{ is a solution to } (\boldsymbol{P_z})\big\}.$$

Then

$$\boxed{b_1(z) = b_2(z) = \Sigma(z) \qquad \text{(so that } \Sigma \text{ is a true minimal energy map).}}$$

This might require some work (although it is well-known for semilinear equations). Implicitely it requires the limit functional I_z to have a Mountain-Pass geometry.

▶ $\Sigma \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^n)$ (important, for our purposes);

▶ The directional derivatives from the left and the right of Σ at every point $z \in \mathbb{R}^n$ along any $w \in \mathbb{R}^n$ exist and it holds:

$$\boxed{\left(\frac{\partial \Sigma}{\partial w}\right)^{-}(z) = \sup_{\phi \in \mathbb{S}(z)} \nabla_z I_z(\phi) \cdot w,}$$

$$\boxed{\left(\frac{\partial \Sigma}{\partial w}\right)^{+}(z) = \inf_{\phi \in \mathbb{S}(z)} \nabla_z I_z(\phi) \cdot w},$$

where $\mathbb{S}(z)$ is the set of the positive radial solutions of $\boldsymbol{P_z}$ at level $\Sigma(z)$.

IV. By the previous step, Σ is always *Clarke differentiable* [4] and we denote by $\mathcal{K} \subset \mathbb{R}^n$ the set of *Clarke critical points* of Σ, namely

$$\boxed{\mathcal{K} = \left\{ z \in \mathbb{R}^n : \ 0 \in \partial_C \Sigma(z) \right\}}.$$

Just for selfcontainedness, we recall that, if $f : \mathbb{R}^n \to \mathbb{R}$ is a $\mathrm{Lip}_{\mathrm{loc}}$ function near a point $z \in \mathbb{R}^n$, then the *Clarke subdifferential* of f at z is defined by

$$\partial_C f(z) = \left\{ \eta \in \mathbb{R}^n : \ f^0(z, w) \geq \eta \cdot w, \ \text{ for every } w \in \mathbb{R}^n \right\},$$

where $f^0(z, w)$ is the generalized derivative of f at z along $w \in \mathbb{R}^n$, defined by

$$f^0(z; w) = \limsup_{\substack{\xi \to z \\ \lambda \to 0+}} \frac{f(\xi + \lambda w) - f(\xi)}{\lambda}.$$

V. Introduce the sets of concentration points. We set

$$\begin{aligned} \mathcal{E} = \big\{ z \in \mathbb{R}^n : \ & \text{there exists a sequence of solutions } u_{\varepsilon_h} \text{ of } \boldsymbol{P_\varepsilon} \text{ with} \\ & |u_{\varepsilon_h}(z)| \geq \delta \text{ for some } \delta > 0, |u_{\varepsilon_h}(z + \varepsilon_h x)| \to 0 \text{ as } |x| \to \infty \\ & \text{uniformly w.r.t. } h, \text{ and } {\varepsilon_h}^{-n} I_{\varepsilon_h}(u_{\varepsilon_h}) \to \Sigma(z) \text{ as } h \to \infty \big\}. \end{aligned}$$

We say that $\mathcal{E}$ is the *energy concentration set* for ($\boldsymbol{P_\varepsilon}$).
Let $m \geq 1$. We set

$$\begin{aligned} \mathcal{E}_m = \big\{ z \in \mathbb{R}^n : \ & \text{there exists a sequence of solutions } u_{\varepsilon_h} \text{ of } \boldsymbol{P_\varepsilon} \text{ with} \\ & |u_{\varepsilon_h}(z)| \geq \delta \text{ for some } \delta > 0, |u_{\varepsilon_h}(z + \varepsilon_h x)| \to 0 \text{ as } |x| \to \infty \\ & \text{uniformly w.r.t. } h, \text{ and } {\varepsilon_h}^{-n} I_{\varepsilon_h}(u_{\varepsilon_h}) \to m \text{ as } h \to \infty \big\}. \end{aligned}$$

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[4] F.H. Clarke, Optimization and nonsmooth analysis, Wiley-Interscience publication, 1983.

We say that $\mathcal{E}_m$ is the *concentration set* for ($\boldsymbol{P_\varepsilon}$) at the energy level m.

