Two solutions for inhomogeneous fully nonlinear elliptic equations at critical growth

Marco Squassina, University of Milan

DCDIS CONFERENCE, LONDON, CANADA, JULY 2001

Overview of the problem

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $1 < p < n$ and $p < q < p^* = \frac{np}{n-p}$. We are concerned with the existence of two nontrivial solutions in $W_0^{1,p}(\Omega)$ of the following problem $(\mathscr{P}_{\varepsilon,\lambda})$

$$-\operatorname{div}\,(\nabla_\xi \mathscr{L}(x,u,\nabla u)) + D_s\mathscr{L}(x,u,\nabla u) = |u|^{p^*-2}u + \lambda|u|^{q-2}u + \varepsilon h\,,$$

with $h \in L^{p'}(\Omega)$, $h \not\equiv 0$, provided that $\varepsilon > 0$ is small and $\lambda > 0$ is large.

Motivations for investigating problems as $(\mathscr{P}_{\varepsilon,\lambda})$ come from various situations in geometry and physics which present lack of compactness. A typical example is Yamabe's problem, i.e. find $u > 0$ such that

$$-4\frac{n-1}{n-2}\Delta_M u = R' u^{(n+2)/(n-2)} - R(x)u \quad \text{on } M,$$

for some constant R', where M is an n-dimensional Riemannian manifold, $R(x)$ its scalar curvature and $-\Delta_M$ is the Laplace–Beltrami operator on M. Since p^* is the critical Sobolev exponent for which the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ fails to be compact, as known, one encounters serious difficulties in applying variational methods to $(\mathscr{P}_{\varepsilon,\lambda})$.

As known, in general, if $h \equiv 0$ and $\lambda = 0$, to obtain a solution of

$$\begin{cases} -\Delta_p u = |u|^{p^*-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

one has to consider in detail the geometry of Ω or has to replace the critical term u^{p^*-1} with $u^{p^*-1-\varepsilon}$ and then investigate the limits of u_ε as $\varepsilon \to 0$.

Let us now assume that $h \equiv 0$ and $\lambda \neq 0$. As we showed by the general Pohozǎev identity of Pucci and Serrin, if

$$p^* \nabla_x \mathscr{L}(x,s,\xi) \cdot x - n D_s \mathscr{L}(x,s,\xi) s \geqslant 0,$$

a.e. in Ω and for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^n$, then $(\mathscr{P}_{\varepsilon,\lambda})$ admits no nontrivial smooth solution for each $\lambda \leqslant 0$ when the domain Ω is star–shaped and $\mathscr{L}$ is sufficiently smooth. Therefore, in this case we are reduced to consider positive λ.

Let us briefly recall the historical background of existence results for problems at critical growth with lower–order perturbations. In 1983, in the pioneering paper

H. BRÉZIS, L. NIRENBERG, Positive solutions of nonlinear elliptic equations involving critical Sobolev exponent, CPAM **36** (1983), 437–477,

Brézis and Nirenberg proved that the problem:

$$\begin{cases} -\Delta u = u^{(n+2)/(n-2)} + \lambda u & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has at least one solution $u \in H_0^1(\Omega)$ provided that:

- $\lambda \in (0, \lambda_1) \qquad \text{if } n \geqslant 4\,,$
- $\lambda \in (\frac{\lambda_1}{4}, \lambda_1) \qquad \text{if } n = 3 \text{ and } \Omega = B(0,R),$

where λ_1 is the first eigenvalue of $-\Delta$ in Ω. The extension to the $p-$Laplacian was achieved by Garcia Azorero and Peral Alonso in:

J. GARCIA AZORERO, I. PERAL ALONSO, Existence and non–uniqueness for the $p-$Laplacian, Comm. PDE **12** (1987), 1389–1430.

In this paper, the existence of a nontrivial solution of:

$$\begin{cases} -\Delta_p u = |u|^{p^*-2}u + \lambda|u|^{q-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

is proven, provided that:

- $\lambda \in (0, \lambda_1)$ if $1 < p = q < p^*$ and $p^2 \leqslant n\,;$
- $\lambda \in (\lambda_0, +\infty)$ if $1 < p < q < p^*$ and $p^2 > n\,;$
- $\lambda \in (0, +\infty)$ if $1 < p < q < p^*$ and $p^2 \leqslant n\,;$
- $\lambda \in (0, +\infty)$ if $\max\left\{p, p^* - \frac{p}{p-1}\right\} < q < p^*\,,$

where λ_1 is the first eigenvalue of $-\Delta_p$ and λ_0 is a suitable positive real number.

