Spike solutions for quasilinear problems

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Let Ω be a possibly unbounded smooth domain of $\mathbb{R}^N$ and consider the problem

$$\begin{cases} -\varepsilon^2 \Delta u + V(x)u = f(u) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where f is superlinear, subcritical with $f(s)/s$ increasing and V is bounded below away from zero. Assume that there exists a compact subset Λ of Ω such that

$$\min_{\Lambda} V < \min_{\partial\Lambda} V.$$

Then there exists $\varepsilon_0 > 0$ such that, for every $\varepsilon \in (0, \varepsilon_0)$, there exists a solution $u_\varepsilon \in H_V(\Omega)$ which admits a unique local (and global) maximum point $x_\varepsilon \in \Lambda$ with

$$\lim_{\varepsilon \to 0} V(x_\varepsilon) = \min_{\Lambda} V.$$

Moreover,

$$u_\varepsilon(x) \leqslant \alpha \exp\left\{-\frac{\beta}{\varepsilon}|x - x_\varepsilon|\right\} \quad \text{for every } x \in \Omega, \text{ for some } \alpha, \beta \in \mathbb{R}.$$

These problems originate from different physical and biological models and, in particular, in the study of *standing waves* for the nonlinear Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + (V(x) - E)\psi - \gamma|\psi|^{p-1}\psi,$$
$$\psi(x,t) = \exp\left\{-\frac{iEt}{\hbar}\right\}u(x).$$

☐ M. DEL PINO, P. FELMER, Local mountain passes for semilinear elliptic problems in unbounded domains, *Calc. Var.* **4** (1996), 121–137.

☐ M. DEL PINO, P. FELMER, Multipeak bound states for nonlinear Schrödinger equations, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **15** (1998), 127–149.

We want to see what type of result can be achieved on the existence of spike solutions for the following singularly perturbed quasilinear elliptic problem

$$\begin{cases} -\varepsilon^2 \sum\limits_{i,j=1}^{N} D_j(a_{ij}(x,u)D_i u) \\ + \frac{\varepsilon^2}{2} \sum\limits_{i,j=1}^{N} D_s a_{ij}(x,u) D_i u D_j u + V(x)u = f(u) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \qquad (*)$$

under suitable assumptions on the functions $a_{ij}(x,u)$, $V(x)$ and $f(u)$.

MAIN DIFFICULTIES W.R.T. THE SEMILINEAR CASE:

► NO uniqueness results for the solutions of the autonomous limiting equation

$$-\sum_{i,j=1}^{N} D_j(b_{ij}(u)D_i u) + \frac{1}{2}\sum_{i,j=1}^{N} b'_{ij}(u) D_i u D_j u + \mu u = f(u) \quad (**)$$

on $\mathbb{R}^N$;

► NO Gidas–Ni–Nirenberg type results for the solutions of $(**)$;

► NO maximization procedure on straight lines, i.e. the fact

$u \in H^1(\mathbb{R}^N)$, $u \geqslant 0$ and u solution of $(**)$ implies that $J(u) = \max_{t \geqslant 0} J(tu)$

is false even if the map $\left\{ s \mapsto \frac{f(s)}{s} \right\}$ is nondecreasing;

► MORE delicate proof that the solution u_ε vanish uniformly on $\partial \Lambda$ as $\varepsilon \to 0$;

► NO suitable results on the exponential decay of the solutions of $(*)$;

► NO smoothness of the functional associated with $(*)$ (mere continuity);

► NO representation results for Palais–Smale sequences.

ASSUMPTIONS AND MAIN RESULT:

Let $f \in C^1(\mathbb{R}^+)$, $1 < p < \frac{N+2}{N-2}$ and $2 < \vartheta \leqslant p+1$ be such that

$$\lim_{s \to +\infty} \frac{f(s)}{s^p} = 0, \qquad \lim_{s \to 0^+} \frac{f(s)}{s} = 0,$$

$$0 < \vartheta F(s) \leqslant f(s)s \quad \text{for every } s \in \mathbb{R}^+,$$

where $F(s) = \int_0^s f(t)\,dt$ for every $s \in \mathbb{R}^+$.
$V : \mathbb{R}^N \to \mathbb{R}$ is a locally Hölder continuous function and there exists $\alpha > 0$ with

$$V(x) \geqslant \alpha \quad \text{for every } x \in \mathbb{R}^N.$$

$a_{ij}(x,s) : \Omega \times \mathbb{R}^+ \to \mathbb{R}$ are measurable in x and of class C^1 with respect to s, $a_{ij}(x,s) = a_{ji}(x,s)$ for every $i, j = 1, \dots, N$ and there exists a positive constant C with

$$|a_{ij}(x,s)| \leqslant C, \quad |D_s a_{ij}(x,s)| \leqslant C$$

for a.e. $x \in \Omega$ and every $s \in \mathbb{R}^+$.

