Singular limit of differential systems with memory

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Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain. We consider the equation

$$u_t - \Delta u + \phi(u) = f, \qquad t > 0.$$

where ϕ and f are a suitable nonlinearity and a time independent source term, respectively. This equation follows by the classical Fourier's law. In some physical contexts, it is more reasonable to take a convolution average of (all or part of) the term $-\Delta u(t)$, namely to replace the above equation with

$$u_t - \omega\Delta u - (1-\omega)\int_0^\infty k(s)\Delta u(t-s)ds + \phi(u) = f, \qquad t > 0,$$

where $\omega \in [0,1)$ and the memory kernel $k : [0,\infty) \to \mathbb{R}$ is a continuous nonnegative function, smooth on $(0,\infty)$, vanishing at infinity and

$$\int_0^\infty k(s)ds = 1.$$

⋆ B.D. Coleman, M.E. Gurtin, ZAMP **18** (1967) & M.E. Gurtin, A.C. Pipkin, ARMA **31** (1968)**.**

Formally, if we choose $k = \delta_0$ (Dirac mass) the new model collapses into the classic heat equation. For $\varepsilon \in (0,1]$, let us set

$$k_\varepsilon(s) = \frac{1}{\varepsilon}k\Big(\frac{s}{\varepsilon}\Big).$$

Then we consider the family of equations

$$u_t - \omega\Delta u - (1-\omega)\int_0^\infty k_\varepsilon(s)\Delta u(t-s)ds + \phi(u) = f, \qquad t > 0.$$

Since $k_\varepsilon \to \delta_0$, the following natural questions arise:

▶ in what terms the new equation converges to the heat equation?

▶ is the dynamics (absorbing sets, attractors,..) uniform in ε?

▶ do we have some stability for the universal attractor $\mathcal{A}_\varepsilon$ as $\varepsilon \to 0$?

▶ do there exist families of robust exponential attractors $\mathcal{E}_\varepsilon$?

1. TRANSFORMATION INTO DYNAMICAL SYSTEM

We introduce the so-called *integrated past history* of u, i.e.

$$\eta^t(\boldsymbol{x}, s) = \int_0^s u(\boldsymbol{x}, t-y)dy \qquad s > 0,\ t > 0.$$

⋆ C.M. Dafermos, ARMA **37** (1970), 297–308.

Keeping in mind the hypotheses on k, and setting

$$\mu(s) = -(1-\omega)k'(s),$$

a formal integration by part yields

$$(1-\omega)\int_0^\infty k_\varepsilon(s)\Delta u(t-s)ds = \int_0^\infty \mu_\varepsilon(s)\Delta\eta^t(s)ds,$$

where

$$\mu_\varepsilon(s) = \frac{1}{\varepsilon^2}\mu\Big(\frac{s}{\varepsilon}\Big).$$

Hence,

$$u_t - \omega\Delta u - \int_0^\infty \mu_\varepsilon(s)\Delta\eta(s)ds + \phi(u) = f, \qquad t > 0.$$

At this point, a further equation ruling the evolution of η is needed. It is

$$\eta^t_t(s) = -\eta^t_s(s) + u(t), \qquad t > 0.$$

The translation into the above system, endowed with appropriate initial and boundary conditions, will allow us to provide a description of the solutions in terms of a *strongly continuous semigroup* of operators (or dynamical system) $S_\varepsilon(t)$, acting on a proper (extended) phase-space.

For a complete bibliographic database of contributions on the dynamics of the above systems (*for* $\varepsilon > 0$ *fixed!*), see

⋆ M. Grasselli, V. Pata, Uniform attractors of nonautonomous systems with memory, in “Evolution Equations, Semigroups and Functional Analysis” (A. Lorenzi and B. Ruf, Eds.), pp.155–178, Progr. Nonlinear Differential Equations Appl. no.50, Birkhäuser, Boston 2002**,**

and, in particular, the large list of references therein.

2. THE FUNCTIONAL SETTING

The symbols $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ stand for the norm and the inner product on $L^2(\Omega)$, respectively. Let $A = -\Delta$ be the Laplace operator on $L^2(\Omega)$ with domain $\mathcal{D}(A) = H_0^1(\Omega) \cap H^2(\Omega)$. We introduce the hierarchy of spaces

$$H^r = \mathcal{D}(A^{r/2}), \qquad r \in \mathbb{R},$$

endowed with the inner products

$$\langle u_1, u_2\rangle_{H^r} = \langle A^{r/2}u_1, A^{r/2}u_2\rangle.$$

Next, let $\mu \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$, with $\mathbb{R}^+ = (0,\infty)$, satisfy:

$$\begin{aligned} &\mu(s) \geq 0, \qquad \forall s \in \mathbb{R}^+, \\ &\mu'(s) + \delta\mu(s) \leq 0, \qquad \forall s \in \mathbb{R}^+, \text{ for some } \delta > 0. \end{aligned}$$