VI. This is the key variational step in order to complete the program. It requires the use of the ***Pohožaev-Pucci-Serrin identity*** [5] [6] applied with suitable truncated test functions. For the rescalings $v_{\varepsilon_h}(x) = u_{\varepsilon_h}(z + \varepsilon_h x)$, We have

$$\begin{aligned}
&\sum_{i,\,l=1}^{n} \int_{\mathbb{R}^n} \partial_i \boldsymbol{q}^l \partial_{\xi_i} \mathcal{L}(z + \varepsilon_h x, v_{\varepsilon_h}, Dv_{\varepsilon_h}) \partial_l v_{\varepsilon_h} \\
&\quad = \int_{\mathbb{R}^n} (\operatorname{div} \boldsymbol{q}) \big(\mathcal{L}(z + \varepsilon_h x, v_{\varepsilon_h}, Dv_{\varepsilon_h}) - G(z + \varepsilon_h x, v_{\varepsilon_h}) \big) \\
&\quad + \int_{\mathbb{R}^n} \boldsymbol{q} \cdot \big(\partial_x \mathcal{L}(z + \varepsilon_h x, v_{\varepsilon_h}, Dv_{\varepsilon_h}) - \partial_x G(z + \varepsilon_h x, v_{\varepsilon_h}) \big),
\end{aligned}$$

for all $\boldsymbol{q} \in C^1_{\mathrm{c}}(\mathbb{R}^n, \mathbb{R}^n)$. Let us take, for $\lambda > 0$,

$$\boldsymbol{q}(x) = \boldsymbol{q}_j(x) = (0, \dots, 0, \Upsilon(\lambda x), 0, \dots, 0), \quad (j\text{-th place nonzero})$$

and $\Upsilon \in C^1_{\mathrm{c}}(\mathbb{R}^n)$ such that $\Upsilon(x) = 1$ if $|x| \leq 1$ and $\Upsilon(x) = 0$ if $|x| \geq 2$. We get

$$\begin{aligned}
&\sum_{i=1}^{n} \int_{\mathbb{R}^n} \lambda \partial_i \Upsilon(\lambda x) \partial_{\xi_i} \mathcal{L}(z + \varepsilon_h x, v_{\varepsilon_h}, Dv_{\varepsilon_h}) \partial_j v_{\varepsilon_h} \\
&\quad = \int_{\mathbb{R}^n} \lambda \partial_j \Upsilon(\lambda x) \big(\mathcal{L}(z + \varepsilon_h x, v_{\varepsilon_h}, Dv_{\varepsilon_h}) - G(z + \varepsilon_h x, v_{\varepsilon_h}) \big) \\
&\quad + \int_{\mathbb{R}^n} \varepsilon_h \Upsilon(\lambda x) \big(\partial_{x_j} \mathcal{L}(z + \varepsilon_h x, v_{\varepsilon_h}, Dv_{\varepsilon_h}) - \partial_{x_j} G(z + \varepsilon_h x, v_{\varepsilon_h}) \big),
\end{aligned}$$

for all $\lambda > 0$ and $h \geq 1$.

We also stress that the C^2 regularity of the solutions is no more needed. Indeed it suffices to have the C^1 regularity [7]. For instance, with ε^p in place of ε^2, for

$$\mathcal{L}(x, s, \xi) = \frac{1}{p} |\xi|^p,$$

one only has [8] $u \in C^{1,\alpha}$.

VII. Prove that if $z \in \mathcal{E}$ or $z \in \mathcal{E}_m$ then there exists $u_* \in H^1(\mathbb{R}^n)$, $u_* \neq 0$ (with $u_* \in \mathbb{S}(z)$ if $z \in \mathcal{E}$) such that v_{ε_h} converges to u_* weakly in H^1 and locally in C^2, and

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[5] ▶ S.I. Pohožaev, *On the eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$, Dokl. Akad. Nauk SSSR* **165** (1965), 36–39.

[6] ▶ P. Pucci, J. Serrin, *A general variational identity*, Indiana Univ. Math. J. **35** (1986), 681–703.

[7] ▶ M. Degiovanni, A. Musesti, M. Squassina, *On the regularity of solutions in the Pucci-Serrin identity*, Calc. Var. Partial Differential Equations **18** (2003), 317–334.

[8] ▶ P. Tolksdorf, *Regularity for a more general class of quasilinear elliptic equations*, J. Differential Equations **51** (1984), 126–150.

it holds

$$\boxed{\int_{\mathbb{R}^n}\left[\frac{\partial\mathcal{L}}{\partial x_j}(x,u_*,Du_*)-\frac{\partial G}{\partial x_j}(x,u_*)\right]\Big|_{x=z}dx=0,\qquad \forall j=1,\dots,n}$$

letting (first) $\lambda\to 0$ and then $h\to\infty$ and exploiting the exponential barriers on v_{ε_h}.

