

*Global solutions and finite time blow up for
damped semilinear wave equations*

Marco Squassina, Politecnico di Milano

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We consider the damped wave equation with superlinear source

$$u_{tt} - \Delta u - \omega \Delta u_t + \mu u_t = |u|^{p-2}u.$$

It is shown that every global solution is uniformly bounded in the natural phase space. Global existence of solutions with initial data in the potential well is obtained. Finite time blow up for solutions starting in the unstable set is proved. High energy initial data for which the solution blows up are constructed.

Results contained in the joint paper:

F. Gazzola, M. Squassina,

Global solutions and finite time blow up
for damped semilinear wave equations,

Ann. Inst. H. Poincaré Anal. Non Linéaire, (2005) to appear.

1. Semilinear Wave Equations

We are concerned with the superlinear hyperbolic problem

$$\begin{cases} u_{tt} - \Delta u - \omega \Delta u_t + \mu u_t = |u|^{p-2}u & \text{in } [0, T] \times \Omega \\ u(0, x) = u_0(x) & \text{in } \Omega \\ u_t(0, x) = u_1(x) & \text{in } \Omega \\ u(t, x) = 0 & \text{on } [0, T] \times \partial\Omega \end{cases}$$

where Ω is an open bounded Lipschitz subset of \mathbb{R}^n ($n \geq 1$), $T > 0$,

$$u_0 \in H_0^1(\Omega), \quad u_1 \in L^2(\Omega),$$

$$\omega \geq 0, \quad \mu > -\omega \lambda_1,$$

λ_1 being the first eigenvalue of $-\Delta$, and

$$2 < p \leq \begin{cases} \frac{2n}{n-2} & \text{for } \omega > 0 \\ \frac{2n-2}{n-2} & \text{for } \omega = 0 \end{cases} \quad \text{if } n \geq 3,$$

$$2 < p < \infty \quad \text{if } n = 1, 2.$$

We study the *behavior of solutions* in the phase space $H_0^1(\Omega)$.

Remark 1. Leaving aside the *well posedness* of the above problem, the constraint $p \leq \frac{2n-2}{n-2}$ for $\omega = 0$ and initial data in $H_0^1 \times L^2$ is up to now unavoidable for the *energy identity* to make sense, i.e. it is not known if formula

$$E(t) + \int_s^t \|u_t(\tau)\|_*^2 d\tau = E(s) \quad \text{for every } 0 \leq s \leq t < T_{\max}$$

holds for $\frac{2n-2}{n-2} < p \leq \frac{2n}{n-2}$: for further comments, see:

• J. Ball,

Global attractors for damped semilinear wave equations,
Discrete Contin. Dyn. Syst. **10** (2004), 31–52.

Some References Related to these Strongly Damped Equations:

- H.A. Levine,
Some additional remarks on the nonexistence of global solutions to nonlinear wave equations,
SIAM J. Math. Anal. **5** (1974), 138–146.
- M. Ohta,
Remarks on blowup of solutions for nonlinear evolution equations of second order,
Adv. Math. Sci. Appl. **8** (1998), 901–910.
- K. Ono,
On global existence, asymptotic stability and blowing up of solutions for some degenerate non-linear wave equations of Kirchhoff type with a strong dissipation,
Math. Meth. Appl. Sci. **20** (1997), 151–177.

2. Potential Well Framework Stuff

When $\omega > 0$ (resp. $\omega = 0$) for $v, w \in H_0^1(\Omega)$ (resp. for $v, w \in L^2(\Omega)$),

$$(v, w)_* = \omega \int_{\Omega} \nabla v \cdot \nabla w + \mu \int_{\Omega} vw, \quad \|v\|_* = (v, v)_*^{1/2};$$

We consider the C^1 functionals $I, J : H_0^1(\Omega) \rightarrow \mathbb{R}$ defined by

$$\begin{aligned} I(u) &= \|\nabla u\|_2^2 - \|u\|_p^p, \\ J(u) &= \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{p}\|u\|_p^p. \end{aligned}$$

