Semiclassical asymptotic analysis for weakly coupled
nonlinear Schrödinger systems
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1. ***A Few Remarks***

A huge work has been devoted to the study of the behavior of solutions to

$$-\varepsilon^2 \Delta u + V(x)u = f(x,u) \qquad \text{in } \mathbb{R}^3$$

as ε goes to zero, where V is a suitable potential function. Typically, there exists a family of solutions u_ε which exhibits a spike shape around the local minima of the function $V(x)$ and decade elsewhere as ε goes to zero. See the works of Ambrosetti, Malchiodi, Del Pino, Felmer, Wei, etc., and the references therein.

Among the others, a physical motivation comes from ***nonlinear optics***, in particular in the study of the pulse in a ***single mode*** optical fiber.

But optical fibers are actually ***bimodal*** due to ***birefringence effects*** which tend to split a pulse into ***two pulses*** in the two polarization directions.

Menyuk (1987) showed that the two polarization components in a birefringence optical fiber are governed by the 1D system (standing waves)

$$\begin{cases} -u_{xx} + u = |u|^2 u + b|v|^2 u & \text{in } \mathbb{R}, \\ -v_{xx} + \omega^2 v = |v|^2 v + b|u|^2 v & \text{in } \mathbb{R}, \end{cases} \qquad \phi(x,t) = e^{i\omega_1^2 t} u(x), \quad \psi(x,t) = e^{i\omega_2^2 t} v(x),$$

where $\omega^2 = \omega_2^2/\omega_1^2$ and $b \geq 0$ is a constant depending on the ***anisotropy of the fiber***.

- great physical interest [1], but few rigorous general results;
- radiality and decay known by the work of Busca-Sirakov (JDE, 2000);
- $b = 0$: two copies of a single nonlinear Schrödinger equation;
- $b = 1$: also known as the Manakov system;
- $b \neq 1$: the situation gets pretty tough;
- if u, v are solutions to $-u_{xx} + u = |u|^2 u$ and $-v_{xx} + \omega^2 v = |v|^2 v$, then the couples $(u, 0)$ and $(0, v)$ are solutions known as ***scalar solitary waves***;
- the existence of a ground state solution $(u, v) \neq (0, 0)$ has been proved by concentration compactness by Cipolatti-Zumpichiatti (Nonlinear Anal, 2000);
- when is a ground state solution (u, v) a ***true-vector soliton***, that is $u > 0$ and $v > 0$? [2]

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[1] ▶ N. Akhmediev, A. Ankiewicz, *Solitons, Nonlinear pulses and beams*, Chapman & Hall, London, 1997

[2] ▶ A. Ambrosetti, E. Colorado, *Bound and ground states of coupled nonlinear Schrödinger equations*, to appear 2005.

▶ A. Ambrosetti, E. Colorado, *Standing waves of some coupled nonlinear Schrödinger equations*, to appear 2005.

▶ L.A. Maia, E. Montefusco, B. Pellacci, *Positive solutions for a weakly coupled nonlinear Schrödinger system*, preprint 2005.

Let $\varepsilon > 0$ and $b > 0$. Motivated by the above physical observations, we consider the system of two weakly coupled nonlinear Schrödinger equations

$$\begin{cases} -\varepsilon^2\Delta u + V(x)u = u^3 + bv^2u & \text{in } \mathbb{R}^3, \\ -\varepsilon^2\Delta v + W(x)v = v^3 + bu^2v & \text{in } \mathbb{R}^3, \end{cases} \tag{S_ε}$$

in the semiclassical limit.

• The (normalized) limiting system

$$\begin{cases} -\Delta u + u = u^3 + bv^2u & \text{in } \mathbb{R}^3, \\ -\Delta v + v = v^3 + bu^2v & \text{in } \mathbb{R}^3, \end{cases}$$

lacks of uniqueness, in general. For instance, if $b = 1$ and U denotes the unique solution to $-\Delta U + U = U^3$ in $\mathbb{R}^3$, then the pairs

$$\big\{(\cos(\theta)U, \sin(\theta)U) :\ 0 \le \theta \le \pi/2\big\}$$

are ground state solutions. If $b < 1$, the system admits the ground state solutions $(0, U)$ and $(U, 0)$. In the case $b > 1$, we suspect that the system admits a unique ground state.

• Unfortunately this produces difficulties, for instance it is not possible to generalize the approaches of Ambrosetti-Badiale-Cingolani, Wang, etc. to the problem.