VIII. Prove that, since $u_*\in\mathbb{S}(z)$, for all $w\in\mathbb{R}^n$ we have

$$\left(\frac{\partial\Sigma}{\partial w}\right)^+(z)=\inf_{\phi\in\mathbb{S}(z)}\nabla_z I_z(\phi)\cdot w\le\int_{\mathbb{R}^n}\left[\frac{\partial\mathcal{L}}{\partial x_j}(x,u_*,Du_*)-\frac{\partial G}{\partial x_j}(x,u_*)\right]\Big|_{x=z}dx=0.$$

Then, by the very definition of $(-\Sigma)^0(z;w)$,

$$(-\Sigma)^0(z;w)\ge\left(\frac{\partial(-\Sigma)}{\partial w}\right)^+(z)\ge 0,\quad\forall w\in\mathbb{R}^n.$$

Then $0\in\partial_C(-\Sigma)(z)$ and, since $\partial_C(-\Sigma)(z)=-\partial_C\Sigma(z)$, we obtain $z\in\mathcal{K}$.

This proves the following:

Theorem 1. *For a reasonable class of problems, $\mathcal{E}\subset\mathcal{K}$.*

Remark 1. *As a straightforward combination of this Theorem with the well known convex hull characterization of $\partial_C\Sigma(z)$, we have*

$$z\in\mathcal{E}\implies 0\in\mathrm{Co}\big\{\lim_j\nabla\Sigma(\xi_j)\,:\,\xi_j\notin\Omega\text{ and }\xi_j\to z\big\},$$

where $\mathrm{Co}\{X\}$ denotes the convex hull of X and Ω is any null set containing the set of points at which Σ fails to be differentiable.

Corollary 1. *If ($\boldsymbol{P_z}$) has a unique solution (up to translation), then $\mathcal{E}\subset\mathrm{Crit}(\Sigma)$.*

Let $m \geq 1$ and $z \in \mathbb{R}^n$. For every $w \in \mathbb{R}^n$ we define $\Gamma^{\mp}_{z,m}(w)$ by

$$\Gamma^{-}_{z,m}(w) := \sup_{\phi \in \mathbb{G}_m(z)} \nabla_z I_z(\phi) \cdot w,$$

$$\Gamma^{+}_{z,m}(w) := -\inf_{\eta \in \mathbb{G}_m(z)} \nabla_z I_z(\phi) \cdot w,$$

where $\mathbb{G}_m(z)$ denotes the set of all the nontrivial, radial, exponentially decaying solutions of ($\boldsymbol{P_z}$) having energy equal to m. Let $m \geq 1$. We set

$$\mathcal{K}_m := \Big\{ z \in \mathbb{R}^n : \; 0 \in \partial\Gamma^{-}_{z,m}(0) \cap \partial\Gamma^{+}_{z,m}(0) \Big\},$$

where ∂ stands for the subdifferential of convex functions,

$$\partial\Gamma^{\mp}_{z,m}(0) = \Big\{ \xi \in \mathbb{R}^n : \; \Gamma^{\mp}_{z,m}(w) \geq \xi \cdot w, \; \text{for every } w \in \mathbb{R}^n \Big\}.$$

Of course, if $\mathbb{G}_m(z) = \{\phi_0\}$ was a singleton (uniqueness!), then $z \in \mathcal{K}_m$ if and only if

$$\Gamma^{-}_{z,m}(w) = \Gamma^{+}_{z,m}(w) = \frac{\partial I_z}{\partial w}(\phi_0) = 0, \qquad \forall w \in \mathbb{R}^n.$$

Arguing as above, we similarly obtain

$$\mathcal{E}_m \subset \mathcal{K}_m.$$

Question: What beyond the $\mathrm{Lip}_{\mathrm{loc}}$ regularity of Σ ?

Do not know. Not even for the simplest model equations without uniqueness features; For instance, on one hand, if we consider the problem

$$-\varepsilon^2 \Delta u + V(x) u = K(x) u^p \quad \text{in } \mathbb{R}^n, \qquad u > 0 \quad \text{in } \mathbb{R}^n,$$

then $\Sigma \in C^m(\mathbb{R}^n)$ provided that both the potentials V and K belong to $C^m(\mathbb{R}^n)$, with $m \geq 1$. On the other hand, if f is not a power (and does not satisfy conditions ensuring uniqueness up to translations), for the equation

$$-\varepsilon^2 \Delta u + V(x) u = K(x) f(u) \quad \text{in } \mathbb{R}^n, \qquad u > 0 \quad \text{in } \mathbb{R}^n,$$

we do not know which regularity beyond $\mathrm{Lip}_{\mathrm{loc}}$ can be achieved by Σ.

