

An Overview on Polyharmonic Problems at Critical Growth

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Plan of the Talk: we briefly report on some recent results by M.S., M.Clapp, F.Gazzola and H.Ch.Grunau (and also on results by other people) on fourth (or higher) order critical problems.

1. A FEW REMARKS: 2nd Vs. 4th

- ▷ motivations from differential geometry (Paneitz operators, '83);
- ▷ as known, maximum principles *fail*, in general ($u(x) = -|x|^2$);
- ▷ symmetry results – Gidas-Ni-Nirenberg – *fail*, in general (Sweers, '03);
- ▷ as known, if $f \in H_0^2(\Omega)$, in general, f^+ , f^- , $|f|$ do *not* belong to $H_0^2(\Omega)$;
- ▷ *no* “trivial” uniform extension operator from $H^2 \cap H_0^1(\Omega)$ into $H^2(\mathbb{R}^n)$;
- ▷ positivity preservation may *fail*, in general; $\Delta^2 u = f$ with $f \geq 0$ does *not* imply $u \geq 0$ (conjectures by Boggio-Hadamard of '08 are false!);
- ▷ the first eigenfunction of $\Delta^2 u = \lambda u$ on Ω can be *negative* on some portion of the domain, for instance long rectangles (negativity around the corners) or very eccentric ellipses (conjecture by Szegő of '50 is false!); the *lack* of information about the sign of the solution comes from the *lack* of information about the sign of the corresponding Green's function;
- ▷ in some situations non-existence results are *less strong* than those for second order equations. For instance $-\Delta u = |u|^{2^*-2}u$ with Dirichlet b.c. in a star-shaped domain Ω has no solution via the Pohozaev identity

$$\int_{\partial\Omega} \left(\frac{\partial u}{\partial n} \right)^2 (x \cdot n) d\sigma = 0,$$

while $\Delta^2 u = |u|^{2^*-2}u$ with Dirichlet b.c. has *no positive* solutions but one cannot exclude *sign-changing* solutions.

2. BIHARMONIC PROBLEM IN CONTRACTIBLE DOMAINS

Let Ω be a bounded smooth domain in \mathbb{R}^n . To solve the problem

$$\begin{cases} -\Delta u = |u|^{2^*-2} u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

either perturb the critical growth nonlinearity by adding subcritical terms or change the geometry (topology) of Ω . We focus on the second approach which has been studied e.g. in the following papers:

- ▶ A. Bahri, J.M. Coron, *On a nonlinear elliptic equation involving the Sobolev exponent: the effect of the topology of the domain*, Comm. Pure Appl. Math. (1988),
- ▶ E.N. Dancer, *A note on an equation with critical exponent*, Bull. London Math. Soc. (1988),
- ▶ W. Ding, *Positive solutions of $\Delta u + u^{(n+2)/(n-2)} = 0$ on contractible domains*, J. Partial Differential Equations (1989),
- ▶ D. Passaseo, *The effect of the domain shape on the existence of positive solutions of the equation $\Delta u + u^{2^*-1} = 0$* , Top. Meth. Nonlin. Anal. (1994).

We consider the following biharmonic critical growth problem

$$\Delta^2 u = |u|^{2^*-2} u \quad \text{in } \Omega$$

either with Navier boundary conditions

$$u = \Delta u = 0 \quad \text{on } \partial\Omega$$

or with Dirichlet boundary conditions

$$u = \nabla u = 0 \quad \text{on } \partial\Omega.$$

Here $\Omega \subset \mathbb{R}^n$ ($n \geq 5$) is an open bounded smooth domain and $2_* = \frac{2n}{n-4}$ is the critical Sobolev exponent for the embedding $H^2(\mathbb{R}^n) \hookrightarrow L^{2_*}(\mathbb{R}^n)$.

Let us set

$$\text{cap}^*(D) = \inf \left\{ \int_{\mathbb{R}^n} |\Delta u|^2 ; u \in \mathcal{D}^{2,2}(\mathbb{R}^n), u = 1 \text{ in } D \right\} (\geq \text{cap}(D)).$$

Different definition since we have to deal with Navier b.c.!