The mountain pass value of J (the potential well depth) is

$$d = \inf_{u \in H_0^1(\Omega) \setminus \{0\}} \max_{\lambda \geq 0} J(\lambda u).$$

If $(n-2)p < 2n$, then

$$d = \frac{p-2}{2p} S_p^{p/(p-2)},$$

being S_p the best Sobolev constant for the embedding $H_0^1 \hookrightarrow L^p$. All nonzero stationary solutions belong to the Nehari manifold

$$\mathcal{N} = \{u \in H_0^1(\Omega) \setminus \{0\} : I(u) = 0\}.$$

Each half line starting from the origin of $H_0^1(\Omega)$ intersects exactly once the manifold \mathcal{N} . \mathcal{N} separates the two unbounded sets

$$\mathcal{N}_+ = \{u \in H_0^1(\Omega) : I(u) > 0\} \cup \{0\},$$

and

$$\mathcal{N}_- = \{u \in H_0^1(\Omega) : I(u) < 0\}.$$

We also consider, for $a \in \mathbb{R}$, the closed sublevels of J

$$J^a = \{u \in H_0^1(\Omega) : J(u) \leq a\}$$

and we introduce the *stable* set \mathcal{W} and the *unstable* set \mathcal{U} :

$$\mathcal{W} = J^d \cap \mathcal{N}_+ \quad \text{and} \quad \mathcal{U} = J^d \cap \mathcal{N}_-.$$

It is readily seen that

$$d = \inf_{u \in \mathcal{N}} J(u).$$

This alternative characterization of d shows that

$$\text{dist}(0, \mathcal{N}) = \inf_{u \in \mathcal{N}} \|\nabla u\|_2 = \sqrt{\frac{2dp}{p-2}} > 0$$

and that, for every $a > d$, we have

$$\mathcal{N}_a = \mathcal{N} \cap J^a \equiv \left\{ u \in \mathcal{N} : \|\nabla u\|_2 \leq \sqrt{\frac{2ap}{p-2}} \right\} \neq \emptyset.$$

Therefore, for every $a > d$, we may define

$$\Lambda_a = \sup \{ \|u\|_2 : u \in \mathcal{N}_a \}.$$

By Poincaré inequality, we have $\Lambda_a < \infty$ for every $a > d$. We put

$$\mathcal{S} = \{ \phi \in H_0^1(\Omega) : \phi \text{ is a stationary solution of the problem} \},$$

$$\mathcal{S}_\ell = \{ \phi \in \mathcal{S} : J(\phi) = \ell \} \quad (\ell \in \mathbb{R}_+).$$

Finally, we consider the energy $\mathcal{E} : H_0^1(\Omega) \times L^2(\Omega) \rightarrow \mathbb{R}$ defined by

$$\mathcal{E}(v, w) = J(v) + \frac{1}{2} \|w\|_2^2$$

and the Lyapunov function $E(t) = \mathcal{E}(u(t), u_t(t))$.

Some Classical References Related to this Framework:

- A. Ambrosetti, P.H. Rabinowitz,
Dual variational methods in critical point theory and applications,
J. Funct. Anal. **14** (1973), 349–381.
- A. Haraux,
Dissipative dynamical systems and applications,
Research in Applied Mathematics, 17 Masson, Paris, 1991.
- Z. Nehari,
On a class of nonlinear second-order differential equations,
Trans. Amer. Math. Soc. **95** (1960), 101–123.
- L.E. Payne, D.H. Sattinger,
Saddle points and instability of nonlinear hyperbolic equations,
Israel Math. J. **22** (1975), 273–303.
- D.H. Sattinger,
On global solution of nonlinear hyperbolic equations,
Arch. Ration. Mech. Anal. **30** (1968), 148–172.
- M. Tsutsumi,
On solutions of semilinear differential equations in a Hilbert space,
Math. Japon. **17** (1972), 173–193.

3. The Local Well-Posedness Result

By *solution* of the problem over $[0, T]$ we mean a function

$$u \in C^0([0, T], H_0^1(\Omega)) \cap C^1([0, T], L^2(\Omega)) \cap C^2([0, T], H^{-1}(\Omega)),$$

with $u_t \in L^2([0, T], H_0^1(\Omega))$ whenever $\omega > 0$, such that

$$u(0) = u_0, \quad u_t(0) = u_1$$

and the equation is satisfied in distributional sense.