Let us now assume $h \not\equiv 0$. Then, a natural question is whether inhomogeneous problems like $(\mathscr{P}_{\varepsilon,\lambda})$ have more than one solution. For bounded domains one of the first answers was given in 1992 in:

G. TARANTELLO, On nonhomogeneous elliptic equations involving critical Sobolev exponent, AIHP–ANL **9** (1992), 281–304,

where it is shown that the problem:

$$\begin{cases} -\Delta u = |u|^{2^*-2}u + h(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

admits two distinct solutions $u_1, u_2 \in H_0^1(\Omega)$ if $\|h\|_2$ is small. The existence of two nontrivial solutions for the $p-$Laplacian problem:

$$\begin{cases} -\Delta_p u = |u|^{p^*-2}u + \lambda|u|^{q-2}u + h(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

for $1 < p < q < p^*$, λ large and $\|h\|_{p'}$ small enough, has been proven in 1995 in:

J. CHABROWSKI, On multiple solutions for the nonhomogeneous $p-$Laplacian with a critical Sobolev exponent, DIE **8** (1995), 705–716.

This achievement has been recently extended by Zhou in:

H.S. ZHOU, Solutions for a quasilinear elliptic equation with critical sobolev exponent and perturbations on $\mathbb{R}^n$, DIE **13** (2000), 595–612,

to the equation:

$$-\Delta_p u + c|u|^{p-2}u = |u|^{p^*-2}u + f(x,u) + h(x)$$

on the entire $\mathbb{R}^n$, where $f(x,u)$ is a lower–order perturbation of $|u|^{p^*-2}u$. This case involves a double loss of compactness, one due to the unboundedness of the domain and the other due to the critical Sobolev exponent. Now, more recently, some results for the more general problem

$$\begin{cases} -\operatorname{div}\,(\nabla_\xi \mathscr{L}(x,u,\nabla u)) + D_s\mathscr{L}(x,u,\nabla u) = g(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

with g subcritical and superlinear have been considered in

D. ARCOYA, L. BOCCARDO, Critical points for multiple integrals of the calculus of variations, ARMA **134** (1996), 249–274,

M. S., Existence of weak solutions to general Euler's equations via nonsmooth critical point theory, AFS Toulouse (6) **9** (2000), 113–131.

M. S., On the existence of positive entire solutions of nonlinear elliptic equations, TMNA **17** (2001), 23–39.

It is therefore natural to see what happens when g has a critical growth.

A first answer was given in:

G. ARIOLI, F. GAZZOLA, Quasilinear elliptic equations at critical growth, NoDEA **5** (1998), 83–97,

where it is proven the existence of a nontrivial solution $u \in H_0^1(\Omega)$ for a class of quasilinear equations of the type:

$$-\sum_{i,j=1}^{n} D_j(a_{ij}(x,u)D_i u) + \frac{1}{2}\sum_{i,j=1}^{n} D_s a_{ij}(x,u) D_i u D_j u = |u|^{2^*-2}u + \lambda u, \tag{1}$$

where the coefficients $(a_{ij}(x,s))$ satisfy some suitable assumptions, including a semilinear asymptotic behaviour as $s \to +\infty$.

Now, in view of the above mentioned results for $-\Delta$, $-\Delta_p$, we expect that problems $(\mathscr{P}_{\varepsilon,\lambda})$ admits at least two nontrivial solutions for ε small and λ large. In order to prove this, we shall argue on the functional $f_{\varepsilon,\lambda} : W_0^{1,p}(\Omega) \to \mathbb{R}$ given by

$$\begin{aligned} f_{\varepsilon,\lambda}(u) = \int_\Omega \mathscr{L}(x,u,\nabla u)\,dx + \qquad\qquad (2) \\ -\frac{1}{p^*}\int_\Omega |u|^{p^*}\,dx - \frac{\lambda}{q}\int_\Omega |u|^q\,dx - \varepsilon\int_\Omega hu\,dx, \end{aligned}$$

where $W_0^{1,p}(\Omega)$ will be endowed with the norm $\|u\|_{1,p} = \left(\int_\Omega |\nabla u|^p\,dx\right)^{1/p}$.

