Exponential stability and singular limit for a linear thermoelastic plate with memory effects

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Let Ω be a bounded planar domain with smooth boundary $\partial\Omega$. Suppose that Ω is occupied, for all time t, by a thin homogeneous isotropic elastic plate. Denoting by u its vertical deflection and by ϑ the temperature variation field, we suppose that the evolution of the pair (u, ϑ) is governed by the following integrodifferential system

$$\begin{cases} u_{tt} - \omega\Delta u_{tt} + \Delta(\Delta u + \vartheta) = 0, \\ \vartheta_t + \displaystyle\int_0^\infty k(s)\big[c\vartheta(t-s) - \Delta\vartheta(t-s)\big]ds - \Delta u_t = 0, \end{cases} \qquad (\mathcal{P}_{\omega,\varepsilon})$$

in $\Omega \times \mathbb{R}^+$, where $\mathbb{R}^+ = (0, \infty)$. Here $\omega \geq 0$, $c \geq 0$, and $k : [0, \infty) \to \mathbb{R}$ is a smooth positive bounded convex function which vanishes at infinity. If k coincides with the Dirac mass at 0, then the above system formally becomes the well-known model of linear thermoelastic plate

$$\begin{cases} u_{tt} - \omega\Delta u_{tt} + \Delta(\Delta u + \vartheta) = 0, \\ \vartheta_t + c\vartheta - \Delta\vartheta - \Delta u_t = 0. \end{cases} \qquad (\mathcal{P}_{0,0})$$

The problems are endowed with Navier boundary conditions

$$u(t) = \Delta u(t) = 0 \qquad \text{on } \partial\Omega,\ t \geq 0,$$
$$\vartheta(t) = 0 \qquad \text{on } \partial\Omega,\ t \in \mathbb{R},$$

and initial conditions

$$(u(0), u_t(0), \vartheta(0)) = (u_0, u_1, \vartheta_0) \qquad \text{in } \Omega,$$
$$\vartheta(-s) = \psi(s) \qquad \text{in } \Omega \times \mathbb{R}^+,$$

where $u_0, u_1, \vartheta_0 : \Omega \to \mathbb{R}$ and $\psi : \Omega \times \mathbb{R}^+ \to \mathbb{R}$ are assigned functions. If we consider the weighted Hilbert space $\mathcal{M}^0_\varepsilon = L^2_{\mu_\varepsilon}(\mathbb{R}^+, H^1)$, endowed with the inner product

$$\langle \eta_1, \eta_2 \rangle_{\mathcal{M}^0_\varepsilon} = \int_0^\infty \mu_\varepsilon(s) \langle B^{1/2} \eta_1(s), B^{r/2} \eta_2(s) \rangle ds$$

and we introduce the product spaces

$$\mathcal{H}^0_{\omega,\varepsilon} = \begin{cases} H^2 \times H^1 \times H^0 \times \mathcal{M}^0_\varepsilon, & \text{if } \varepsilon > 0,\ \omega > 0, \\ H^2 \times H^0 \times H^0 \times \mathcal{M}^0_\varepsilon, & \text{if } \varepsilon > 0,\ \omega = 0, \\ H^2 \times H^1 \times H^0, & \text{if } \varepsilon = 0,\ \omega > 0, \\ H^2 \times H^0 \times H^0, & \text{if } \varepsilon = 0,\ \omega = 0, \end{cases}$$

endowed with the norms

$$\|(u, u_t, \vartheta, \eta)\|^2_{\mathcal{H}^0_{\omega,\varepsilon}} = \begin{cases} \|u\|^2_{H^2} + \|u_t\|^2_{H^0} + \omega\|u_t\|^2_{H^1} + \|\vartheta\|^2_{H^0} + \|\eta\|^2_{\mathcal{M}^0_\varepsilon}, & \text{if } \varepsilon > 0, \\ \|u\|^2_{H^2} + \|u_t\|^2_{H^0} + \omega\|u_t\|^2_{H^1} + \|\vartheta\|^2_{H^0}, & \text{if } \varepsilon = 0. \end{cases}$$

It is by now standard to prove that

In absence of rotational inertia effects, i.e., $\omega = 0$, system (**??**) is essentially parabolic and its exponential stability was proved with various kinds of boundary conditions

We already mentioned that, in absence of the rotational inertia term ($\omega = 0$), if the memory kernel is not allowed to grow too rapidly around the origin, although each trajectory of $\mathcal{P}_{0,\varepsilon}$ squeezes to zero as time goes to infinity, the associated semigroup $S_{0,\varepsilon}(t)$ on $\mathcal{H}^0_{0,\varepsilon}$ lacks of exponentially stability.