We first establish local existence and uniqueness for solutions.

Theorem 1. *There exist $T > 0$ and a unique solution to the problem over $[0, T]$. Moreover, if*

$$T_{\max} = \sup \{T > 0 : u = u(t) \text{ exists on } [0, T]\} < \infty$$

then

$$\lim_{t \rightarrow T_{\max}} \|u(t)\|_q = \infty \quad \text{for all } q \geq 1 \text{ such that } q > \frac{n(p-2)}{2};$$

if $n \geq 3$ and $p = 2^*$ (so that $\omega > 0$), this also holds for $q = \frac{n(p-2)}{2} = 2^*$.

Definition 1. If $T_{\max} < \infty$, we say that the solution u blows up and that T_{\max} is the blow up time. If $T_{\max} = \infty$, we say that u is global.

Remark 2. As it might be expected, for *fixed* initial data, we have $T_{\max} \rightarrow \infty$ as $\omega \rightarrow \infty$.

Some in Focus References:

- A.N. Carvalho, J.W. Cholewa,
Local well posedness for strongly damped wave equations with critical nonlinearities,
Bull. Austral. Math. Soc. **66** (2002), 443–463.
- A. Haraux,
Dissipative dynamical systems and applications,
Research in Applied Mathematics, 17 Masson, Paris, 1991.

4. Boundedness of Global Solutions

In the strongly damped case we have the following

Theorem 2. *Assume that $\omega > 0$. Then, every global solution $u(t)$ satisfies*

$$u \in L^\infty(\mathbb{R}_+, H_0^1(\Omega)) \cap W^{1,\infty}(\mathbb{R}_+, L^2(\Omega)).$$

Moreover, if $n = 1, 2$ or if

$$n \geq 3 \quad \text{and} \quad 2 < p < 2^*,$$

then there exists $\ell \in \mathbb{R}_+$ such that $\mathcal{S}_\ell \neq \emptyset$,

$$\lim_{t \rightarrow \infty} E(t) = \ell, \quad \lim_{t \rightarrow \infty} \text{dist}_{H_0^1}(u(t), \mathcal{S}_\ell) = 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} \|u_t(t)\|_2 = 0,$$

and there exist $\{t_j\} \subset \mathbb{R}_+$ with $t_j \rightarrow \infty$ and $\phi \in \mathcal{S}_\ell$ such that

$$\lim_{j \rightarrow \infty} \|\nabla u(t_j) - \nabla \phi\|_2 = 0.$$

Remark 3. Under the assumptions of the previous Theorem,

$$\begin{aligned} \lim_{t \rightarrow \infty} \|\nabla u(t) - \nabla u(t + \kappa)\|_2 &= 0 & \text{for } \omega > 0, \\ \lim_{t \rightarrow \infty} \|u(t) - u(t + \kappa)\|_2 &= 0 & \text{for } \omega = 0, \end{aligned}$$

for every $\kappa > 0$. Simple but *pretty fruitful stabilization property*.
Indeed, fixed $\kappa > 0$, for every $t > 0$ we have

$$\begin{aligned} \int_{\Omega} |\nabla u(t) - \nabla u(t + \kappa)|^2 &= \int_{\Omega} \left| \int_t^{t+\kappa} \nabla u_t(\tau) d\tau \right|^2 \\ &\leq \kappa \int_{\Omega} \int_t^{t+\kappa} |\nabla u_t(\tau)|^2 d\tau \leq \kappa c \int_t^{t+\kappa} \|u_t(\tau)\|_*^2 d\tau \\ &= \kappa c (E(t) - E(t + \kappa)). \end{aligned}$$

Since $E(t)$ is nonincreasing and lower bounded, $E(t)$ admits finite limit as $t \rightarrow \infty$. This immediately yields the assertion by letting $t \rightarrow \infty$ in the previous inequality. For $\omega = 0$ the proof is similar.