The first solution is obtained via a local minimization argument while the second solution will follow by the mountain pass theorem without Palais–Smale condition in its non–smooth version.

In general, under reasonable assumptions on $\mathscr{L}$, $f_{\varepsilon,\lambda}$ is continuous but not even locally Lipschitzian unless $\mathscr{L}$ does not depend on u or is subjected to some very restrictive growth conditions. Then, we shall refer to the non–smooth critical point theory developed in:

A. CANINO, M. DEGIOVANNI, Nonsmooth critical point theory and quasilinear elliptic equations, *Topological Methods in Differential Equations and Inclusions*, 1–50 – A. Granas, M. Frigon, G. Sabidussi Eds. – Montreal (1994).

J.N. CORVELLEC, M. DEGIOVANNI, M. MARZOCCHI, Deformation properties for continuous functionals and critical point theory, TMNA **1** (1993), 151–171,

M. DEGIOVANNI, M. MARZOCCHI, A critical point theory for nonsmooth functionals, AMPA (4) **167** (1994), 73–100.

We assume that $\mathscr{L}(x,s,\xi) : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is measurable in x for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^n$, of class C^1 in s and of class C^2 in ξ and that $\mathscr{L}(x,s,\cdot)$ is strictly convex and $p-$homogeneous with $\mathscr{L}(x,s,0)=0$. Moreover:

$(\mathscr{H}_1)$ there exists $\nu > 0$ such that:

$$\mathscr{L}(x,s,\xi) \geqslant \frac{\nu}{p}|\xi|^p$$

a.e. in Ω and for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^n$;

$(\mathscr{H}_2)$ there exists $c_1, c_2 \in \mathbb{R}$ such that:

$$|D_s \mathscr{L}(x,s,\xi)| \leqslant c_1 |\xi|^p$$

a.e. in Ω and for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^n$ and

$$\left|\nabla^2_{\xi\xi} \mathscr{L}(x,s,\xi)\right| \leqslant c_2 |\xi|^{p-2} \tag{3}$$

a.e. in Ω and for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^n$;

$(\mathscr{H}_3)$ there exist $R > 0$ and $\gamma \in]0, q-p[$ such that:

$$|s| \geqslant R \Longrightarrow D_s \mathscr{L}(x,s,\xi)s \geqslant 0 \tag{4}$$

a.e. in Ω and for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^n$ and

$$D_s \mathscr{L}(x,s,\xi)s \leqslant \gamma \mathscr{L}(x,s,\xi) \tag{5}$$

a.e. in Ω and for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^n$.

Under the previous assumptions, the following is our main result:

Theorem 0.1. *There exists* $\lambda_0 > 0$ *such that for all* $\lambda > \lambda_0$ *there exists* $\varepsilon_0 > 0$ *such that* $(\mathscr{P}_{\varepsilon,\lambda})$ *has at least two nontrivial solutions in* $u_1, u_2 \in W_0^{1,p}(\Omega)$ *for each* $0 < \varepsilon < \varepsilon_0$ *.*

Remark 0.2. *We stress that no asymptotic behaviour is assumed on* $\mathscr{L}(x, s, \xi)$ *and* $D_s\mathscr{L}(x, s, \xi)s$ *as* s *goes to* $+\infty$*, while in A–G's paper, to prove that problem (1) has a solution, it was assumed that*

$$\lim_{s\to+\infty} a_{ij}(x, s) = \delta_{ij}, \quad \lim_{s\to+\infty} sD_s a_{ij}(x, s) = 0, \quad (i, j = 1, .., n)$$

uniformly with respect to $x \in \Omega$*, namely problem (1) converges "in some sense" to the semilinear equation* $-\Delta u = |u|^{2^*-2}u + \lambda u$ *.*

Remark 0.3. *We point out that we assumed (4) just for* $|s| \geqslant R$*, while in A–G's paper, for problem (1), it was assumed that:*

$$\forall s \in \mathbb{R} : \quad \sum_{i,j=1}^{n} sD_s a_{ij}(x, s)\xi_i\xi_j \geqslant 0.$$