Theorem 1. *Let $\tilde{\Omega}$ be a smooth bounded domain of \mathbb{R}^n ($n \geq 5$) and let H be a closed subset contained in $\tilde{\Omega}$. Then, there exists $\varepsilon > 0$ such that if $\Omega \subset \tilde{\Omega}$ is a smooth domain with $\text{cap}^*(\tilde{\Omega} \setminus \Omega) < \varepsilon$ and such that H cannot be deformed in $\tilde{\Omega}$ into a subset of Ω , then there exists a nontrivial solution of*

$$\begin{cases} \Delta^2 u = |u|^{2^*-2}u & \text{in } \Omega \\ u = \nabla u = 0 & \text{on } \partial\Omega. \end{cases}$$

Theorem 2. *Let $\tilde{\Omega}$ be a smooth bounded domain of \mathbb{R}^n ($n \geq 5$) and let H be a closed subset contained in $\tilde{\Omega}$. Then, there exists $\varepsilon > 0$ such that if $\Omega \subset \tilde{\Omega}$ is a smooth domain with $\text{cap}^*(\tilde{\Omega} \setminus \Omega) < \varepsilon$ and such that H cannot be deformed in $\tilde{\Omega}$ into a subset of Ω , then there exists a positive solution of*

$$\begin{cases} \Delta^2 u = u^{(n+4)/(n-4)} & \text{in } \Omega \\ u = \Delta u = 0 & \text{on } \partial\Omega. \end{cases}$$

► F. Gazzola, H.-Ch. Grunau, M. Squassina, *Existence and non-existence results for critical growth biharmonic elliptic equations*, Calc. Var. Partial Differential Equations (2003).

For Navier b.c., one has to study very carefully the behaviour of suitable sequences $u_h \in H^2 \cap H_0^1(\Omega_h)$ for varying domains Ω_h . In contrast with the spaces $H_0^2(\Omega_h)$, there is no “trivial” extension operator into $H^2(\mathbb{R}^n)$. Only as positivity of solutions is concerned, the situation with respect to Dirichlet boundary conditions is more involved than with respect to Navier boundary conditions.

3. MAIN TOOLS IN THE PROOF

▷ Variational methods, critical point theory.

Constrained critical points of the functional

$$f(u) = \int_{\Omega} |\Delta u|^2 \quad \text{on} \quad \mathcal{V}(\Omega) = \{u \in H_0^2(\Omega) : \|u\|_{2_*} = 1\};$$

▷ Nodal solutions “double” the energy (in the space \mathbb{R}^N , in the half space \mathbb{R}_+^N): **no test with u^\pm** , decomposition in **dual cones**;

Lemma 1. *Let $u \in D^{2,2}(\mathbb{R}^n)$ be a nodal solution of the equation*

$$\Delta^2 u = |u|^{8/(n-4)} u \quad \text{in } \mathbb{R}^n. \quad (1)$$

Then $\|\Delta u\|_2^2 \geq 2^{4/n} S \|u\|_{2_}^2$.*

Proof. Consider the convex closed cone

$$\mathcal{K} = \{u \in D^{2,2}(\mathbb{R}^n) : u \geq 0 \text{ a.e. in } \mathbb{R}^n\},$$

and its dual cone

$$\mathcal{K}' = \{u \in D^{2,2}(\mathbb{R}^n) : (u, v)_{2,2} \leq 0 \quad \forall v \in \mathcal{K}\}.$$

Let us show that $\mathcal{K}' \subseteq -\mathcal{K}$. For each $h \in C_c^\infty(\mathbb{R}^n) \cap \mathcal{K}$ consider the solution u_h of the problem

$$\Delta^2 u_h = h \quad \text{in } \mathbb{R}^n.$$

Then by the positivity of the fundamental solution of Δ^2 in \mathbb{R}^n , it follows $u_h \in \mathcal{K}$ and thus if $v \in \mathcal{K}'$,

$$\forall h \in C_c^\infty(\mathbb{R}^n) \cap \mathcal{K} : \int_{\mathbb{R}^n} h v \, dx = \int_{\mathbb{R}^n} \Delta^2 u_h v \, dx = (u_h, v)_{2,2} \leq 0.$$

By a density argument one obtains $v \leq 0$ a.e. in \mathbb{R}^n , i.e. $v \in -\mathcal{K}$. Now, by results of