For the following class of NSE with potentials

$$-\varepsilon^2 \Delta u + V(x)u = K(x)f(u) \quad \text{in } \mathbb{R}^n, \qquad u > 0 \quad \text{in } \mathbb{R}^n,$$

the - previously outlined - program can be carried on [9] [10].

Concerning the sufficient conditions in order to have existence of solutions concentrating around a suitable critical points of Σ there is a *huge* amount of papers. We just refer to the various contributions cited in:

M. Del Pino, P. Felmer, *Semi-classical states for nonlinear Schrodinger equations,*
J. Functional Anal. (1997).

In the particular case where $f(u)$ is a pure power,

$$f(u) = u^p,$$

then we are allowed to give an explicit representation for the ground state function, merely depending on the potentials V and K: by the results of [11], we know that there is uniqueness (up to translation) of positive solutions for

$$-\Delta u + u = u^p, \qquad \text{in } \mathbb{R}^n,$$

and, by a suitable variable rescaling, also for the "limit" problem at $x = z$

$$-\Delta u + V(z)u = K(z)u^p, \qquad \text{in } \mathbb{R}^n.$$

This allows to give an explicit representation for the ground state function, merely depending on the potentials V and K:

$$\boxed{\Sigma(z) = \Gamma \, \frac{V^{\frac{p+1}{p-1} - \frac{n}{2}}(z)}{K^{\frac{2}{p-1}}(z)}}$$

for a suitable constant $\Gamma > 0$. Notice that, since

$$\frac{p+1}{p-1} - \frac{n}{2} \sim 0$$

if and only if

$$p \sim \frac{n+2}{n-2} = 2^* - 1,$$

then $V(x)$ has a weak influence in the location as $p \to 2^* - 1$.

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[9] ▶ S. Secchi, M. Squassina, *On the location of concentration points for singularly perturbed elliptic equations,* Adv. Differential Equations **9** (2004), 221–239.

[10] ▶ S. Secchi, M. Squassina, *On the location of spikes for the Schrödinger equation with electromagnetic field,* Commun. Contemp. Math. **9** (2005), 251–268.

[11] ▶ M.K. Kwong, *Uniqueness of positive solutions of $\Delta u - u + u^p = 0$ in $\mathbb{R}^N$*, Arch. Ration. Mech. Anal. **105** (1989), 243–266.

4. *Class II: System of weakly coupled semilinear equations*

The program can also be carried on for the semilinear system [12]

$$\begin{cases} -\varepsilon^2\Delta u + u = K(x)v^q, & \text{in } \mathbb{R}^n, \\ -\varepsilon^2\Delta v + v = Q(x)u^p, & \text{in } \mathbb{R}^n, \\ \ u,\, v > 0, & \text{in } \mathbb{R}^n, \end{cases} \tag{S_ε}$$

where $p, q > 1$ are lying *below* the so called "critical hyperbola"

$$\mathscr{C}_n = \big\{(p,q) \in (1,\infty)\times(1,\infty) : \tfrac{1}{p+1} + \tfrac{1}{q+1} = 1 - \tfrac{2}{n}\big\}, \qquad n \geq 3,$$

which naturally arises in the study of this problem and constitutes the borderline between existence and nonexistence results.

The interest in looking for conditions for the spike location of the solutions to (S_ε) is mainly motivated by the following simple observation:
contrary to the scalar case, there is *no uniqueness result* available in the literature for the (radial) solutions to the (limiting) system associated with (S_ε)

$$\begin{cases} -\Delta u + u = K(z)v^q, & \text{in } \mathbb{R}^n, \\ -\Delta v + v = Q(z)u^p, & \text{in } \mathbb{R}^n, \\ \ u,\, v > 0, & \text{in } \mathbb{R}^n, \end{cases} \tag{S_z}$$

where $z \in \mathbb{R}^n$ is frozen and acts as a parameter.

As a consequence, in the vectorial case, we do not know whether the (suitably defined) ground energy map Σ is C^1*-smooth* and admits an *explicit representation formula*. Hence, the necessary conditions in terms of Clarke subdifferential (or weaker) appear here even more natural than in the case of a single equation.