Finally, let $R, \nu > 0$ and $0 < \gamma < \vartheta - 2$ be such that

$$\sum_{i,j=1}^{N} a_{ij}(x,s)\xi_i\xi_j \geqslant \nu|\xi|^2,$$

$$\sum_{i,j=1}^{N} sD_s a_{ij}(x,s)\xi_i\xi_j \leqslant \gamma \sum_{i,j=1}^{N} a_{ij}(x,s)\xi_i\xi_j$$

a.e. in Ω, for every $s \in \mathbb{R}^+$ and $\xi \in \mathbb{R}^N$, and

$$s \geqslant R \implies \sum_{i,j=1}^{N} D_s a_{ij}(x,s)\xi_i\xi_j \geqslant 0$$

a.e. in Ω, for every $s \in \mathbb{R}^+$ and $\xi \in \mathbb{R}^N$.

Notice that we require:

▷ NO monotonicity assumptions on $\left\{ s \mapsto \frac{f(s)}{s} \right\}$;

▷ NO uniqueness assumption on the "limiting" equations on $\mathbb{R}^N$;

▷ NO restrictions on the behaviour of V at infinity;

▷ Standard conditions on the nonlinearity f and on the potential V;

▷ Standard conditions on the coefficients $a_{ij}(x, s)$.

Let $H_V(\Omega)$ be the weighted Hilbert space defined by

$$H_V(\Omega) = \left\{ u \in H_0^1(\Omega) : \int_\Omega V(x) u^2 < +\infty \right\},$$

and denote by $\|\cdot\|_{H_V(\Omega)}$ the corresponding norm.

Under the previous assumptions, we have the following result.

Theorem 1. *Let Λ be a compact subset of Ω such that there exists $x_0 \in \Lambda$ with*

$$V(x_0) = \min_\Lambda V < \min_{\partial\Lambda} V,$$

$$\sum_{i,j=1}^N a_{ij}(x_0, s)\xi_i\xi_j = \min_{x\in\Lambda} \sum_{i,j=1}^N a_{ij}(x, s)\xi_i\xi_j$$

for $s \in \mathbb{R}^+$ and $\xi \in \mathbb{R}^N$.

Then there exists $\varepsilon_0 > 0$ such that, for every $\varepsilon \in (0, \varepsilon_0)$, there exist $u_\varepsilon \in H_V(\Omega) \cap C(\overline{\Omega})$ and $x_\varepsilon \in \Lambda$ satisfying the following properties:

(a) u_ε *is a weak solution of the problem*

$$\begin{cases} -\varepsilon^2 \sum\limits_{i,j=1}^{N} D_j(a_{ij}(x,u)D_i u) \\ +\frac{\varepsilon^2}{2} \sum\limits_{i,j=1}^{N} D_s a_{ij}(x,u) D_i u D_j u + V(x)u = f(u) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega\,; \end{cases} \qquad (P_\varepsilon)$$

(b) $u_\varepsilon(x_\varepsilon) = \sup\limits_{\Omega} u_\varepsilon$*;*

(c) *if we set*

$$\sigma = \sup\left\{s > 0 : \ f(t) \leqslant tV(x_0) \text{ for every } t \in [0,s]\right\},$$

we have $u_\varepsilon(x_\varepsilon) > \sigma > 0$*;*

(d) *for every neighbourhood* $\mathscr{N}$ *of the set*

$$\mathscr{M} = \left\{x \in \Lambda : \; V(x) = V(x_0)\right\},$$

we have

$$\lim_{\varepsilon \to 0} u_\varepsilon = 0 \quad \textit{uniformly on } \Omega \setminus \mathscr{N} \; ;$$

(e) $\lim_{\varepsilon \to 0} d(x_\varepsilon, \mathscr{M}) = 0$ *and*

$$\lim_{\varepsilon \to 0} \|u_\varepsilon\|_{H_V(\Omega)} = 0, \quad \lim_{\varepsilon \to 0} \|u_\varepsilon\|_{L^q(\Omega)} = 0 \;\; \textit{for every } 2 \leqslant q < +\infty.$$

M. SQUASSINA, Spike solutions for a class of singularly perturbed quasilinear elliptic equations, *preprint*, 2002.

MAIN STEPS OF THE PROOF:

It is in the spirit of the del Pino–Felmer penalization scheme.