► J.J. Moreau, *Décomposition orthogonale d'un espace hilbertien selon deux cônes mutuellement polaires*, C. R. Acad. Sci. Paris 1962

for each $u \in D^{2,2}(\mathbb{R}^n)$ there exists a unique pair (u_1, u_2) in $\mathcal{K} \times \mathcal{K}'$ such that

$$u = u_1 + u_2, \quad (u_1, u_2)_{2,2} = 0. \quad (2)$$

Let u be a nodal solution of (1) and let $u_1 \in \mathcal{K}$ and $u_2 \in \mathcal{K}'$ be the components of u according to this decomposition. We obtain that $u_i \not\equiv 0$ and

$$|u(x)|^{2^*-2}u(x)u_i(x) \leq |u_i(x)|^{2^*}, \quad i = 1, 2, \quad (3)$$

for a.e. $x \in \mathbb{R}^n$. Indeed, if $i = 1$ and $u(x) \leq 0$ then (3) is trivial, while if $u(x) \geq 0$, since $u_2 \in -\mathcal{K}$ one has $u(x) = u_1(x) + u_2(x) \leq u_1(x)$ and again (3) holds. The case $i = 2$ is similar. By combining the Sobolev inequality with (2) and (3), we get for $i = 1, 2$

$$\begin{aligned} S \|u_i\|_{2_*}^2 &\leq \|u_i\|_{2,2}^2 = \int_{\mathbb{R}^n} \Delta u \Delta u_i \, dx = \int_{\mathbb{R}^n} \Delta^2 u u_i \, dx = \\ &= \int_{\mathbb{R}^n} |u|^{2^*-2} u u_i \, dx \leq \int_{\mathbb{R}^n} |u_i|^{2^*} \, dx = \|u_i\|_{2_*}^{2^*}, \end{aligned}$$

which yields $\|u_i\|_{2_*}^2 \geq S^{(n-4)/4}$. This will imply the assertion. \square

▷ *PS* condition holds at level $c \in (S, 2^{4/N}S)$;

▷ Construct a *PS* sequence at level $c \in (S, 2^{4/N}S)$: here we need the *topological constraint*; More precisely, construction of two levels $c_1, c_2 > S$ with $c_2 < 2^{4/N}S$ such that f^{c_2} cannot be deformed into f^{c_1} .

– the energy remains relatively large if we prevent the functions from concentrating “too close” to their domain of definition. Proof involved due to the lack of Uniform Extension Operator in varying domains with Navier boundary conditions!

▷ Similarly, the global compactness result (Struwe, Math. Z. 1984) does not extend in a straightforward way to Navier boundary condition, some work needs to be done.

4. NONEXISTENCE RESULTS

We collect here a number of nonexistence results for the equation

$$\boxed{\Delta^2 u = \lambda u + |u|^{8/(n-4)} u \text{ in } \Omega}$$

under either Navier or Dirichlet boundary conditions. Here, we include the linear term λu . It turns out that the discussion of the borderline case $\lambda = 0$, where the purely critical equation has to be considered, is somehow **involved** and **still not exhaustive**.

- Nonexistence under Navier b.c. $u = \Delta u = 0$ on $\partial\Omega$

It turns out that nonexistence questions in this case are relatively hard and that only restricted results are available.

The case $\lambda = 0$

– no *positive* solution may exist, provided Ω is star shaped.

The key ingredient is a Pohožaev type identity. By means of techniques developed by Pucci and Serrin, we also have

– no *nontrivial radial* solutions in the ball.

– more difficult seems this result for *any* nontrivial solution.

In this case the Pohožaev type identity (Pucci-Serrin) is not powerful enough to exclude sign changing solutions.

The case $\lambda > 0$

We prove the following result.

Theorem 3. *Let $n \in \{5, 6, 7\}$ and $B \subset \mathbb{R}^n$ be the unit ball. Then there exists a number $\lambda_* > 0$ such that the problem*

$$\begin{cases} \Delta^2 u = \lambda u + u^{(n+4)/(n-4)} & \text{in } B \\ u = \Delta u = 0 & \text{on } \partial B \end{cases} \quad (4)$$

admits no positive solution if $\lambda \in (0, \lambda_]$.*

Moreover, there are complementary *existence* results implying that such a nonexistence result can hold *at most* in the dimensions 5, 6, 7. These

are the critical dimensions for the problem, at least, when restricting to *positive* solutions in the ball.