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[12] ▶ A. Pomponio, M. Squassina, *Locating the peaks of semilinear elliptic systems*, Adv. Nonlinear Stud. **5** (2005), to appear.

For the above semilinear system, see e.g. [13]

Remark 2. *As already observed, up to our knowledge there is no (known) uniqueness result for the elliptic system*

$$-\Delta\xi+\xi=\zeta^q, \qquad -\Delta\zeta+\zeta=\xi^p, \qquad \textit{in } \mathbb{R}^n,$$

and so, in general, we cannot provide an explicit expression for Σ*. Slightly more in general, if* V *is smooth and* $\alpha \le V(x) \le \beta$*, consider the system*

$$\begin{cases} -\Delta u + V(z)u = K(z)v^q, & \textit{in } \mathbb{R}^n, \\ -\Delta v + V(z)v = Q(z)u^p, & \textit{in } \mathbb{R}^n, \\ \ u,\, v > 0, & \textit{in } \mathbb{R}^n. \end{cases}$$

Assuming *that the previous system has a unique solution* (ξ,ζ)*, then we claim that*

$$\boxed{\Sigma(z)=\Gamma\frac{V^{\frac{(p+1)(q+1)}{pq-1}-\frac{n}{2}}(z)}{Q^{\frac{q+1}{pq-1}}(z)\,K^{\frac{p+1}{pq-1}}(z)}}, \qquad (\textit{subordinated to uniqueness})$$

for a suitable positive constant Γ*. Indeed, by rescaling*

$$u(x)=\varpi_1\xi(\mu x) \quad \textit{and} \quad v(x)=\varpi_2\zeta(\mu x),$$

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[13] ▶ C.O. Alves, P.C. Carrião, O.H. Miyagaki, *On the existence of positive solutions of a perturbed Hamiltonian system in* $\mathbb{R}^n$, J. Math. Anal. Appl. **276** (2002), 673–690.

▶ C.O. Alves, S.H.M. Soares, J. Yang, *On existence and concentration of solutions for a class of Hamiltonian systems in* $\mathbb{R}^N$, Adv. Nonlinear Stud. **3** (2003), 161–180.

▶ Ph. Clément, D.G. De Figueiredo, E. Mitidieri, *Positive solutions of semilinear elliptic systems,* Comm. Partial Differential Equations **17** (1992), 923–940.

▶ D.G. Costa, C.A. Magalhães, *A variational approach to noncooperative elliptic systems,* Nonlinear Anal. **25** (1995), 699–715.

▶ D.G. De Figueiredo, P. Felmer, *On superquadratic elliptic systems,* Trans. Amer. Math. Soc. **343** (1994), 99–116.

▶ D.G. De Figueiredo, J. Yang, *Decay, symmetry and existence of solutions of semilinear elliptic systems,* Nonlinear Anal. **33** (1998), 211–234.

▶ D.G. De Figueiredo, C.A. Magalhães, *On nonquadratic Hamiltonian elliptic systems,* Adv. Differential Equations **1** (1996), 881–898.

▶ J. Hulshof, .C.A.M. van der Vorst, *Differential systems with strongly indefinite variational structure,* J. Functional Anal. **114** (1993), 32–58.

▶ A. Pistoia, M. Ramos, *Locating the peaks of the least energy solutions to an elliptic system with Neumann boundary conditions,* J. Differential Equations **201** (2004), 160–176.

▶ A. Pistoia, M. Ramos, *Spike-layered solutions of singularly perturbed elliptic systems,* NoDEA Nonlinear Differential Equations Appl., to appear.

▶ M. Ramos, J. Yang, *Spike-layered solutions for an elliptic system with Neumann boundary conditions,* Trans. Amer. Math. Soc. **357** (2005), 3265–3284.

▶ J. Yang, *Nontrivial solutions of semilinear elliptic systems in* $\mathbb{R}^N$, Electron. J. Differ. Equ. Conf., **6** (2001), 343–357.