1 Recalls of non–smooth critical point theory

Definition 1.1. *Let* $(\mathscr{X}, d)$ *be a metric space,* $f : \mathscr{X} \to \mathbb{R}$ *a continuous function and* $u \in \mathscr{X}$*. We denote by* $|df|(u)$ *the supremum of* $\sigma \in [0, +\infty[$ *such that there exist* $\delta > 0$ *and a continuous map*

$$\mathscr{H} : B_\delta(u) \times [0, \delta] \to \mathscr{X}$$

such that for all $(v, t) \in B_\delta(u) \times [0, \delta]$

$$d(\mathscr{H}(v, t), v) \leqslant t, \quad f(\mathscr{H}(v, t)) \leqslant f(v) - \sigma t.$$

We say that the extended real number $|df|(u)$ *is the weak slope of* f *at* u*.*

Definition 1.2. *Let* $(\mathscr{X}, d)$ *be a metric space,* $f : \mathscr{X} \to \mathbb{R}$ *a continuous function and* $u \in \mathscr{X}$*. We say that* u *is a critical point of* f *if* $|df|(u) = 0$*.*

Definition 1.3. *Let* $(\mathscr{X}, d)$ *be a metric space,* $f : \mathscr{X} \to \mathbb{R}$ *a continuous function and* $c \in \mathbb{R}$*. We say that* f *satisfies the Palais–Smale condition at level* c *if every*

$(u_h) \subset \mathscr{X}$ with $f(u_h) \to c$ and $|df|(u_h) \to 0$ admits a convergent subsequence.

Definition 1.4. *We say that u is a weak solution to $(\mathscr{P}_{\varepsilon,\lambda})$, if $u \in W_0^{1,p}(\Omega)$ and*

$$-\mathrm{div}\ (\nabla_\xi \mathscr{L}(x,u,\nabla u)) + D_s\mathscr{L}(x,u,\nabla u) = |u|^{p^*-2}u + \lambda|u|^{q-2}u + \varepsilon h(x)$$

in $\mathscr{D}'(\Omega)$.

By the growth conditions on $\mathscr{L}$ this definition is well posed.

Definition 1.5. *We say that $(u_h) \subset W_0^{1,p}(\Omega)$ is a concrete Palais Smale sequence at level $c \in \mathbb{R}$ ($(CPS)_c-$sequence, in short) for $f_{\varepsilon,\lambda}$, if $f_{\varepsilon,\lambda}(u_h) \to c$,*

$$-\mathrm{div}\ (\nabla_\xi \mathscr{L}(x,u_h,\nabla u_h)) + D_s\mathscr{L}(x,u_h,\nabla u_h) \in W^{-1,p'}(\Omega)\,,$$

eventually as $h \to +\infty$ and

$$\begin{aligned} &-\mathrm{div}\ (\nabla_\xi \mathscr{L}(x,u_h,\nabla u_h)) + D_s\mathscr{L}(x,u_h,\nabla u_h) + \\ &-|u_h|^{p^*-2}u_h - \lambda|u_h|^{q-2}u_h - \varepsilon h(x) \to 0 \end{aligned}$$

strongly in $W^{-1,p'}(\Omega)$. We say that $f_{\varepsilon,\lambda}$ satisfies the concrete Palais–Smale condition at level c ($(CPS)_c$ in short), if every $(CPS)_c-$sequence for $f_{\varepsilon,\lambda}$ admits a strongly convergent subsequence.

Theorem 1.6. *Assume that $u \in W_0^{1,p}(\Omega)$ is such that $|df_{\varepsilon,\lambda}|(u) < +\infty$. Then*

$$w_u = -\mathrm{div}\,(\nabla_\xi \mathscr{L}(x,u,\nabla u)) + D_s\mathscr{L}(x,u,\nabla u) + \\ - |u|^{p^*-2}u - \lambda|u|^{q-2}u - \varepsilon h(x) \in W^{-1,p'}(\Omega)$$

and $\|w_u\|_{-1,p'} \leqslant |df_{\varepsilon,\lambda}|(u)$. In particular, if u is a critical point of $f_{\varepsilon,\lambda}$ then u is a weak solution to $(\mathscr{P}_{\varepsilon,\lambda})$.