• In the scalar (single peaked) case, the techniques of del Pino and Felmer (CoV 1996) ***does not require uniqueness*** assumptions on the limiting problem.
It is based on a suitable ***penalization*** $\tilde{J}_\varepsilon$ of the functional J_ε associated with the problem outside the region where the potential functions achieve the minimum, so that the critical points $(u_\varepsilon, v_\varepsilon)$ of $\tilde{J}_\varepsilon$ are actually solutions to the original problem with the desired features.

• We need a vectorial penalization. In general *truncation* with cut-off functions *does not work* (cannot control many of the needed estimates).

• Due to the particular structure of the nonlinearity,

$$F(u, v) = \frac{1}{4}(|u|^4 + 2b|uv|^2 + |v|^4),$$

for $\gamma > 0$ to be chosen small enough, the idea is to ***replace*** F by:

$$\chi(x)F(s) + (1 - \chi(x)) \begin{cases} \frac{1}{4}\left(|s|^4 + 2b|st|^2 + |t|^4\right) & \text{if } |s|^4 + 2b|st|^2 + |t|^4 \le \gamma^2 \\ \frac{\gamma}{2}\sqrt{|s|^4 + 2b|st|^2 + |t|^4} - \frac{\gamma^2}{4} & \text{if } |s|^4 + 2b|st|^2 + |t|^4 \ge \gamma^2, \end{cases}$$

being $\chi = \chi_{B(p,r)}$ for some r and for p local minimum of the potentials, i.e.

$$V(p) = \inf_{\xi \in B(p,r)} V(\xi) < \min_{\xi \in \partial B(p,r)} V(\xi),$$
$$W(p) = \inf_{\xi \in B(p,r)} W(\xi) < \min_{\xi \in \partial B(p,r)} W(\xi).$$

2. *Existence of Solutions (Attractive Case)*

Let us set

$$\mathscr{H} = \Big\{(u,v) \in H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3) : \ \textstyle\int_{\mathbb{R}^3} V(x)u^2 + W(x)v^2 < \infty\Big\}.$$

The following result provides the desired family of spike solutions [3]

Theorem 1. *Assume that V and W are continuous on $\mathbb{R}^3$ and there exists $\alpha > 0$ with*

$$V(x) \geq \alpha, \ \ W(x) \geq \alpha, \quad \textit{for all } x \in \mathbb{R}^3.$$

Furthermore, assume that there exist $p \in \mathbb{R}^3$ and $r > 0$ such that

$$V(p) = \inf_{B(p,r)} V < \min_{\partial B(p,r)} V,$$

$$W(p) = \inf_{B(p,r)} W < \min_{\partial B(p,r)} W.$$

Then, for every $\varepsilon > 0$ sufficiently small, the system

$$\begin{cases} -\varepsilon^2 \Delta u + V(x)u = u^3 + bv^2u & \textit{in } \mathbb{R}^3 \\ -\varepsilon^2 \Delta v + W(x)v = v^3 + bu^2v & \textit{in } \mathbb{R}^3 \end{cases}$$

admits a nontrivial nonnegative solution $(u_\varepsilon, v_\varepsilon) \in \mathscr{H}$ such that:

(a) *$u_\varepsilon + v_\varepsilon$ admits exactly one local maximum point $x_\varepsilon \in B(p,r)$ with*

$$\lim_{\varepsilon \to 0} V(x_\varepsilon) = \inf_{B(p,r)} V \quad \textit{or} \quad \lim_{\varepsilon \to 0} W(x_\varepsilon) = \inf_{B(p,r)} W \ ;$$

(b) *there exist $\Lambda_1, \Lambda_2 > 0$ such that, for every $x \in \mathbb{R}^3$,*

$$u_\varepsilon(x) + v_\varepsilon(x) \leq \Lambda_1 \exp\left\{ -\Lambda_2 \frac{|x - x_\varepsilon|}{\varepsilon} \right\};$$

(c) *assuming that $b \in (0, b_p)$, where*

$$b_p = \sqrt[4]{\max\left\{ \tfrac{W(p)}{V(p)}, \tfrac{V(p)}{W(p)} \right\}},$$

then the following dichotomy holds:

$$\textit{either} \quad \begin{cases} \lim\limits_{\varepsilon \to 0} u_\varepsilon(x_\varepsilon) > 0, \\ \lim\limits_{\varepsilon \to 0} v_\varepsilon(x_\varepsilon) = 0, \end{cases} \quad \textit{or} \quad \begin{cases} \lim\limits_{\varepsilon \to 0} u_\varepsilon(x_\varepsilon) = 0, \\ \lim\limits_{\varepsilon \to 0} v_\varepsilon(x_\varepsilon) > 0. \end{cases}$$

In particular, if $V = W$, the dichotomy holds for every $b \in (0,1)$.