Step I. The “original” functional $I_\varepsilon : H_V(\Omega) \to \mathbb{R}$ gets “penalized” outside Λ, i.e. we construct a new functional J_ε

$$J_\varepsilon(u) = \frac{\varepsilon^2}{2} \sum_{i,j=1}^{N} \int_\Omega a_{ij}(x,u) D_i u D_j u + \frac{1}{2} \int_\Omega V(x) u^2 - \int_\Omega G(x,u)$$

being G a suitable modification of F such that critical points of J_ε which are uniformly small outside Λ are solutions of the original problem.

Step II. J_ε satisfies the (concrete) Palais–Smale condition at every level (I_ε does not, in general) and the MP theorem can be directly applied to get a critical point u_ε of J_ε (with precise energy estimates).

Step III. We prove that $\|u_\varepsilon\|_{\infty,\partial\Lambda} \to 0$ as $\varepsilon \to 0$ (hardest step, here we repeatedly use the Pucci–Serrin identity to estimate the critical values). For instance, the following nice proposition is used.

Proposition 2. *Let $h : \mathbb{R} \to \mathbb{R}$ be a continuous and odd function such that*

$$\lim_{s\to 0} \frac{h(s)}{s} = -\nu < 0, \quad \lim_{s\to\infty} \frac{|h(s)|}{s^{\frac{N+2}{N-2}}} = 0, \quad H(s_0) > 0 \text{ for some } s_0 > 0,$$

where $H(s) = \int_0^s h(t)\,dt$. Let $b_{ij} \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ with $b'_{ij} \in L^\infty(\mathbb{R})$ and assume that there exists $\nu > 0$ and $R > 0$ with

$$\sum_{i,j=1}^{N} b_{ij}(s)\xi_i\xi_j \geqslant \nu|\xi|^2, \quad |s| \geqslant R \Rightarrow \sum_{i,j=1}^{N} b'_{ij}(s)\xi_i\xi_j \geqslant 0$$

for every $s \in \mathbb{R}$ and $\xi \in \mathbb{R}^N$.

Let $w \in H^1(\mathbb{R}^N)$ be any nontrivial solution of the equation

$$-\sum_{i,j=1}^{N} D_j(b_{ij}(u)D_iu) + \frac{1}{2}\sum_{i,j=1}^{N} b'_{ij}(u)D_iuD_ju = h(u) \quad \text{in } \mathbb{R}^N.$$

We denote by J *the associated functional*

$$J(u) = \frac{1}{2} \sum_{i,j=1}^{N} \int_{\mathbb{R}^N} b_{ij}(u) D_i u D_j u - \int_{\mathbb{R}^N} H(u).$$

Then it results

$$J(w) \geqslant \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} J(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0,1], H^1(\mathbb{R}^N)) : \ \gamma(0) = 0, \ J(\gamma(1)) < 0\}$.

<u>SKETCH OF THE PROOF</u>. It is readily seen that J has a MP geometry and the MP value is well defined. Let us consider the dilation path

$$\gamma(t)(x) = \begin{cases} w\left(\frac{x}{t}\right) & \text{if } t > 0 \\ 0 & \text{if } t = 0. \end{cases}$$

Notice that $\|\gamma(t)\|_{H^1}^2 = t^{N-2}\|Dw\|_2^2 + t^N\|w\|_2^2$ for every $t \in \mathbb{R}^+$, thus the curve γ belongs to $C([0,+\infty[, H^1(\mathbb{R}^N))$. For every $t \in \mathbb{R}^+$ it results that

$$J(\gamma(t)) = \frac{1}{2}\sum_{i,j=1}^{N}\int_{\mathbb{R}^N} b_{ij}(\gamma(t))D_i\gamma(t)D_j\gamma(t) - \int_{\mathbb{R}^N} H(\gamma(t))$$

$$= \frac{t^{N-2}}{2}\sum_{i,j=1}^{N}\int_{\mathbb{R}^N} b_{ij}(w)D_iwD_jw - t^N\int_{\mathbb{R}^N} H(w)$$

which yields, for every $t \in \mathbb{R}^+$

$$\frac{d}{dt}J(\gamma(t)) = \frac{N-2}{2}t^{N-3}\sum_{i,j=1}^{N}\int_{\mathbb{R}^N} b_{ij}(w)D_iwD_jw - Nt^{N-1}\int_{\mathbb{R}^N} H(w).$$