Theorem 1.7. *There exists $\varrho > 0$ such that $f_{\varepsilon,\lambda}$ is weakly lower semicontinuous on $\left\{u \in W_0^{1,p}(\Omega) :\ \|u\|_{1,p} \leqslant \varrho\right\}$, for each $\lambda \in \mathbb{R}$ and $\varepsilon > 0$.*

Lemma 1.8. *For each $\lambda \in \mathbb{R}$ there exist $\varepsilon > 0$ and $\varrho, \eta > 0$ such that:*

$$\forall u \in W_0^{1,p}(\Omega) :\ \|u\|_{1,p} = \varrho \Longrightarrow f_{\varepsilon,\lambda}(u) > \eta .$$

Proposition 1.9. *For each $\lambda \in \mathbb{R}$ there exists $\varepsilon_0 > 0$ such that $(\mathscr{P}_{\varepsilon,\lambda})$ admits at* *least one nontrivial solution* $u_1 \in W_0^{1,p}(\Omega)$ *for each* $\varepsilon < \varepsilon_0$ *. Moreover* $f_{\varepsilon,\lambda}(u_1) < 0$ *.*

Lemma 1.10. *Let* $c \in \mathbb{R}$ *. Then each* $(CPS)_c-$*sequence for* $f_{\varepsilon,\lambda}$ *is bounded.*

Theorem 1.11. *There exist* $K > 0$ *and* $\varepsilon_0 > 0$ *such that* $f_{\varepsilon,\lambda}$ *satisfies* $(CPS)_c$ *with*

$$0 < c < \frac{p^* - \gamma - p}{p^*(\gamma + p)} (\nu S)^{\frac{n}{p}} - K\varepsilon \tag{6}$$

for each $\varepsilon < \varepsilon_0$ *and* $\lambda > 0$ *.*

Proof. Let (u_h) be a concrete Palais–Smale sequence for $f_{\varepsilon,\lambda}$ at level c. Since (u_h) is bounded in $W_0^{1,p}(\Omega)$, up to a subsequence we have:

$$u_h \to u \quad \text{in } L^p(\Omega)\,, \qquad \nabla u_h \rightharpoonup \nabla u \quad \text{in } L^p(\Omega)\,.$$

Note that by the results of L. Boccardo and F. Murat, we also have:

$$\text{for a.e. } x \in \Omega : \quad \nabla u_h(x) \to \nabla u(x)\,.$$

Moreover, we have

$$\langle w_{\varepsilon,\lambda}, u\rangle + \|u\|_{p^*}^{p^*} = \int_\Omega \nabla_\xi \mathscr{L}(x,u,\nabla u)\cdot\nabla u\,dx + \\ + \int_\Omega D_s\mathscr{L}(x,u,\nabla u)u\,dx\,,$$

where $w_{\varepsilon,\lambda} \in W^{-1,p'}(\Omega)$ is defined by

$$\langle w_{\varepsilon,\lambda}, v\rangle = \lambda\int_\Omega |u|^{q-2}uv\,dx + \varepsilon\int_\Omega hv\,dx\,.$$

This, (nontrivial!) yields the existence of $d \in \mathbb{R}$ with

$$\limsup_h \left\{\int_\Omega \nabla_\xi \mathscr{L}(x,u_h,\nabla u_h)\cdot\nabla u_h - \int_\Omega |u_h|^{p^*}\,dx\right\} \leqslant d \leqslant \\ \leqslant \left\{\int_\Omega \nabla_\xi \mathscr{L}(x,u,\nabla u)\cdot\nabla u - \int_\Omega |u|^{p^*}\,dx\right\}\,. \tag{7}$$

Of course, we have:

$$\Big\{\nabla_\xi \mathscr{L}(x,u_h,\nabla u_h)-\nabla_\xi \mathscr{L}(x,u_h,\nabla(u_h-u))\Big\}\rightharpoonup \nabla_\xi \mathscr{L}(x,u,\nabla u)$$

in $L^{p'}(\Omega)$. Let us note that it actually holds the strong limit

$$\Big\{\nabla_\xi \mathscr{L}(x,u_h,\nabla u_h)-\nabla_\xi \mathscr{L}(x,u_h,\nabla(u_h-u))\Big\}\to \nabla_\xi \mathscr{L}(x,u,\nabla u)$$

in $L^{p'}(\Omega)$, since by (3) there exist $\tau\in]0,1[$ and $c>0$ with:

$$\begin{aligned}
&|\nabla_\xi \mathscr{L}(x,u_h,\nabla u_h)-\nabla_\xi \mathscr{L}(x,u_h,\nabla(u_h-u))|\leqslant\\
&\leqslant \big|\nabla^2_{\xi\xi}\mathscr{L}(x,u_h,\nabla u_h+(\tau-1)\nabla u)\big|\,|\nabla u|\leqslant\\
&\leqslant c|\nabla u_h|^{p-2}|\nabla u|+c|\nabla u|^{p-1}\,.
\end{aligned}$$