E. Montefusco, B. Pellacci, M. Squassina, *Semiclassical asymptotic analysis for weakly coupled nonlinear Schrödinger systems*, 2005.

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[3] We deal with the **attractive** case $b > 0$, for the **repulsive** case $b < 0$ see the recent work of A. Pomponio, Coupled nonlinear Schrödinger systems with potentials, preprint (2005). See also J. Wei, T.-C. Lin, Spikes in two coupled nonlinear Schrodinger equations, ANIHPC **22** (2005) for the case of **constant potentials**.

3. ***Some Open Stuff***

- Case $p \neq q$ with

$$\inf_{B(p,r)} V < \min_{\partial B(p,r)} V, \qquad \inf_{B(q,r)} W < \min_{\partial B(q,r)} W;$$

- Case of multi-peak solutions (try new approach of Jeanjean requiring no uniqueness?);
- Case of singular (single/multi-polar) potentials;
- Case of potentials vanishing at ∞;
- Tentative proof of uniqueness for:

$$\begin{cases} -\Delta\xi + \xi = \xi^3 + b\zeta^2\xi & \text{in } \mathbb{R}^3, \\ -\Delta\zeta + \zeta = \zeta^3 + b\xi^2\zeta & \text{in } \mathbb{R}^3, \\ \xi > 0, \ \zeta > 0 & \text{in } \mathbb{R}^3, \\ \xi, \zeta \text{ radial, ground state}, & \\ b > 1. & \end{cases}$$

4. ***On Least Energy Functions***

Contrary to the scalar case with power nonlinearity, we cannot in general derive an explicit representation of the so called ground energy function, defined as

$$\Sigma(p) = \inf_{(u,v)\in\mathcal{N}_p} I_p(u,v),$$

where I_p is the functional associated with the limiting system

$$\begin{cases} -\Delta u + V(p)u = u^3 + bv^2u & \text{in } \mathbb{R}^3, \\ -\Delta v + W(p)v = v^3 + bu^2v & \text{in } \mathbb{R}^3, \end{cases}$$

and $\mathcal{N}_p$ is the Nehari manifold.

Since the limiting problem lacks of uniqueness, then the ground energy function Σ may lack of regularity and representation formulas.

5. ***Scalar Observations***

Consider the singularly perturbed equation

$$-\varepsilon^2\Delta u + u = f(x,u) \quad \text{in } \mathbb{R}^n, \qquad u > 0 \quad \text{in } \mathbb{R}^n, \tag{P_ε}$$

and, for every fixed $p \in \mathbb{R}^n$, the limiting autonomous equation

$$-\Delta u + u = f(p,u) \quad \text{in } \mathbb{R}^n, \qquad u > 0 \quad \text{in } \mathbb{R}^n. \tag{P_0}$$

There are (classical) choices of $f(x,u)$ such that the *ground energy function* Σ

$$\Sigma(p) = \inf_{\phi\in\mathcal{N}_p} I_p(\phi), \qquad I_p(u) = \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u|^2 + u^2 - \int_{\mathbb{R}^n} F(p,u)$$

is C^1-smooth with an explicit formula. For instance, for some $\Gamma > 0$,

$$\Sigma(p) = \Gamma\,\frac{V^{\frac{q+1}{q-1}-\frac{n}{2}}(p)}{K^{\frac{2}{q-1}}(p)}, \qquad \text{in the case } f(x,u) = K(x)u^q,\ 1 < q < \tfrac{n+2}{n-2}.$$

This Σ admits ***representation***, is ***smooth*** (as V, K are) and rules the concentration:[4]

$$\forall\{u_\varepsilon\} \text{ concentrating at } p \quad \Rightarrow \quad \nabla\Sigma(p) = 0$$

$$\nabla\Sigma(p) = 0 \text{ and } p \text{ is nondegenerate} \quad \Rightarrow \quad \exists\{u_\varepsilon\} \text{ concentrating at } p.$$

This follows by exploiting the ***uniqueness of solutions*** up to translations of (P_0). ***If not?***

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[4] ► X. Wang, B. Zeng, *On concentration of positive bound stated of nonlinear Schrödinger equations with competing potential functions*, SIAM J. Math. Anal. **28** (1997), 633–655.