– again, nonexistence of *any nontrivial* solution seems out of reach.

The case $\lambda < 0$

In the preceding nonexistence results, the positivity of a solution, the nonexistence of which had to be shown, is intensively exploited. For $\lambda \geq 0$ one can conclude from $u \geq 0$ that even $-\Delta u \geq 0$ in Ω .

For λ negative, one can only argue as before, when λ is close enough to 0. If $\lambda \ll 0$, the framework of positive solutions is no longer adequate. Even nonexistence for nontrivial radial solutions in the ball is an open stuff!

• Nonexistence under Dirichlet b.c. $u = \nabla u = 0$ on $\partial\Omega$

In this situation, existence results for positive solutions are more involved than under Navier boundary conditions and cannot even be expected in general domains. On the other hand, we know much more about nonexistence.

The case $\lambda = 0$

Also in this case, the nonexistence of any nontrivial solutions has to be left open, only more restricted results are available. Oswald proved the nonexistence of *positive* solutions in strictly star shaped domains by means of a Pohožaev type identity and strong maximum principles for superharmonic functions.

Moreover, we can also exclude nontrivial radial solutions:

Theorem 4. *Let $B \subset \mathbb{R}^n$ ($n \geq 5$) be the unit ball. Then problem*

$$\begin{cases} \Delta^2 u = |u|^{8/(n-4)} u & \text{in } B, \\ u = \nabla u = 0 & \text{on } \partial B \end{cases}$$

admits no nontrivial radial solution.

The discussion of nonexistence of *nontrivial radial solutions* for polyharmonic problems in the borderline case $\lambda = 0$ remains essentially open.

The case $\lambda < 0$

Nonexistence of any nontrivial solution is known, provided the domain Ω is star shaped. This proof is based on a generalized Pohožaev identity and covers also the general polyharmonic case $(-\Delta)^K$.

The case $\lambda > 0$

The following result holds:

Let $n \in \{5, 6, 7\}$ and assume that Ω is the unit ball B . Then there exists a number $\lambda_* > 0$ such that a necessary condition for a *nontrivial radial* solution to exist is $\lambda > \lambda_*$.

Again this special behaviour of the critical growth Dirichlet problem may be observed only in space dimensions 5, 6 and 7 (Pucci-Serrin critical dimensions). Thus the critical dimensions seem to be *independent* of the boundary condition.

Possible explanation in terms of *summability of the fundamental solution* corresponding to Δ^2 :

► E. Jannelli, *The role played by space dimension in elliptic critical problems*, J. Differential Equations (1999).

5. WHAT ABOUT BAHRI-CORON AT HIGHER ORDER?

It is natural to ask: once we have existence results in contractible domains (which are perturbation of domain with nontrivial topology), do Bahri-Coron type results hold for higher order equations? **Yes.**

Theorem 5. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 5$, be a smooth bounded domain and assume that $H_k(\Omega, \mathbb{Z}_2) \neq \{0\}$ for some positive integer k . Then the problem*

$$\begin{cases} \Delta^2 u = u^{(n+4)/(n-4)} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = \Delta u = 0 & \text{on } \partial\Omega. \end{cases}$$

admits a solution $u \in H^2(\Omega) \cap H_0^1(\Omega)$.

► F. Ebobisse, M.O. Ahmedou, *On a nonlinear fourth order elliptic equation involving the critical Sobolev exponent*, Nonlinear Anal. (2003)

Theorem 6. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 5$, be a smooth bounded domain. There exists $q > 1$ such that the problem*

$$\begin{cases} \Delta^2 u = |u|^{2^*-2}u & \text{in } \Omega \\ u = \nabla u = 0 & \text{on } \partial\Omega \end{cases}$$

admits a nontrivial solution $u \in H_0^2(\Omega)$ provided that

$$\{x \in \mathbb{R}^n : \rho \leq |x - a| \leq \rho q\} \subset \Omega$$

and

$$\{x \in \mathbb{R}^n : |x - a| < \rho\} \not\subset \Omega$$

for some $\rho > 0$ and $a \in \mathbb{R}^n$.