Therefore we have

$$\nabla_\xi \mathscr{L}(x,u_h,\nabla u_h)\cdot\nabla u_h = \nabla_\xi \mathscr{L}(x,u_h,\nabla(u_h-u))\cdot\nabla u_h +$$
$$\nabla_\xi \mathscr{L}(x,u,\nabla u)\cdot\nabla u_h + o(1) = \nabla_\xi \mathscr{L}(x,u_h,\nabla(u_h-u))\cdot\nabla(u_h-u) +$$
$$\nabla_\xi \mathscr{L}(x,u,\nabla u)\cdot\nabla u + o(1) \quad \text{in } L^1(\Omega)\,,$$

as $h \to +\infty$, namely

$$\begin{aligned}&\nabla_\xi \mathscr{L}(x,u_h,\nabla u_h)\cdot\nabla u_h - \nabla_\xi \mathscr{L}(x,u,\nabla u)\cdot\nabla u = \\ &= \nabla_\xi \mathscr{L}(x,u_h,\nabla(u_h-u))\cdot\nabla(u_h-u) + o(1) \quad \text{in } L^1(\Omega)\,,\end{aligned} \tag{8}$$

as $h \to +\infty$. In a similar way, since there exists $\widetilde{c} > 0$ with

$$\Big||u_h|^{p^*} - |u_h|^{p^*-p}|u_h-u|^p\Big| \leqslant \widetilde{c}\Big[|u_h|^{p^*-p}(|u_h|^{p-1}+|u|^{p-1})\Big]\,|u|,$$

one obtains

$$\Big\{|u_h|^{p^*} - |u_h|^{p^*-p}|u_h-u|^p\Big\} \to |u|^{p^*} \quad \text{in } \; L^1(\Omega)\,. \tag{9}$$

In particular, by combining (7), (8) and (9), it results:

$$\limsup_h \int_\Omega \Big[\nabla_\xi \mathscr{L}(x,u_h,\nabla(u_h-u))\cdot\nabla(u_h-u)+ \tag{10}$$

$$-|u_h|^{p^*-p}|u_h-u|^p\Big]\,dx \leqslant 0\,.$$

On the other hand, by Hölder and Sobolev inequalities, we get:

$$\int_\Omega \Big[\nabla_\xi \mathscr{L}(x,u_h,\nabla(u_h-u))\cdot\nabla(u_h-u)-|u_h|^{p^*-p}|u_h-u|^p\Big]\,dx \geqslant$$

$$\geqslant \nu\|\nabla(u_h-u)\|_p^p-\frac{1}{S}\|u_h\|_{p^*}^{p^*-p}\|\nabla(u_h-u)\|_p^p=$$

$$=\left\{\nu-\frac{1}{S}\|u_h\|_{p^*}^{p^*-p}\right\}\|\nabla(u_h-u)\|_p^p\,, \tag{11}$$

which turns out to be coercive if:

$$\limsup_h \|u_h\|_{p^*}^{p^*}<(\nu S)^{\frac{n}{p}}\,. \tag{12}$$

Now, from $f_{\varepsilon,\lambda}(u_h) \to c$ we deduce

$$\int_\Omega \mathscr{L}(x, u_h, \nabla u_h)\, dx - \frac{1}{p^*}\|u_h\|_{p^*}^{p^*} = \frac{\lambda}{q}\|u\|_q^q + \varepsilon \int_\Omega hu\, dx + c + o(1)\,, \tag{13}$$

as $h \to +\infty$. On the other hand, by using (5), from $f'_{\varepsilon,\lambda}(u_h)(u_h) \to 0$ we obtain

$$\frac{\gamma+p}{p}\int_\Omega \mathscr{L}(x, u_h, \nabla u_h)\, dx - \frac{1}{p}\|u_h\|_{p^*}^{p^*} \geqslant \frac{\lambda}{p}\|u\|_q^q + \frac{\varepsilon}{p}\int_\Omega hu\, dx + o(1)\,, \tag{14}$$

as $h \to +\infty$. Multiplying (13) by $\frac{\gamma+p}{p}$, we obtain

$$\frac{\gamma+p}{p}\int_\Omega \mathscr{L}(x, u_h, \nabla u_h)\, dx - \frac{\gamma+p}{pp^*}\|u_h\|_{p^*}^{p^*} = \tag{15}$$