Theorem 1. *Let $\varepsilon \in (0,1]$ and assume that μ satisfies*

$$\lim_{s\to 0} \sqrt{s}\, \mu_\varepsilon(s) = 0.$$

Then the semigroup $S_{0,\varepsilon}(t)$ on $\mathcal{H}^0_{0,\varepsilon}$ is not exponentially stable.

Remark 2. *If the set of initial data is restricted to null initial past histories, i.e., $\psi \equiv 0$, it can be proved that the energy of the system $\mathcal{P}_{\omega,\varepsilon}$ with $\omega \geq 0$ exponentially decays to 0 provided that k satisfies reasonable assumptions. This was done by Fabrizio, Lazzari and Muñoz Rivera for a clamped plate, assuming $c = 0$. Therefore the presence of nonvanishing initial past history plays a discriminating role in the stability of the system with memory effects.*

Remark 3. *One might initially think that the system could be exponentially stable at least for small values of the time rescaling* ε *(when the kernel is very peaked around the origin,) but, as the above theorem shows, there is instead a breaks down from* $\varepsilon = 0$ *to* $\varepsilon > 0$.

■ M. Grasselli, J.E. Muñoz Rivera, V. Pata, JMAA, (2005).

For $\omega \in (0, 1]$, we have the following

Theorem 4. *There exist two positive constants* Θ *and* ς*, both independent of* $\varepsilon \in [0, 1]$ *and* $\omega \in (0, 1]$*, such that*

$$\mathcal{E}(t) \leq \varsigma \mathcal{E}(0) e^{-\omega \Theta t}, \qquad \forall\, t \geq 0.$$

where $\mathcal{E}(t) = \|S_{\omega,\varepsilon}(t)\|^2_{\mathcal{H}^0_{\omega,\varepsilon}}$.

Remark 5. *The energy exponential decay rate breaks down in the case* $\omega = 0$*, according to the lack of exponential stability*

Remark 6. *The result follows by constructing a suitably defined small perturbation of the first order energy functional.*

Next, we get a singular limit estimate on the line initiated in

■ M. Conti, V. Pata, M. Squassina, Indiana UMJ (2005).

■ M. Conti, V. Pata, M. Squassina, DCDS-A (2005).

We introduce the lifting and projection maps

$$\mathbb{L}_{\omega,\varepsilon} : \mathcal{H}^0_{\omega,0} \to \mathcal{H}^0_{\omega,\varepsilon}, \qquad \mathbb{P}_{\omega} : \mathcal{H}^0_{\omega,\varepsilon} \to \mathcal{H}^0_{\omega,0}, \qquad \mathbb{Q}_{\omega,\varepsilon} : \mathcal{H}^0_{\omega,\varepsilon} \to \mathcal{M}^0_{\varepsilon},$$

defined, respectively, by

$$\mathbb{L}_{\omega,\varepsilon}(u, u_t, \vartheta) = \begin{cases} (u, u_t, \vartheta, 0), & \text{if } \varepsilon > 0, \\ (u, u_t, \vartheta), & \text{if } \varepsilon = 0, \end{cases}$$

and

$$\mathbb{P}_{\omega}(u, u_t, \vartheta, \eta) = (u, u_t, \vartheta) \qquad \text{and} \qquad \mathbb{Q}_{\omega,\varepsilon}(u, u_t, \vartheta, \eta) = \eta.$$