► A. Ambrosetti, A. Malchiodi, S. Secchi, *Multiplicity results for some nonlinear Schrödinger equations with potentials*, Arch. Rational Mech. Anal., **159** (2001), 253–271.

6. ***Nonuniqueness***

As known, the ***uniqueness feature*** for

$$-\Delta u + u = f(p,u) \quad \text{in}\, \mathbb{R}^n, \qquad u > 0 \quad \text{in}\, \mathbb{R}^n.$$

is a ***delicate matter***, it is currently available only under ***rather restrictive assumptions*** on f. For (other, limiting) systems (of interest for the study of the corresponding singularly perturbed problem) the situation is ***even worse***, e.g.:

$$\begin{cases} -\Delta\xi + \xi = \zeta^q, & \text{in}\, \mathbb{R}^3, \\ -\Delta\zeta + \zeta = \xi^p, & \text{in}\, \mathbb{R}^3, \\ \xi > 0,\ \zeta > 0, & \text{in}\, \mathbb{R}^3, \end{cases} \qquad \begin{cases} -\Delta\xi + \xi = \xi^3 + b\zeta^2\xi & \text{in}\, \mathbb{R}^3, \\ -\Delta\zeta + \zeta = \zeta^3 + b\xi^2\zeta & \text{in}\, \mathbb{R}^3, \\ \xi > 0,\ \zeta > 0 & \text{in}\, \mathbb{R}^3, \end{cases}$$

No uniqueness result available.

7. ***In General***

For many scalar and vectorial Lagrangians, there holds:

- Σ is a ***locally Lipschitz continuous*** function (what beyond the $\mathrm{Lip}_{\mathrm{loc}}$?) ;
- Σ has ***representation formulas for the left and right derivatives*** $\left(\frac{\partial\Sigma}{\partial z_j}\right)^-(p)$, $\left(\frac{\partial\Sigma}{\partial z_j}\right)^+(p)$.

Motivated by these facts, recently, some conditions for locating the concentration points for P_ε in presence of a nonlinearity f, not necessarily a power, have been investigated.

$$\forall\{u_\varepsilon\}\ \textit{of minimal energy solutions concentrating at}\ p \quad \Rightarrow \quad 0 \in \partial_C\Sigma(p).$$

where ∂_C denotes the ***Clarke subdifferential*** of Σ, see,

S. Secchi, M. Squassina, *On the location of concentration points for singularly perturbed elliptic equations,* Adv. Differential Equations **9** (2004), 221–239.

S. Secchi, M. Squassina, *On the location of spikes for the Schrödinger equation with electromagnetic field*, Commun. Contemp. Math. **9** (2005), 251–268.

8. ***Sufficiency?***

$$0 \in \partial_C\Sigma(p),\ \text{plus more} \quad \overset{?}{\Longrightarrow} \quad \exists\{u_\varepsilon\}\ \textit{of minimal energy solutions concentrating at}\ p.$$

9. *Back to the Schrödinger System*

Let us set, for $b > 0$,

$$\mathcal{O}_b = \Big\{p \in \mathbb{R}^3 :\ \max\Big\{\tfrac{W(p)}{V(p)}, \tfrac{V(p)}{W(p)}\Big\} \geq b^4\Big\}, \qquad \Gamma = \tfrac{1}{4}\|U\|^4_{L^4(\mathbb{R}^3)},$$

being U the unique ground state solution to $-\Delta u + u = u^3$ in $\mathbb{R}^3$. J_ε denotes the functional associated with the system. Consider the sets:

$$\begin{aligned}\mathcal{E} = \Big\{p \in \mathbb{R}^3 :\ &\text{there exists a sequence of solutions } (u_\varepsilon, v_\varepsilon) \text{ of} \\ &(S_\varepsilon) \text{ with } u_\varepsilon(p+\varepsilon x) + v_\varepsilon(p+\varepsilon x) \to 0 \text{ as } |x| \to \infty \\ &\text{uniformly w.r.t. } \varepsilon, \text{ and } \varepsilon^{-3} J_\varepsilon(u_\varepsilon, v_\varepsilon) \to \Sigma(p) \text{ as } \varepsilon \to 0\Big\},\end{aligned}$$