Notice that μ is decreasing, and

$$\mu(s) \leq \mu(s_0)e^{-\delta(s-s_0)}, \qquad \forall s \geq s_0 > 0.$$

For any given $\varepsilon \in (0,1]$, we define the function

$$\mu_\varepsilon(s) = \frac{1}{\varepsilon^2}\mu\left(\frac{s}{\varepsilon}\right),$$

and we consider the weighted Hilbert spaces

$$\mathcal{M}_\varepsilon^r = L^2_{\mu_\varepsilon}(\mathbb{R}^+, H^{r+1}), \qquad r \in \mathbb{R},$$

endowed with the inner products

$$\langle \eta_1, \eta_2\rangle_{\mathcal{M}_\varepsilon^r} = \int_0^\infty \mu_\varepsilon(s)\langle A^{(1+r)/2}\eta_1(s), A^{(1+r)/2}\eta_2(s)\rangle ds.$$

Finally, for $\varepsilon \in [0,1]$, we define the product Banach spaces

$$\mathcal{H}_\varepsilon^r = \begin{cases} H^r \times \mathcal{M}_\varepsilon^r, & \text{if } \varepsilon > 0, \\ H^r, & \text{if } \varepsilon = 0, \end{cases}$$

normed by $\|(u,\eta)\|^2_{\mathcal{H}_\varepsilon^r} = \|u\|^2_{H^r} + \|\eta\|^2_{\mathcal{M}_\varepsilon^r}$.

3. RIGOROUS PROBLEM SETTING

Let $\phi \in C^1(\mathbb{R})$, with $\phi(0) = 0$, be such that

$$\liminf_{|x|\to\infty} \phi'(x) > -\omega\lambda_1,$$

where λ_1 is the first eigenvalue of A. In particular, ϕ' is bounded below,

$$\inf_{x\in\mathbb{R}} \phi'(x) \geq -\ell, \qquad \forall x \in \mathbb{R},$$

for some $\ell \geq 0$. Moreover,

$$|\phi'(x)| \leq c(1 + |x|^\gamma), \qquad \forall x \in \mathbb{R}, \;\; \gamma \leq 4.$$

Notice that the derivative of the double-well potential, $\phi(x) = x^3 - x$, is an allowed nonlinearity. Finally, we assume

$$f \in H^0 \quad \text{independent of time}.$$

We have now all the ingredients to introduce the following

Problem $\boldsymbol{P_\varepsilon}$. *Given* $(u_0, \eta_0) \in \mathcal{H}^0_\varepsilon$, *find* $(u, \eta) \in C([0, \infty), \mathcal{H}^0_\varepsilon)$ *solution to*

$$\boxed{\begin{cases} u_t + \omega A u + \displaystyle\int_0^\infty \mu_\varepsilon(s) A\eta(s) ds + \phi(u) = f, \\ \eta_t = T_\varepsilon \eta + u, \end{cases}}$$

for $t > 0$, *satisfying the initial conditions* $u(0) = u_0$ *and* $\eta^0 = \eta_0$.

Problem $\boldsymbol{P_0}$. *Given* $u_0 \in \mathcal{H}^0_0$, *find* $u \in C([0, \infty), \mathcal{H}^0_0)$ *solution to*

$$\boxed{u_t + Au + \phi(u) = f},$$

for $t > 0$, *satisfying the initial condition* $u(0) = u_0$.

Both $\boldsymbol{P_\varepsilon}$ and $\boldsymbol{P_0}$ are well posed and generate corresponding semigroups.

4. SEMIGROUPS COMPARISON

Of course, the heat equation

$$u_t - \Delta u + \phi(u) = f, \qquad t > 0,$$

generates a dynamical system

$$S_0(t) : H^0 \to H^0,$$

while

$$\begin{cases} u_t + \omega A u + \displaystyle\int_0^\infty \mu_\varepsilon(s) A\eta(s) ds + \phi(u) = f, \\ \eta_t = T_\varepsilon \eta + u, \end{cases}$$

generates a dynamical system

$$S_\varepsilon(t) : \mathcal{H}^0_\varepsilon \to \mathcal{H}^0_\varepsilon, \qquad \mathcal{H}^0_\varepsilon = H^0 \times \mathcal{M}^0_\varepsilon.$$

In which sense

$$\boxed{S_\varepsilon(t) \quad \overset{?}{\mathbf{Vs.}} \quad S_0(t)} \qquad\qquad \boxed{\mathcal{H}^0_\varepsilon \quad \overset{?}{\mathbf{Vs.}} \quad H^0}.$$