The goal if to show that, for every $R \geq 0$ and $T > \tau > 0$

$$\lim_{\substack{\omega \to 0^+ \\ \varepsilon \to 0^+}} \sup_{z \in B_{\mathcal{H}^2_{\omega,\varepsilon}}(R)} \sup_{t \in [\tau, T]} \| S_{\omega,\varepsilon}(t) z - \mathbb{L}_{\omega,\varepsilon} S_{0,0}(t) \mathbb{P}_\omega z \|_{\mathcal{H}^0_{\omega,\varepsilon}} = 0,$$

and, for $\omega = 0$, for every $R \geq 0$ and $\tau > 0$

$$\lim_{\varepsilon \to 0^+} \sup_{z \in B_{\mathcal{H}^2_{0,\varepsilon}}(R)} \sup_{t \in [\tau, \infty]} \| S_{0,\varepsilon}(t) z - \mathbb{L}_{0,\varepsilon} S_{0,0}(t) \mathbb{P}_0 z \|_{\mathcal{H}^0_{0,\varepsilon}} = 0,$$

If z denotes the initial data, we prove the convergence of $S_{\omega,\varepsilon}(t)z$ towards $\mathbb{L}_{\omega,\varepsilon} S_{0,0}(t) \mathbb{P}_\omega z$ in the $\mathcal{H}^0_{\omega,\varepsilon}$-norm. To be more precise, the first three components of the solution $\mathbb{P}_\omega S_{\omega,\varepsilon}(t)z$ are shown to converge to $S_{0,0}(t)\mathbb{P}_\omega z$ in the $\mathcal{H}^0_{\omega,0}$-norm, whereas the history component η^t vanishes in the $\mathcal{M}^0_\varepsilon$-norm on all time intervals $[\tau, T]$, with $\tau > 0$, due to the presence of a possibly nonvanishing initial history.

More precisely, we prove that

Theorem 7. *For every $R \geq 0$, $T > 0$, and $z \in B_{\mathcal{H}^2_{\omega,\varepsilon}}(R)$, there exist $K_R \geq 0$, independent of T, and $Q_{R,T} \geq 0$ such that*

$$\|S_{\omega,\varepsilon}(t)z - \mathbb{L}_{\omega,\varepsilon}S_{0,0}(t)\mathbb{P}_\omega z\|_{\mathcal{H}^0_{\omega,\varepsilon}} \leq \|\eta_0\|_{\mathcal{M}^0_\varepsilon} e^{-\frac{\delta t}{4\varepsilon}} + Q_{R,T}\sqrt{\omega} + K_R \sqrt[4]{\varepsilon},$$

for every $t \in [0, T]$.

Remark 8. *On one hand the rotational inertia ($\omega > 0$) effects help in restoring the exponential stability but, on the other, the singular limit estimate holds on finite-time intervals which are bounded away from zero.*

Remark 9. *The $\mathcal{H}^2_{\omega,\varepsilon}$-regularity of the initial data is needed in order to exploit the exponential stability of the limiting system as well as to tackle the following term coming from rotational inertia $-\omega\langle u_{tt}(t), A\bar{u}_t(t)\rangle$ due to rotational inertia effects (we set $\bar{u} = \hat{u} - u$, where u refers to the limiting problem and $\hat{u}$ refers to the moving problem).*

A meaningful and straightforward byproduct is the following

Corollary 10. *For every $R \geq 0$ and $z \in B_{\mathcal{H}^2_{0,\varepsilon}}(R)$, there exists $K_R \geq 0$ with*

$$\|S_{0,\varepsilon}(t)z - \mathbb{L}_{0,\varepsilon}S_{0,0}(t)\mathbb{P}_0 z\|_{\mathcal{H}^0_{0,\varepsilon}} \leq \|\eta_0\|_{\mathcal{M}^0_\varepsilon} e^{-\frac{\delta t}{4\varepsilon}} + K_R \sqrt[4]{\varepsilon},$$

for every $t \geq 0$.

Remark 11. *Observe that in this case the non decaying system (for $\varepsilon > 0$) is arbitrarily closed to the exponentially stable limiting problem. The coefficients appearing in the estimate no longer depend on the time interval, so that in turn we obtain a closeness control over the whole $\mathbb{R}^+$.*

■ M. Grasselli, M. Squassina, Preprint, (2005).