$$= \frac{\gamma+p}{pq}\lambda\|u\|_q^q + \frac{\gamma+p}{p}\varepsilon\int_\Omega hu + \frac{\gamma+p}{p}c + o(1)\,,$$

as $h \to +\infty$. Therefore, by combining (15) with (14), one gets

$$
\begin{aligned}
\frac{p^*-\gamma-p}{pp^*}\|u_h\|_{p^*}^{p^*} &\leqslant -\frac{q-\gamma-p}{pq}\lambda\|u\|_q^q + \\
&+ c'\varepsilon\int_\Omega hu\,dx + \frac{\gamma+p}{p}c + o(1) \leqslant \\
&\leqslant c'\varepsilon\int_\Omega hu\,dx + \frac{\gamma+p}{p}c + o(1)\,,
\end{aligned}
\tag{16}
$$

as $h \to +\infty$. Now, we deduce that

$$
\|u_h\|_{p^*}^{p^*} \leqslant \frac{p^*(\gamma+p)}{p^*-\gamma-p}c + \widetilde{K}\varepsilon + o(1),
$$

as $h \to +\infty$ for some $\widetilde{K} > 0$. In particular, condition (12) is fulfilled if

$$
\frac{p^*(\gamma+p)}{p^*-\gamma-p}c + \widetilde{K}\varepsilon < (\nu S)^{\frac{n}{p}}
$$

which yields range (6) for ε small and a suitable $K > 0$. By combining (10) and

(11) we conclude that u_h goes to u strongly in $W_0^{1,p}(\Omega)$. ☐

Proof. Let us choose $\phi \in W_0^{1,p} \cap L^\infty(\Omega)$ such that $\|\phi\|_{p^*} = 1$ and $\int_\Omega h\phi\, dx < 0$. It is easily seen that

$$\lim_{t\to+\infty} f_{\varepsilon,\lambda}(t\phi) = -\infty\,.$$

After some computations, one gets

$$\lim_{\lambda\to+\infty} \sup_{t\geqslant 0} f_{\varepsilon,\lambda}(t\phi) = 0\,,$$

so that there exists $\lambda_0 > 0$ such that:

$$0 < \sup_{t\geqslant 0} f_{\varepsilon,\lambda}(t\phi) < \frac{p^*-\gamma-p}{p^*(\gamma+p)}\,(\nu S)^{\frac{n}{p}} - K\varepsilon \qquad (17)$$

for each $\lambda \geqslant \lambda_0$ and $\varepsilon < \varepsilon_0$. Let $w = t\phi$ with t so large that $f_{\varepsilon,\lambda}(w) < 0$ and set

$$\Phi = \left\{\gamma \in C([0,1], W_0^{1,p}(\Omega)) :\ \gamma(0) = 0,\ \gamma(1) = w\right\}$$

and

$$\beta_{\varepsilon,\lambda} = \inf_{\gamma \in \Phi} \max_{t \in [0,1]} f_{\varepsilon,\lambda}(\gamma(t))$$

By the Mountain–Pass theorem one finds a Palais–Smale sequence (u_h) in $W_0^{1,p}(\Omega)$ at level $\beta_{\varepsilon,\lambda}$ with

$$\lambda \geqslant \lambda_0 \Longrightarrow 0 < \beta_{\varepsilon,\lambda} < \frac{p^* - \gamma - p}{p^*(\gamma + p)} (\nu S)^{\frac{n}{p}} - K\varepsilon$$

for each $\varepsilon < \varepsilon_0$. Therefore there exist a subsequence of (u_h) strongly convergent to some u_2 which solves $(\mathscr{P}_{\varepsilon,\lambda})$. Of course $u_1 \not\equiv u_2$. ☐

Theorem 1.12. *Let Ω be star–shaped with respect to the origin and*

$$\nabla_x \mathscr{L}(x,s,\xi) \cdot x - \frac{n}{p^*} D_s \mathscr{L}(x,s,\xi)s + \left\{ \frac{n}{p^*} - \frac{n}{q} \right\} \lambda |s|^q \geqslant 0 \,, \qquad (18)$$

for a.e. $x \in \Omega$ and all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^n$. Then $(\mathscr{P}_{0,\lambda})$ has no nontrivial solution $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$.