Let us introduce the *lifting map*

$$\mathbb{L}_\varepsilon : \mathcal{H}^0_0 \to \mathcal{H}^0_\varepsilon,$$

and of the *projection map*

$$\mathbb{P} : \mathcal{H}^0_\varepsilon \to \mathcal{H}^0_0,$$

given by

$$\mathbb{L}_\varepsilon u = \begin{cases} (u, 0), & \text{if } \varepsilon > 0, \\ u, & \text{if } \varepsilon = 0, \end{cases} \qquad \mathbb{P}(u, \eta) = u.$$

Then a reasonable (and actually fruitful!) comparison (now between semigroups working on the **same** phase-space) is the following

$$\boxed{S_\varepsilon(t) \quad \mathbf{Vs.} \quad \mathbb{L}_\varepsilon \circ S_0(t) \circ \mathbb{P}}$$

$$\mathcal{H}^0_\varepsilon \ni z = (u_0, \eta^0) \overset{\mathbb{P}}{\mapsto} u_0 \overset{S_0(t)}{\mapsto} S_0(t)u_0 \overset{\mathbb{L}_\varepsilon}{\mapsto} (S_0(t)u_0, 0) \in \mathcal{H}^0_\varepsilon.$$

5. THE CONVERGENCE ESTIMATE

In addition to the *lifting map*

$$\mathbb{L}_\varepsilon : \mathcal{H}_0^0 \to \mathcal{H}_\varepsilon^0,$$

and of the *projection maps*

$$\mathbb{P} : \mathcal{H}_\varepsilon^0 \to \mathcal{H}_0^0$$

consider a second projection

$$\mathbb{Q}_\varepsilon : \mathcal{H}_\varepsilon^0 \to \mathcal{M}_\varepsilon^0,$$

given by

$$\mathbb{Q}_\varepsilon(u,\eta) = \eta.$$

Theorem 1. *For every $R \geq 0$ there exist $K_R \geq 0$ such that, for any $z = (u_0, \eta_0) \in B_{\mathcal{H}_\varepsilon^1}(R)$ and every $t \geq 0$, there hold*

$$\|\mathbb{P}S_\varepsilon(t)z - S_0(t)\mathbb{P}z\|_{H^0} \leq K_R\, h(t)\sqrt[8]{\varepsilon}\,,$$

$$\|\mathbb{P}S_\varepsilon(t)z - S_0(t)\mathbb{P}z\|_{L^2(0,t;H^1)} \leq K_R\, h(t)\sqrt[8]{\varepsilon}\,,$$

$$\|\mathbb{Q}_\varepsilon S_\varepsilon(t)z\|_{\mathcal{M}_\varepsilon^0} \leq \|\eta_0\|_{\mathcal{M}_\varepsilon^0} e^{-\frac{\delta t}{4\varepsilon}} + K_R\sqrt{\varepsilon}\,,$$

where

$$h(t) = (1+t)^{3/4} e^{\ell t}.$$

[Comments about the convergence estimates]

Theorem 2. *If in addition u_0 belongs to a bounded subset of H^2, then the above term $\sqrt[8]{\varepsilon}$ can be replaced by $\sqrt[4]{\varepsilon}$ times a constant depending on the H^2-bound of u_0.*

Collecting the above estimate, we obtain

$$\boxed{\|S_\varepsilon(t)z - \mathbb{L}_\varepsilon S_0(t)\mathbb{P}z\|_{\mathcal{H}_\varepsilon^0} \leq \|\eta_0\|_{\mathcal{M}_\varepsilon^0} e^{-\frac{\delta t}{4\varepsilon}} + K_R\, h(t)\sqrt[8]{\varepsilon}}$$

for every time interval $[\tau, T]$, with $\tau > 0$.

6. ROBUST EXPONENTIAL ATTRACTORS

Let $\mathcal{H}$ be a Banach space. Given $\mathcal{B}_1, \mathcal{B}_2 \subset \mathcal{H}$, we denote by

$$\mathrm{dist}_{\mathcal{H}}\big(\mathcal{B}_1, \mathcal{B}_2\big) = \sup_{z_1 \in \mathcal{B}_1} \inf_{z_2 \in \mathcal{B}_2} \|z_1 - z_2\|_{\mathcal{H}}$$

the *Hausdorff semidistance* in $\mathcal{H}$ from $\mathcal{B}_1$ to $\mathcal{B}_2$, and by

$$\mathrm{dist}^{\mathrm{sym}}_{\mathcal{H}}\big(\mathcal{B}_1, \mathcal{B}_2\big) = \max\big\{\mathrm{dist}_{\mathcal{H}}\big(\mathcal{B}_1, \mathcal{B}_2\big), \mathrm{dist}_{\mathcal{H}}\big(\mathcal{B}_2, \mathcal{B}_1\big)\big\}$$

the *symmetric Hausdorff distance* in $\mathcal{H}$ between $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively.