▶ H. Zou, *A priori estimates for a semilinear elliptic system without variational structure and their applications,* Math. Ann. **323** (2002), 713–735.

where we have set

$$\mu = \mu(z) = V^{\frac{1}{2}}(z),$$

$$\varpi_1 = \varpi_1(z) = \frac{V^{\frac{q+1}{pq-1}}(z)}{Q^{\frac{q}{pq-1}}(z)\, K^{\frac{1}{pq-1}}(z)},$$

$$\varpi_2 = \varpi_2(z) = \frac{V^{\frac{p+1}{pq-1}}(z)}{Q^{\frac{1}{pq-1}}(z)\, K^{\frac{p}{pq-1}}(z)},$$

we see that (u, v) *is the unique solution of the original system, yielding the expression of* Σ*.*

Let us observe that

$$\frac{(p+1)(q+1)}{pq-1} - \frac{n}{2} \sim 0$$

if and only if

$$\mathrm{dist}_{\mathbb{R}^2}((p,q), \mathscr{C}_n) \sim 0.$$

Then, for problems with powers (p, q) *close to the set* $\mathscr{C}_n$*, the potential* $V(x)$ *would be expected to have a weak influence in the location of concentration points (as in the scalar case).*

If e.g. p and q are both less than $\frac{n+2}{n-2}$, then system (S_ε) admits a *natural variational structure* (of Hamiltonian type) which is based on the strongly indefinite functional $f_\varepsilon : H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \to \mathbb{R}$,

$$f_\varepsilon(u,v) = \int_{\mathbb{R}^n} \varepsilon^2 \nabla u \cdot \nabla v + uv - \tfrac{1}{q+1} \int_{\mathbb{R}^n} K(x)|v|^{q+1} - \tfrac{1}{p+1} \int_{\mathbb{R}^n} Q(x)|u|^{p+1}.$$

However, for our purposes, as well as for dealing with possibly supercritical values of p or q, we consider a corresponding *dual variational structure*, mainly relying on the Legendre-Fenchel transformation. In the following, we just briefly recall some of the core ingredients. For $\frac{1}{p+1} + \frac{1}{q+1} > \frac{n-2}{n}$, consider the linear operators

$$\begin{aligned} T_1 &: L^{\frac{q+1}{q}}(\mathbb{R}^n) \to W^{2,\frac{q+1}{q}}(\mathbb{R}^n) \hookrightarrow L^{p+1}(\mathbb{R}^n), \\ T_2 &: L^{\frac{p+1}{p}}(\mathbb{R}^n) \to W^{2,\frac{p+1}{p}}(\mathbb{R}^n) \hookrightarrow L^{q+1}(\mathbb{R}^n), \end{aligned}$$

defined as $T_1 = T_2 = (-\Delta + \mathrm{Id})^{-1}$. Notice that T_1 and T_2 are continuous. Then, we consider the linear operator (take into account the proper Sobolev embeddings)

$$T : L^{\frac{p+1}{p}}(\mathbb{R}^n) \times L^{\frac{q+1}{q}}(\mathbb{R}^n) \to L^{p+1}(\mathbb{R}^n) \times L^{q+1}(\mathbb{R}^n), \qquad T = \begin{bmatrix} 0 & T_1 \\ & \\ T_2 & 0 \end{bmatrix},$$

explicitly defined by

$$\langle T\eta, \xi \rangle = \xi_1 T_1 \eta_2 + \xi_2 T_2 \eta_1, \qquad \forall \eta = (\eta_1, \eta_2), \ \forall \xi = (\xi_1, \xi_2).$$

Finally we introduce the Banach space $(\mathscr{H}, \|\cdot\|_{\mathscr{H}})$,

$$\mathscr{H} = L^{\frac{p+1}{p}}(\mathbb{R}^n) \times L^{\frac{q+1}{q}}(\mathbb{R}^n), \qquad \|\eta\|^2_{\mathscr{H}} = \|\eta_1\|^2_{L^{\frac{p+1}{p}}(\mathbb{R}^n)} + \|\eta_2\|^2_{L^{\frac{q+1}{q}}(\mathbb{R}^n)}$$

and the (dual) C^1 functional $J_\varepsilon : \mathscr{H} \to \mathbb{R}$ defined as

$$J_\varepsilon(\eta) = \tfrac{p}{p+1} \int_{\mathbb{R}^n} \frac{|\eta_1|^{\frac{p+1}{p}}}{Q^{\frac{1}{p}}(\varepsilon x)} + \tfrac{q}{q+1} \int_{\mathbb{R}^n} \frac{|\eta_2|^{\frac{q+1}{q}}}{K^{\frac{1}{q}}(\varepsilon x)} - \tfrac{1}{2} \int_{\mathbb{R}^n} \langle T\eta, \eta \rangle.$$