Since w is locally C^1 (by the sign condition on b'_{ij}), we can use the Pucci–Serrin identity (in the following form, it suffices w locally Lipschitz!!!)

$$\forall h \in C_c^1(\mathbb{R}^N, \mathbb{R}^N): \quad \sum_{i,j=1}^N \int_{\mathbb{R}^N} D_i h^j D_{\xi_i} \mathscr{L}(x, w, Dw) D_j w$$
$$- \int_{\mathbb{R}^N} \left[(\operatorname{div} h)\, \mathscr{L}(x, w, Dw) + h \cdot D_x \mathscr{L}(x, w, Dw) \right]$$
$$= \int_{\mathbb{R}^N} (h \cdot Dw) \varphi(x, w)$$

where we choose $\varphi = 0$ and

$$\mathscr{L}(s, \xi) = \frac{1}{2} \sum_{i,j=1}^N b_{ij}(s) \xi_i \xi_j - H(s) \quad \text{for every } s \in \mathbb{R} \text{ and } \xi \in \mathbb{R}^N,$$
$$h(x) = h_k(x) = T\left(\frac{x}{k}\right) x \quad \text{for every } x \in \mathbb{R}^N \text{ and } k \geqslant 1,$$

being $T \in C_c^1(\mathbb{R}^N)$ such that $T(x) = 1$ if $|x| \leqslant 1$ and $T(x) = 0$ if $|x| \geqslant 2$. In particular it results that $h_k \in C_c^1(\mathbb{R}^N, \mathbb{R}^N)$ for every $k \geqslant 1$ and

$$D_i h_k^j(x) = D_i T\left(\frac{x}{k}\right)\frac{x_j}{k} + T\left(\frac{x}{k}\right)\delta_{ij} \quad \text{for every } i, j = 1, \dots, N.$$

Then, since $D_x \mathscr{L}(w, Dw) = 0$, it follows that

$$\forall k \geqslant 1: \quad \sum_{i,j=1}^{n} \int_{\mathbb{R}^N} D_i T\left(\frac{x}{k}\right)\frac{x_j}{k} D_j w D_{\xi_i}\mathscr{L}(w, Dw) + \int_{\mathbb{R}^N} T\left(\frac{x}{k}\right) D_\xi \mathscr{L}(w, Dw)\cdot Dw - \int_{\mathbb{R}^N} DT\left(\frac{x}{k}\right)\cdot\frac{x}{k}\,\mathscr{L}(w, Dw) - \int_{\mathbb{R}^N} T\left(\frac{x}{k}\right) N\mathscr{L}(w, Dw) = 0.$$

Since there exists $C > 0$ with $D_i T\left(\frac{x}{k}\right)\frac{x_j}{k} \leqslant C$ for $k \geqslant 1$ and $i, j = 1, \dots, N$, by the Dominated Convergence Theorem, letting $k \to +\infty$, we obtain

$$\int_{\mathbb{R}^N} \Big[N\mathscr{L}(w, Dw) - D_\xi \mathscr{L}(w, Dw)\cdot Dw\Big] = 0,$$

namely,

$$\frac{N-2}{2}\sum_{i,j=1}^{N}\int_{\mathbb{R}^N} b_{ij}(w)D_iwD_jw = N\int_{\mathbb{R}^N} H(w).$$

By plugging this equation into the formula of $\frac{d}{dt}J(\gamma(t))$, we get

$$\frac{d}{dt}J(\gamma(t)) = N(1-t^2)t^{N-3}\int_{\mathbb{R}^N} H(w)$$

which yields $\frac{d}{dt}J(\gamma(t)) > 0$ for $t < 1$ and $\frac{d}{dt}J(\gamma(t)) < 0$ for $t > 1$, i.e.

$$\sup_{t\in[0,1]} J(\gamma(t)) = J(\gamma(1)) = J(w).$$

Then, after a suitable scale change in t, the assertion follows. ■

Step IV. for ε small, u_ε solves the problem with the required properties (w.r.t. Λ).