Remark 4. Assume that $n \geq 3$ and that Ω is star shaped. Then, in the limiting case $p = 2^*$, the well known Pohőzaev identity combined with the unique continuation property for elliptic equations yields

$$\mathcal{S} = \{0\}.$$

Then, it is possible to show that from every global solution $u = u(t)$ we may extract a subsequence $\{u(t_j)\}$ with $u(t_j) \rightharpoonup 0$ weakly in $H_0^1(\Omega)$, while the strong convergence $u(t_j) \rightarrow 0$ seems to be out of reach.

Theorem 3 (J. Esquivel-Avila, JMAA, 279 (2003)). Let $\omega = 0$ and

$$2 < p \leq \begin{cases} \frac{2n-2}{n-2} & \text{for } n \geq 3 \\ 6 & \text{for } n = 2 \end{cases}, \quad 2 < p < \infty \quad \text{for } n = 1.$$

Then, every global solution $u(t)$ satisfies

$$u \in L^\infty(\mathbb{R}_+, H_0^1(\Omega)) \cap W^{1,\infty}(\mathbb{R}_+, L^2(\Omega)).$$

In absence of damping ($\omega = \mu = 0$), see

• T. Cazenave,

Uniform estimates for solutions of nonlinear Klein-Gordon equations,

J. Funct. Anal. **60** (1985), 36–55.

Remark 5. In order to prove the boundedness of global solutions we make use of a delicate analysis of all the terms involved in the equation. The corresponding statement for the weakly damped case $\omega = 0$ has recently been obtained by Esquivel-Avila. Based on the just mentioned delicate analysis we provide a different proof.

Remark 6. By combining the boundedness of global solutions that we obtained in the above Theorems with some well known convergence results one can prove stabilization of the *whole flow*.

- in one space dimension, for any $p > 2$, one obtains

$$\lim_{t \rightarrow \infty} \|u_t(t)\|_2 + \|\nabla u(t) - \phi\|_2 = 0 \quad (\text{Stabilization})$$

for some equilibrium ϕ , as a consequence of Theorem 2.4 in

• J.K. Hale, G. Raugel,

Convergence in gradient-like systems with applications to PDE,
Z. Angew. Math. Phys. **43** (1992), 63–124.

It is sufficient to follow step by step the arguments therein (performed on a different equation!), with the only difference that the orbit precompactness is free by the afore mentioned results and not as a byproduct of the existence of the global attractor (our equation does not possess the global attractor!).

In two space dimensions, the situation is different for weak and strong damping:

- if $\omega = 0$ one has stabilization for $p = 4, 6$;

- if $\omega > 0$ one has stabilization for any *even integer* $p \geq 4$.

For both the assertions, see

• A. Haraux, M.A. Jendoubi,

Convergence of bounded weak solutions of the wave equation with dissipation and analytic nonlinearity,

Calc. Var. Partial Differential Equations **9** (1999), 95–124.

5. Global Existence Results

We turn to the global existence result.

Theorem 4. *Assume that there exists $\bar{t} \in [0, T_{\max})$ such that*

$$u(\bar{t}) \in \mathcal{W} \quad \text{and} \quad \mathcal{E}(u(\bar{t}), u_t(\bar{t})) \leq d.$$

Then $T_{\max} = \infty$ and, for every $t > \bar{t}$,

$$\|\nabla u(t)\|_2^2 + \|u_t(t)\|_2^2 \leq \frac{\Theta(\omega, \mu)}{t}$$

where

$$\Theta(\omega, \mu) = \begin{cases} C_\mu(1 + \frac{1}{\omega} + \omega) & \text{for } \omega > 0 \\ C(1 + \frac{1}{\mu} + \mu) & \text{for } \omega = 0 \end{cases}$$

and C is independent of μ , whereas C_μ only depends on μ .