$$\begin{aligned}\mathcal{E}_V = \Big\{p \in \mathcal{O}_b :\ &\text{there exists a sequence of solutions } (u_\varepsilon, v_\varepsilon) \text{ of } (S_\varepsilon) \text{ with} \\ &u_\varepsilon(p) \geq \delta \text{ for some } \delta > 0,\ u_\varepsilon(p+\varepsilon x) + v_\varepsilon(p+\varepsilon x) \to 0 \text{ as } |x| \to \infty \\ &\text{uniformly w.r.t. } \varepsilon, \text{ and } \varepsilon^{-3} J_\varepsilon(u_\varepsilon, v_\varepsilon) \to \Gamma\sqrt{V(p)} \text{ as } \varepsilon \to 0\Big\},\end{aligned}$$

$$\begin{aligned}\mathcal{E}_W = \Big\{p \in \mathcal{O}_b :\ &\text{there exists a sequence of solutions } (u_\varepsilon, v_\varepsilon) \text{ of } (S_\varepsilon) \text{ with} \\ &v_\varepsilon(p) \geq \delta \text{ for some } \delta > 0,\ u_\varepsilon(p+\varepsilon x) + v_\varepsilon(p+\varepsilon x) \to 0 \text{ as } |x| \to \infty \\ &\text{uniformly w.r.t. } \varepsilon, \text{ and } \varepsilon^{-3} J_\varepsilon(u_\varepsilon, v_\varepsilon) \to \Gamma\sqrt{W(p)} \text{ as } \varepsilon \to 0\Big\},\end{aligned}$$

and

$$\mathcal{E}_\Sigma = \mathcal{E} \setminus \mathcal{O}_b.$$

If ∂_C stands for the Clarke subdifferential, we shall also set:

$$\begin{aligned}\mathrm{Crit}(V) &= \big\{p \in \mathcal{O}_b :\ \nabla V(p) = 0\big\}, \\ \mathrm{Crit}(W) &= \big\{p \in \mathcal{O}_b :\ \nabla W(p) = 0\big\}, \\ \mathrm{Crit}_C(\Sigma) &= \big\{p \in \mathbb{R}^3 \setminus \mathcal{O}_b :\ 0 \in \partial_C \Sigma(p)\big\}.\end{aligned}$$

We recall a useful property of ground states justifying the introduction of $\mathcal{O}_b$.

Proposition 1. *Let $p \in \mathcal{O}_b$ and assume that (φ_p, ψ_p) is a least energy solution to*

$$\begin{cases} -\Delta u + V(p)u = u^3 + bv^2u & \text{in } \mathbb{R}^3, \\ -\Delta v + W(p)v = v^3 + bu^2v & \text{in } \mathbb{R}^3. \end{cases}$$

Then either $\varphi_p = 0$ and $\psi_p \neq 0$ or $\varphi_p \neq 0$ and $\psi_p = 0$.

10. *Necessary Conditions*

Theorem 2. *Assume that $V, W \in C^1(\mathbb{R}^3)$ with*

$$|\nabla V(x)| \leq \beta e^{\gamma|x|} \qquad \textit{and} \qquad |\nabla W(x)| \leq \beta e^{\gamma|x|},$$

for all $x \in \mathbb{R}^3$ and for some constants $\beta > 0$ and $\gamma \geq 0$.

Then Σ is locally Lipschitz continuous and the following facts hold:

(a) *if $V \neq W$, then $\mathcal{E}_V \cap \mathcal{E}_W = \emptyset$ and*

$$\mathcal{E} = \mathcal{E}_V \cup \mathcal{E}_W \cup \mathcal{E}_\Sigma,$$

where

$$\mathcal{E}_V \times \mathcal{E}_W \times \mathcal{E}_\Sigma \subset \mathrm{Crit}(V) \times \mathrm{Crit}(W) \times \mathrm{Crit}_C(\Sigma).$$

(b) *if $V \neq W$ are bounded, there exists $b_\star > 0$ with*

$$\mathcal{E} = \mathcal{E}_V \cup \mathcal{E}_W \quad \textit{for all } b \in (0, b_\star),$$

where

$$\mathcal{E}_V \times \mathcal{E}_W \subset \mathrm{Crit}(V) \times \mathrm{Crit}(W).$$

In particular, if $V = W$, then

$$\mathcal{E} = \mathcal{E}_V \subset \mathrm{Crit}(V) \quad \textit{for all } b \in (0, 1).$$