Step V. via a boot argument also assertion (d) of the theorem (which involves arbitrariness in the choice of the neighbourhood of the set of minima of V) is proved.

SOME RELATED OPEN PROBLEMS:

Problem 3. *Under suitable assumptions on* b_{ij} *and* f*, does a* ***GNN*** *type result (radial symmetry) hold for the solutions of the equation in* $\mathbb{R}^N$

$$-\sum_{i,j=1}^{N} D_j(b_{ij}(u)D_i u) + \frac{1}{2}\sum_{i,j=1}^{N} b'_{ij}(u)D_i u D_j u + \mu u = f(u) \quad (***)$$

Problem 4. *Under suitable assumptions on* b_{ij} *and* f*, is it possible to prove, as in the semilinear case, a* ***uniqueness*** *result of the positive smooth solutions of* $(***)$*?*

Problem 5. *Under suitable assumptions on* b_{ij} *and* f*, is it possible to prove, as in the semilinear case, that there exists a* ***least energy*** *solution of* $(***)$*? i.e. is there a positive solution* $\omega \in H^1(\mathbb{R}^N)$ *such that*

$$\widetilde{J}(\omega) = \inf\left\{\widetilde{J}(u) : \; u \in H^1(\mathbb{R}^N) \setminus \{0\} \text{ is a solution of } (***)\right\},$$

being $\widetilde{J}$ *the functional associated with* $(***)$*? We believe so, and in particular that* *this solution correspond exactly to the MP solution.*

Problem 6. *Is it true or false that the sequence $(u_\varepsilon)_{\varepsilon>0}$ of solutions of problem (P_ε) is **bounded** in $L^\infty(\Omega)$?*

Problem 7. *Is it true or false that for each $\varepsilon > 0$ the solution u_ε of the problem (P_ε) admit a **unique** local maximum point inside Λ?*

Problem 8. *Is it true or false that the solutions u_ε of problem (P_ε) **decay exponentially** like in the semilinear case?*

Problem 9. *Does our result extend to the **multi–peak** case? i.e. if there are k compact disjoint subsets $\Lambda_1, \dots, \Lambda_k$ of Ω with*

$$\min_{\Lambda_j} V < \min_{\partial \Lambda_j} V \quad \text{for every } j = 1, \dots, k,$$

then, for ε small, can one find k solutions $u_\varepsilon^{(1)}, \dots, u_\varepsilon^{(k)} \in H_V(\Omega)$ having maxima which concentrate around the minima of $V|_{\Lambda_1}, \dots, V|_{\Lambda_k}$ respectively?

Problem 10. *Let $V \equiv 1$ and assume that $\{x \mapsto d(x, \partial\Omega)\}$ admits a global maximum at x_0. Can one exhibit solutions u_ε of (P_ε) concentrating around x_0?*

Definition 11. *Let $\varepsilon > 0$ and $c \in \mathbb{R}$. We say that $(u_h) \subset H_V(\Omega)$ is a concrete Palais–Smale sequence at level c ($(CPS)_c-$sequence, in short) for the functional J_ε, if $J_\varepsilon(u_h) \to c$ and*

$$\sum_{i,j=1}^{N} D_s a_{ij}(x,u_h) D_i u_h D_j u_h \in (H_V(\Omega))' \quad \text{as } h \to +\infty,$$

$$-\varepsilon^2 \sum_{i,j=1}^{N} D_j(a_{ij}(x,u_h) D_i u_h)$$

$$+\frac{\varepsilon^2}{2} \sum_{i,j=1}^{N} D_s a_{ij}(x,u_h) D_i u_h D_j u_h + V(x) u_h - g(x,u_h) \to 0$$

strongly in $(H_V(\Omega))'$. We say that J_ε satisfies the concrete Palais–Smale condition at level c ($(CPS)_c$ condition), if every $(CPS)_c-$sequence for J_ε admits a strongly convergent subsequence in $H_V(\Omega)$.

See e.g. the papers:

☐ A. Canino, Multiplicity of solutions for quasilinear elliptic equations,
Topol. Methods Nonlinear Anal. **6** (1995), 357–370.

☐ J.N. Corvellec, M. Degiovanni, M. Marzocchi, Deformation properties for continuous functionals and critical point theory,
Topol. Methods. Nonlinear Anal. **1** (1993), 151–171.