Remark 7. Let $\omega > 0$ and $\mu = 0$. Although the above inequality gives only a one-sided control, since $\Theta(\omega, 0) \rightarrow \infty$ both for $\omega \rightarrow 0$ and $\omega \rightarrow \infty$, the best dissipation rate for the energy norm (with respect to the damping coefficient ω) seems to be achieved at the minimum point of $\Theta(\omega, 0)$, which occurs at $\omega = 1$. Physically, as $\omega \rightarrow 0$ the dissipation gets lost, whereas for $\omega \rightarrow \infty$ the system tends to freeze since ω acts only on the velocity u_t . A similar phenomenon has popped up in

- V. Pata, M. Squassina,

On the strongly damped wave equation,

Comm. Math. Phys. **253** (2005), 511–533

for a (different) class of strongly damped wave equations in discussing the size of the Universal Attractor as ω moves.

Remark 8. In the *sublinear case* $p \in (1, 2]$, it is straightforward to see that $T_{\max} = \infty$ for *arbitrary choices* of the initial data. For any fixed $T > 0$, define the functional $\Upsilon : [0, T] \rightarrow \mathbb{R}^+$,

$$\Upsilon(t) = \frac{1}{2} \|\nabla u(t)\|_2^2 + \frac{1}{2} \|u_t(t)\|_2^2 + \frac{1}{p} \|u(t)\|_p^p.$$

Then, simple computations show that, for some $c > 0$,

$$\Upsilon'(t) \leq c \Upsilon^{\frac{2p-2}{p}}(t) \quad \text{for every } t \in [0, T].$$

Since $p \in (1, 2]$, we have $2(p-1)/p \in (0, 1]$ and

$$\|\nabla u(t)\|_2^2 + \|u_t(t)\|_2^2 \leq 2\Upsilon(t) \leq \begin{cases} c(T+c)^{\frac{p}{2-p}} & \text{for } p < 2 \\ ce^{cT} & \text{for } p = 2. \end{cases}$$

By the continuation principle, u is global.

Some Related References:

- R. Ikehata, T. Suzuki,

Stable and unstable sets for evolution equations of parabolic and hyperbolic type,

Hiroshima Math. J. **26** (1996), 475–491.

- K. Ono,

On global existence, asymptotic stability and blowing up of solutions for some degenerate non-linear wave equations of Kirchhoff type with a strong dissipation,

Math. Meth. Appl. Sci. **20** (1997), 151–177.

- D.H. Sattinger,

On global solution of nonlinear hyperbolic equations,

Arch. Ration. Mech. Anal. **30** (1968), 148–172.

6. Blow-up Results

We come to the blow up results.

Theorem 5. *Assume that u is the unique local solution to the problem. Then $T_{\max} < \infty$ if and only if there exists $\bar{t} \in [0, T_{\max})$ such that*

$$u(\bar{t}) \in \mathcal{U} \quad \text{and} \quad \mathcal{E}(u(\bar{t}), u_t(\bar{t})) \leq d.$$

This is known for $\omega = 0$, see Vitillaro, ARMA **149** (1999), 155–182.

Remark 9. Basically, use a refined *concavity method* on a function like

$$L^2\text{-norm squared of } u(t) + L^2(0, t, (H_0^1, \|\cdot\|_*))\text{-norm squared of } u,$$

the second addendum accounting for *damping effects*.

Precisely, consider the positive continuous function $\theta : [0, T] \rightarrow \mathbb{R}_+$

$$\theta(t) = \|u(t)\|_2^2 + \int_0^t \|u(\tau)\|_*^2 d\tau + (T - t)\|u_0\|_*^2.$$

θ has a.e. second derivative. Show that there exists $\delta > 0$ with

$$\theta(t)\theta''(t) - \frac{p+2}{4}\theta'(t)^2 \geq \delta \quad \text{for a.e. } t \in [0, T],$$

implying the blow up of θ at some T^* , and in turn of u at T^* .

Remark 10. As a byproduct of the proof it is clear that $T_{\max} < \infty$ if and only if $E(t) \rightarrow -\infty$ as $t \rightarrow T_{\max}$. In particular, the *solution blow up occurs* if and only if the *energy blow up occurs*.

In the weakly damped case we state the blow up of solutions with suitable initial data having energy larger than the mountain pass level.