► T. Bartsch, T. Weth, M. Willem, *A Sobolev inequality with remainder term and critical equations on domains with topology for the polyharmonic operator*, Calc. Var. Partial Differential Equations, (2003)

6. INFINITE SIGN CHANGING ENTIRE SOLUTIONS

Consider the equation ($n \geq 3$)

$$-\Delta u = |u|^{2^*-2}u \quad \text{in } \mathbb{R}^n.$$

Positive solutions are radially symmetric and have the form

$$u(x) = [n(n-2)]^{(n-2)/4} \left(\frac{d}{d^2 + |x - x_0|^2} \right)^{(n-2)/2}.$$

► G. Talenti, *Best constant in Sobolev inequality*, Ann. Mat. Pura Appl. 1976

There exist infinitely many inequivalent nodal solutions

► W. Ding, *On a conformally invariant elliptic equation on \mathbb{R}^n* , Comm. Math. Phys. 1986

Consider the equation ($n \geq 5$)

$$\Delta^2 u = |u|^{2^*-2}u \quad \text{in } \mathbb{R}^n.$$

Positive solutions are radially symmetric and have the form

$$u(x) = [(n+2)n(n-2)(n-4)]^{(n-4)/8} \left(\frac{d}{d^2 + |x - x_0|^2} \right)^{(n-4)/2}.$$

► C.S. Lin, *A classification of solutions of a conformally invariant fourth order equation in \mathbb{R}^n* , Comm. Math. Helvetici 1998

Problem: does there exist nodal solutions? **Yes**

► T. Bartsch, M. Schneider, T. Weth, *Multiple solutions to a critical polyharmonic equations*, preprint, 2003

7. SYMMETRY OF SOLUTIONS

Consider the problem ($B \subset \mathbb{R}^n$, $n \geq 2$)

$$\begin{cases} -\Delta u = f(u) & \text{in } B \\ u > 0 & \text{in } B \\ u = 0 & \text{on } \partial B \end{cases}$$

- Assume that $f \in \text{Lip}$, then all the smooth solutions of the above problem are radially symmetric and radially decreasing.

Consider the problem ($B \subset \mathbb{R}^n$, $n \geq 5$)

$$\begin{cases} \Delta^2 u = f(u) & \text{in } B \\ u > 0 & \text{in } B \\ u = \Delta u = 0 & \text{on } \partial B \end{cases}$$

- Assume that $f \in C^1$ and $f, f' \geq 0$, then all the smooth solutions u of the above problem are radially symmetric and radially decreasing.

With Dirichlet b.c. there exist at least one radially symmetric and radially decreasing solution in the ball, for instance if

$$f \in \text{Lip}, \quad f(0) > 0, \quad f'(s) \geq 0, \quad f'(0) \text{ small.}$$

But in general the GNN **fails!** Counterexamples by Sweers on

$$\begin{cases} \Delta^2 u + u = 0 & \text{in } B \\ u > 0 & \text{in } B \\ u = \nabla u = 0 & \text{on } \partial B. \end{cases}$$

There may exist *radially symmetric* solutions $u(r) = u(|x|)$ which are *not radially decreasing*: $u'(r) \not\leq 0$ (also with Navier b.c.).

► B. Gidas, W.M. Ni, L. Nirenberg, *Symmetry and related properties via the maximum principle*, Comm. Math. Phys. 1979

- ▶ W. Troy, *Symmetry properties in systems of semilinear elliptic equations*, JDE 1981
- ▶ G. Sweers, *No Gidas-Ni-Nirenberg for biharmonic equations*, Math. Nachr. 2003

8. OPEN STUFF

- CRITICAL CASE:

- ▷ non-existence in star-shaped domain: OK (sign change??);
- ▷ existence in domains with nontrivial topology: OK for order 4 (≥ 4 ??);
- ▷ existence in some contractible domains: OK for order 4 (≥ 4 ??);

- NEAR (SUPER) CRITICAL CASE:

- ▷ existence in domains with nontrivial topology: NOT known;
- ▷ existence in some contractible domains: NOT known;

To our knowledge no result is known on problems with nearly critical growth for the equation

$$\Delta^2 u = u^{(n+4)/(n-4)+\varepsilon} \quad \text{in } \Omega, \quad \varepsilon > 0 \text{ small,}$$

with Dirichlet b.c.

$$u = \nabla u = 0 \quad \text{on } \partial\Omega$$

or with Navier b.c.

$$u = \Delta u = 0 \quad \text{on } \partial\Omega$$

in contractible or non-contractible domains $\Omega \subset \mathbb{R}^n$.