If $\eta^\varepsilon = (\eta_1^\varepsilon, \eta_2^\varepsilon)$ is a critical point of J_ε, then $(u_\varepsilon(x), v_\varepsilon(x)) = (\bar{u}_\varepsilon(\frac{x}{\varepsilon}), \bar{v}_\varepsilon(\frac{x}{\varepsilon}))$, with

$$(\bar{u}_\varepsilon, \bar{v}_\varepsilon) = (T_1 \eta_2^\varepsilon, T_2 \eta_1^\varepsilon) \in W^{2,\frac{q+1}{q}} \cap L^{p+1} \times W^{2,\frac{p+1}{p}} \cap L^{q+1},$$

corresponds to a solution to (S_ε) with $u_\varepsilon(x), v_\varepsilon(x) \to 0$ for $|x| \to \infty$. In light of the above summability, we have $f_\varepsilon(u_\varepsilon, v_\varepsilon) \in \mathbb{R}$ for all $\varepsilon > 0$. Analogously, associated with the limit system, we introduce the limiting functional $I_z : \mathscr{H} \to \mathbb{R}$

$$I_z(\eta) = \tfrac{p}{p+1} \int_{\mathbb{R}^n} \frac{|\eta_1|^{\frac{p+1}{p}}}{Q^{\frac{1}{p}}(z)} + \tfrac{q}{q+1} \int_{\mathbb{R}^n} \frac{|\eta_2|^{\frac{q+1}{q}}}{K^{\frac{1}{q}}(z)} - \tfrac{1}{2} \int_{\mathbb{R}^n} \langle T\eta, \eta \rangle.$$

From the viewpoint of our investigation, the main advantage of exploiting the dual variational functional I_z is that it admits a *mountain-pass geometry* and the mountain-pass value corresponds to the *least possible energy* of system (S_z).

The (dual) *ground energy function* $\Sigma : \mathbb{R}^n \to \mathbb{R}$ of (S_z) is given by

$$\Sigma(z) := \inf_{\eta \in \mathcal{N}_z} I_z(\eta),$$

where $\mathcal{N}_z$ is the *Nehari manifold* of I_z, that is

$$\mathcal{N}_z = \big\{\eta \in \mathscr{H} : \ \eta \neq (0,0) \text{ and } I_z'(\eta)[\eta] = 0\big\}.$$

We shall denote by $\mathcal{K} \subset \mathbb{R}^n$ the set of *Clarke critical points* of Σ.

We say that the pair $(u_\varepsilon, v_\varepsilon)$ is a *strong solution* to system $(\boldsymbol{P_\varepsilon})$ if it is a distributional solution and $(u_\varepsilon, v_\varepsilon) \in W^{2,(q+1)/q}(\mathbb{R}^n) \times W^{2,(p+1)/p}(\mathbb{R}^n)$. We set

$$\begin{aligned}\mathcal{E} := \big\{ z \in \mathbb{R}^n :\ &\text{there exists a sequence of strong solutions } (u_{\varepsilon_h}, v_{\varepsilon_h}) \text{ of } (\boldsymbol{P_\varepsilon}) \text{ with} \\ &|u_{\varepsilon_h}(z)|, |v_{\varepsilon_h}(z)| \geq \delta \text{ for some } \delta > 0,\ |u_{\varepsilon_h}(z+\varepsilon_h x)|,\ |v_{\varepsilon_h}(z+\varepsilon_h x)| \to 0 \\ &\text{as } |x| \to \infty \text{ uniformly w.r.t. } h, \text{ and } {\varepsilon_h}^{-n} f_{\varepsilon_h}(u_{\varepsilon_h}, v_{\varepsilon_h}) \to \Sigma(z) \text{ as } h \to \infty \big\}.\end{aligned}$$

We say that $\mathcal{E}$ is the *energy concentration set* for $(\boldsymbol{P_\varepsilon})$.

Assume that $K, Q \in C^1(\mathbb{R}^n)$ and

$$\alpha \leq K(x) \leq \beta, \quad \alpha \leq Q(x) \leq \beta, \qquad \text{for all } x \in \mathbb{R}^n,$$

$$|\nabla K(x)|, |\nabla Q(x)| \leq Ce^{M|x|}, \qquad \text{for all } x \in \mathbb{R}^n \text{ with } |x| \text{ large.}$$

for some positive constants α, β, C and M.