Theorem 6. *Assume that $\omega = 0$ and $\mu \geq 0$. In addition, assume that $(u_0, u_1) \in \mathcal{N}_- \times L^2(\Omega)$ are such that*

$$\mathcal{E}(u_0, u_1) > d, \quad \|u_0\|_2 \geq \Lambda_{\mathcal{E}(u_0, u_1)}, \quad \int_{\Omega} u_0 u_1 \geq 0.$$

Then $T_{\max} < \infty$ for the corresponding solution u .

As a consequence of Theorem 6, we obtain arbitrarily high energy initial data for which the solution of blows up in finite time.

Theorem 7. *Assume that $\omega = 0$ and $\mu \geq 0$. Then, for every $m > 0$, there exist initial data*

$$(u_0^m, u_1^m) \in \mathcal{N}_- \times L^2(\Omega)$$

such that $\mathcal{E}(u_0^m, u_1^m) \geq m$ and $T_{\max} < \infty$ for the corresponding solution.

These Theorems are new also for the undamped wave equation.

Some Related References:

• R. Ikehata,

Some remarks on the wave equations with nonlinear damping and source terms,

Nonlinear Anal. **27** (1996), 1165–1175.

• R. Ikehata, T. Suzuki,

Stable and unstable sets for evolution equations of parabolic and hyperbolic type,

Hiroshima Math. J. **26** (1996), 475–491.

• H.A. Levine,

Instability and nonexistence of global solutions to nonlinear wave equations of the form $Pu_{tt} = -Au + \mathcal{F}(u)$,

Trans. Amer. Math. Soc. **192** (1974), 1–21.

• H.A. Levine, J. Serrin,

Global nonexistence theorems for quasilinear evolution equations with dissipation,

Arch. Ration. Mech. Anal. **137** (1997), 341–361.

• M. Tsutsumi,

On solutions of semilinear differential equations in a Hilbert space,

Math. Japon. **17** (1972), 173–193.

7. Sketch of the Proof of Theorem 6

We may now prove the following weak antidissipativity property.

Lemma 1. *Assume that $\omega = 0$ and $\mu \geq 0$. In addition, assume that $u_0 \in \mathcal{N}_-$ and $u_1 \in L^2(\Omega)$ are such that*

$$\int_{\Omega} u_0 u_1 \geq 0.$$

Let u be the solution of the problem with initial data (u_0, u_1) . Then the map $\{t \mapsto \|u(t)\|_2\}$ is strictly increasing as long as $u(t) \in \mathcal{N}_-$.

Proof. Let $F(t) = \|u(t)\|_2^2$ and $G(t) = F'(t) = 2 \int_{\Omega} u u_t$. The function G is Lipschitzian. Note also that $G(0) = 2 \int_{\Omega} u_0 u_1 \geq 0$ and that G satisfies

$$G'(t) + \mu G(t) > 0 \quad \text{for a.e. } t \in [0, T_{\max})$$

since $u(t) \in \mathcal{N}_-$ (so that $I(u(t)) < 0$). Indeed, if u solves the problem, it is standard to show that the map

$$\left\{ t \mapsto \frac{d^2}{dt^2} \|u(t)\|_2^2 \right\}$$

is defined for a.e. t . Hence, if $\omega = 0$ and $\mu \geq 0$, the identity reads as

$$\frac{d^2}{dt^2} \|u(t)\|_2^2 + \mu \frac{d}{dt} \|u(t)\|_2^2 = 2 [\|u_t(t)\|_2^2 - I(u(t))]$$

for a.e. $t \in [0, T_{\max})$. It is easy to check that this implies that

$$\frac{d}{dt} \|u(t)\|_2^2 > 0 \quad \text{as long as } u(t) \in \mathcal{N}_-,$$

namely the assertion. □

The solution u satisfies

$$\bigcup_{t \in [0, T_{\max})} u(t) \subset \mathcal{N}_-.$$

Indeed, if this was not the case, then there would exist an exit time $T \in (0, T_{\max})$ where $u(t)$ exits \mathcal{N}_- , that is, $u(T) \in \mathcal{N}$. By the Lemma

$$\|u(T)\|_2 > \|u_0\|_2 \geq \Lambda_{E(0)}.$$

Moreover (notice that E is constant if $\omega = \mu = 0$) we simply have

$$J(u(T)) \leq E(T) \leq E(0).$$

This shows that

$$u(T) \in \mathcal{N} \cap J^{E(0)} = \mathcal{N}_{E(0)}.$$

Together with the definition of $\Lambda_{E(0)}$, this leads to a contradiction.