9. DOMAIN SYMMETRY AND MULTIPLICITY

Let $K \geq 1$ and let Ω be a bounded smooth domain in \mathbb{R}^N with $N > 2K$. Consider the polyharmonic elliptic problem

$$\begin{cases} (-\Delta)^K u = |u|^{K_*-2}u + f & \text{in } \Omega \\ \left(\frac{\partial}{\partial \nu}\right)^j u \Big|_{\partial\Omega} = 0 \\ j = 0, \dots, K-1, \end{cases} \quad (P_{\Omega, f})$$

where $f \in H^{-K}(\Omega)$ and $K_* = \frac{2N}{N-2K}$ denotes the critical exponent for the Sobolev embedding $H_0^K(\Omega) \hookrightarrow L^{K_*}(\Omega)$.

Theorem 7. *There exists a $\kappa > 0$ such that, if $f \neq 0$ and $\|f\|_{H^{-K}} < \kappa$, then problem $(P_{\Omega, f})$ has at least 2 solutions.*

Theorem 8. *If $G \neq \{1\}$ acts freely on Ω then there exists a $\kappa > 0$ with the property that, for every $f \neq 0$ which is G -invariant and such that $\|f\|_{H^{-K}} < \kappa$, problem $(P_{\Omega, f})$ has at least 3 solutions one of which is G -invariant and one of which is not.*

For example, if Ω is symmetric with respect to the origin (i.e. $x \in \Omega$ if $-x \in \Omega$) and $0 \notin \Omega$ then, for every even function $f \neq 0$ with $\|f\|_{H^{-K}}$ small enough, problem $(P_{\Omega, f})$ has at least 3 solutions, one of which is even and one of which is not.

Denote by S_K the best Sobolev constant for the embedding $H_0^K(\Omega) \hookrightarrow L^{K_*}(\Omega)$,

$$S_K = \inf \left\{ \|u\|_{K,2}^2 : u \in H_0^K(\Omega), \int_{\Omega} |u|^{K_*} = 1 \right\}.$$

We write $\sharp G_j$ for the cardinality of G_j . Our main result is the following.

Theorem 9. *Let $\{1\} = G_1, \dots, G_m$ be closed subgroups of $\mathcal{O}(N)$ acting freely on Ω such that $\#G_1 < \dots < \#G_m$ and $G_{m-1} \subset G_m$. Then at least one of the following assertions holds:*

(a) *$m > 1$ and for $f = 0$ problem $(P_{\Omega,0})$ has a nontrivial solution u such that*

$$\|u\|_{K,2}^2 \leq (\#G_{m-1})(S_K)^{N/2K};$$

(b) *$m \geq 1$ and there exists a $\kappa > 0$ with the property that, if f is G_i -invariant for every $i = 1, \dots, m$, $f \neq 0$ and $\|f\|_{H^{-K}} < \kappa$, then the problem*

$$\begin{cases} (-\Delta)^K u = |u|^{K^*-2}u + f & \text{in } \Omega \\ \left(\frac{\partial}{\partial \nu}\right)^j u \Big|_{\partial\Omega} = 0 \\ j = 0, \dots, K-1 \end{cases}$$

has at least $m+1$ solutions u_0, u_1, \dots, u_m such that u_i is G_i -invariant but not G_{i+1} -invariant for $i = 1, \dots, m-1$, and u_m is G_m -invariant.

► M. Clapp, M. Squassina, *Nonhomogeneous polyharmonic elliptic problems at critical growth with symmetric data*, Commun. Pure Appl. Anal. (2003).

In the case $K = 1$ stronger results hold, see

► M. Clapp, O. Kavian, B. Ruf, *Multiple solutions of nonhomogenous elliptic equations with critical nonlinearity on symmetric domains*, Commun. Contemporary Math. (2003).

This is due to stronger nonexistence result than in the case $K > 1$!