Definition 1. *A compact set $\mathcal{E} \subset \mathcal{H}$ is called an exponential attractor or inertial set for a semigroup $S(t)$ if the following conditions hold:*

(i) *$\mathcal{E}$ is invariant of $S(t)$, that is, $S(t)\mathcal{E} \subset \mathcal{E}$ for every $t \geq 0$;*

(ii) $\dim_{\mathbb{F}}\mathcal{E} < \infty$, *that is, $\mathcal{E}$ has finite fractal dimension;*

(iii) *there exist an increasing function $J : \mathbb{R}^+ \to \mathbb{R}^+$ and $\kappa > 0$ such that, for any set $\mathcal{B} \subset \mathcal{H}$ with $\sup_{z_0 \in \mathcal{B}} \|z_0\|_0 \leq R$ there holds*

$$\mathrm{dist}_{\mathcal{H}}(S(t)\mathcal{B}, \mathcal{E}) \leq J(R)e^{-\kappa t}.$$

Theorem 3. *Assume that $\gamma < 4$. Then for every $\varepsilon \in [0, 1]$ there exists a set $\mathcal{E}_\varepsilon$, compact in $\mathcal{H}^0_\varepsilon$, which satisfies the following conditions.*

(i) *$\mathcal{E}_\varepsilon$ is positively invariant for $S_\varepsilon(t)$, that is,*

$$S_\varepsilon(t)\mathcal{E}_\varepsilon \subset \mathcal{E}_\varepsilon, \qquad \forall t \geq 0.$$

(ii) *There exist $\kappa > 0$ and a positive increasing function M (both independent of ε) such that, for every bounded set $\mathcal{B} \subset B_{\mathcal{H}^0_\varepsilon}(R)$,*

$$\mathrm{dist}_{\mathcal{H}^0_\varepsilon}(S_\varepsilon(t)\mathcal{B}, \mathcal{E}_\varepsilon) \leq M(R)e^{-\kappa t}, \qquad \forall t \geq 0.$$

(iii) *The fractal dimension of $\mathcal{E}_\varepsilon$ in $\mathcal{H}^0_\varepsilon$ is uniformly bounded in ε.*

(iv) *There exist $\Theta \geq 0$ and $\tau \in (0, \frac{1}{8}]$ such that*

$$\mathrm{dist}^{\mathrm{sym}}_{\mathcal{H}^0_\varepsilon}(\mathcal{E}_\varepsilon, \mathbb{L}_\varepsilon\mathcal{E}_0) \leq \Theta\varepsilon^\tau.$$

The last property (iv) witnesses the robustness of the family $\{\mathcal{E}_\varepsilon\}$ with respect to the singular limit $\varepsilon \to 0$, and it is equivalent to

$$\mathrm{dist}^{\mathrm{sym}}_{H^0}\big(\mathbb{P}\mathcal{E}_\varepsilon, \mathcal{E}_0\big) + \sup_{z \in \mathcal{E}_\varepsilon} \|\mathbb{Q}_\varepsilon z\|_{\mathcal{M}^0_\varepsilon} \leq \Theta \varepsilon^\tau.$$

7. GLOBAL ATTRACTOR AND UPPER SEMICONTINUITY

Theorem 4. *For every $\varepsilon \in [0,1]$, the strongly continuous semigroup $S_\varepsilon(t)$ acting on the phase-space $\mathcal{H}^0_\varepsilon$ possesses a connected global attractor $\mathcal{A}_\varepsilon$ (that is, the unique compact, fully invariant, and attracting set) which is bounded in $\mathcal{H}^1_\varepsilon$, uniformly with respect to ε. Moreover,*

$$\lim_{\varepsilon \to 0} \mathrm{dist}_{\mathcal{H}^0_\varepsilon}\big(\mathcal{A}_\varepsilon, \mathbb{L}_\varepsilon \mathcal{A}_0\big) = 0.$$

Equivalently,

$$\lim_{\varepsilon \to 0} \Big[\mathrm{dist}_{H^0}\big(\mathbb{P}\mathcal{A}_\varepsilon, \mathcal{A}_0\big) + \sup_{z \in \mathcal{A}_\varepsilon} \|\mathbb{Q}_\varepsilon z\|_{\mathcal{M}^0_\varepsilon}\Big] = 0.$$

Since $\mathcal{E}_\varepsilon$ is a compact attracting set, it follows that $\mathcal{A}_\varepsilon \subset \mathcal{E}_\varepsilon$. Hence,

Corollary 1. *If $\gamma < 4$, the global attractor $\mathcal{A}_\varepsilon$ has finite fractal dimension, uniformly with respect to ε.*

8. CONCLUDING REMARKS..