By contradiction, we assume that u is global. By energy arguments analogous to those exploited for the global boundedness Theorems, there exist a function $\phi \in \mathcal{N}_{E(0)}$ and a diverging sequence $\{t_j\}$ such that $u(t_j) \rightharpoonup \phi$ in $H_0^1(\Omega)$, so that

$$\|\phi\|_2 \leq \Lambda_{E(0)}.$$

But the monotonicity Lemma and the above inequalities we get

$$\|\phi\|_2 > \|u_0\|_2 \geq \Lambda_{E(0)},$$

a contradiction. Theorem 6 is proved.

8. Sketch of the Proof of Theorem 7

We first recall a simple property of \mathcal{N} .

Lemma 2. For any $\sigma \geq \sqrt{\frac{2dp}{p-2}}$ and for any $k \geq 1$ there exists $u \in \mathcal{N}$ with

$$\text{supp}(u) = \overline{\Omega/k} \quad \text{and} \quad \|\nabla u\|_2 = \sigma.$$

Proof. For $k = 1$ it is immediate to find u ; for $k > 1$ just rescale u as

$$u_k(x) = \begin{cases} k^{\frac{2}{p-1}} u(kx) & \text{for } x \in \frac{\text{supp}(u)}{k} \\ 0 & \text{for } x \notin \frac{\text{supp}(u)}{k}. \end{cases}$$

□

Fix $m > 0$ sufficiently large and take $u_1^m \equiv 0$. Take any $v \in H_0^1(\Omega) \setminus \{0\}$ such that $\text{supp}(v) \subset \Omega \setminus \frac{\Omega}{2}$. Then, take $\alpha_m > 0$ so large that

$$\alpha_m \|v\|_2 \geq \Lambda_m, \quad J(\alpha_m v) < 0.$$

By the Lemma, we may find $w_m \in \mathcal{N}$ with $\text{supp}(w_m) \subset \overline{\Omega/2}$ and

$$J(w_m) = \frac{p-2}{2p} \|\nabla w_m\|_2^2 = m - J(\alpha_m v) > m.$$

Finally, let $u_0^m = w_m + \alpha_m v$. Since $\text{supp}(w_m) \cap \text{supp}(v) = \emptyset$, it holds

$$\begin{aligned} \|u_0^m\|_2 &= \|w_m\|_2 + \alpha_m \|v\|_2 > \Lambda_m, \\ J(u_0^m) &= J(w_m) + J(\alpha_m v) = m, \\ I(u_0^m) &= I(w_m) + I(\alpha_m v) = I(\alpha_m v) < 0, \end{aligned}$$

the latter inequality following since J is nonnegative in $\mathcal{N}_+ \cup \mathcal{N}$. Moreover, $\mathcal{E}(u_0^m, u_1^m) = J(u_0^m) = m$. The proof is complete. □

9. Some Open Stuff

We collect here a few questions and open problems connected with the statements of our results:

- Do our blow up Theorems for high energy data extend to the case of strong damping $\omega > 0$? do they extend to the case of nonlinear (weak) damping such as $\mu|u_t|^{m-2}u_t$ with $m > 2$ in place of μu_t ?

- Many authors have obtained both global existence and blow up results for equations which present nonlinear damping terms such as $|u_t|^{m-2}u_t$ with $m > 2$, enlightening the interaction that pops up with the corresponding power source $|u|^{p-2}u$. In analogy with these extensions, one could wonder whether it is possible to obtain some results for a nonlinear strong damping such as $-\Delta_m u_t$.

We stress that the blow up Theorems, being based on a kind of concavity argument for which the linearity of the dissipation terms is particularly helpful in performing the reduction to an ordinary differential inequality in time, would become too involved. Moreover, testing the equation with u generates hard to manage terms which may also *lack summability* if $m > 2$.