The main result of the paper, linking the energy concentration set $\mathcal{E}$ with the set $\mathcal{K}$ of Clarke critical set of Σ, is provided by the following

Theorem 2. *$\mathcal{E} \subset \mathcal{K}$.*

Remark 3. *Conversely, under suitable assumptions, if there exists an absolute minimum (or maximum) point z_* for Σ, then $z_* \in \mathcal{E} \neq \emptyset$.*

Corollary 2. *Under the (unproved) assumption that, for all $z \in \mathbb{R}^n$, system (S_ε) admits a unique positive solution (up to translations), Σ is C^1-smooth and*

$$\mathcal{E} \subset \mathrm{Crit}\big(Q^{\frac{q+1}{pq-1}}\, K^{\frac{p+1}{pq-1}}\big),$$

where Crit*(f) denotes the set of (classical) critical points of f.*

Theorem 3. *Then $\mathcal{E}_m \subset \mathcal{K}_m$.*

6. *Class III: System of Schrodinger equations (just a starting remark)*

For $\beta_{ij} > 0$ for $i, j = 1, \ldots, N$ (attractive case), consider the system:

$$-\varepsilon^2 \Delta u_j + V_j(x) u_j = K_j(x) u_j^3 + \sum_{i \neq j} \beta_{ij} u_i^2 u_j \qquad \text{in } \mathbb{R}^3,$$

with $u_j > 0$ in $\mathbb{R}^3$, $j = 1, \ldots, N$. For $z \in \mathbb{R}^3$, consider the limiting system:

$$-\Delta u_j + V_j(z) u_j = K_j(z) u_j^3 + \sum_{i \neq j} \beta_{ij} u_i^2 u_j \qquad \text{in } \mathbb{R}^3,$$

and the associated limiting functional $I_z : H^1(\mathbb{R}^3) \times \cdots \times H^1(\mathbb{R}^3) \to \mathbb{R}$

$$I_z(\mathbf{u}) = \sum_{j=1}^{N} \left(\frac{1}{2} \int_{\mathbb{R}^3} |\nabla u_j|^2 + \frac{V_j(z)}{2} \int_{\mathbb{R}^3} u_j^2 - \frac{K_j(z)}{4} \int_{\mathbb{R}^3} u_j^4 \right) - \frac{1}{4} \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \beta_{ij} \int_{\mathbb{R}^3} u_i^4 u_j^4,$$

where $\mathbf{u} = (u_1, \ldots, u_N)$.

Main Aims:

- Sufficient conditions to have solutions concentrating around a (suitable) point.
- Necessary conditions to have solutions concentrating a (given) point.

Consider, for $z \in \mathbb{R}^3$, the $N \times N$ matrix A_z

$$A_z := \begin{bmatrix} K_1(z) & \beta_{12} & \dots & \beta_{1N} \\ \beta_{21} & K_2(z) & \dots & \beta_{2N} \\ \vdots & \vdots & \vdots & \vdots \\ \beta_{N1} & \dots & \beta_{NN-1} & K_N(z) \end{bmatrix}$$

We introduce the following set

$$\mathscr{P} = \big\{ z \in \mathbb{R}^3 : \text{ the matrix } A_z \text{ is positive definite} \big\}.$$

The next property, classical in the scalar case, sounds like a good starting point.

Theorem 4 (see [14] [15]). *Let us set*

$$\Sigma(z) = \inf_{\mathbf{u} \in \mathcal{N}_z} I_z(\mathbf{u}),$$

where $\mathcal{N}_z$ is the set of all the $\mathbf{u} \in (H^1(\mathbb{R}^3))^N$ with $u_j \geq 0$, $u_j \neq 0$, and

$$\begin{cases} \int_{\mathbb{R}^3} |\nabla u_j|^2 + V_j(z) \int_{\mathbb{R}^3} u_j^2 - K_j(z) \int_{\mathbb{R}^3} u_j^4 = \sum_{i \neq j} \beta_{ij} \int_{\mathbb{R}^3} u_i^4 u_j^4 \\ \text{for all } j = 1, \dots, N \end{cases}$$

Assume that

$$z \in \mathscr{P}.$$

Then $\Sigma(z)$ is achieved by a positive, radially symmetric and stricty decreasing solution $\mathbf{u}_z$.

Hence, referring to previous notations,

$$\Sigma_* = \Sigma|_{\mathscr{P}}, \quad \mathcal{E}_* = \mathcal{E} \cap \mathscr{P}, \quad \mathcal{K}_* = \mathcal{K} \cap \mathscr{P}$$

might do the job...

[Work in progress jointly with E. Montefusco and B. Pellacci].

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[15] ▶ LA Maia, E. Montefusco, B. Pellacci, *Positive solutions for a weakly coupled nonlinear Schrodinger system*, preprint, 2005.