Cycle cover property and $CPP = SCC$ property are not equivalent

Romeo Rizzi*

June 4, 2002

Dipartimento di Informatica e Telecomunicazioni, Università di Trento
via Sommarive 14, 38050 Povo, Italy
`romeo@science.unitn.it`

Abstract

Let G be an undirected graph. The *Chinese Postman Problem (CPP)* asks for a shortest postman tour in G, i.e. a closed walk using each edge at least once. The *Shortest Cycle Cover Problem (SCC)* asks for a family $\mathcal{C}$ of circuits of G such that each edge is in some circuit of $\mathcal{C}$ and the total length of all circuits in $\mathcal{C}$ is as small as possible. Clearly, an optimal solution of CPP can not be greater than a solution of SCC. A graph G has the *$CPP = SCC$ property* when the solutions to the two problems have the same value.

Graph G is said to have the *cycle cover property* if for every Eulerian 1, 2-weighting $w : E(G) \mapsto \{1, 2\}$ there exists a family $\mathcal{C}$ of circuits of G such that every edge e is in precisely w_e circuits of $\mathcal{C}$. The cycle cover property implies the $CPP = SCC$ property.

We give a counterexample to a conjecture of Zhang [8, 9, 2, 10] stating the equivalence of the cycle cover property and the $CPP = SCC$ property for 3-connected graphs. This is also a counterexample to the stronger conjecture of Lai and Zhang, stating that every 3-connected graph with the $CPP = SCC$ property has a nowhere-zero 4-flow. We actually obtain infinitely many cyclically 4-connected counterexamples to both conjectures.

Key words: cycle cover, faithful cover, Petersen graph, 4-flow, counterexample.

1 Introduction

Let $G = (V, E)$ be an undirected graph, possibly with parallel edges. A *postman tour* (*Euler tour*) in G is a closed walk using each edge at least (exactly) once. The *Chinese Postman Problem (CPP)* asks for a shortest postman tour in G. We denote by $V_o(G)$ the set of nodes with odd degree in G. Mei Gu Guan [4] observed that CPP is equivalent to the problem of finding a minimum $V_o(G)$*-join* in G, i.e. a subgraph J of G with $V_o(J) = V_o(G)$, since the graph obtained by G duplicating the edges in J will be Eulerian, hence will admit an Euler tour. The first to efficiently solve CPP were Edmonds and Johnson [3]. (See [1] for a simpler method inspired by results of Sebő [7]).

*Research carried out with financial support of the project TMR-DONET nr. ERB FMRX-CT98-0202 of the European Community and partially supported by a post-doc fellowship by the "Dipartimento di Matematica Pura ed Applicata" of the University of Padova.

A *cycle* is a closed walk C where repetition of nodes is forbidden. Denote by $|C|$ the *length* of C, i.e. the number of nodes in C. The *Shortest Cycle Cover Problem (SCC)* asks for a family $\mathcal{C}$ of cycles of G with $\sum_{C \in \mathcal{C}} |C|$ as small as possible and such that each edge of G is in some cycle of $\mathcal{C}$. An optimal solution of CPP can not be greater than a solution of SCC, since, when G is connected, it is always possible to read out a postman tour of G from a cycle cover of G. A graph G has the *$CPP = SCC$ property* when the solutions to the two problems have the same value. A well known graph without the $CPP = SCC$ property is the Petersen graph $\mathcal{P}$, shown in Figure 1 on the left. Indeed, the 1-factors of $\mathcal{P}$ are the minimum $V_o(\mathcal{P})$-joins in $\mathcal{P}$ and, since they are all isomorphic, we essentially have to consider only the 1-factor shown in Figure 1 in the middle. To do so, just check that the edge weighting shown in Figure 1 on the right is *bad* in the sense that no family $\mathcal{C}$ of cycles exists in $\mathcal{P}$ such that every edge is taken precisely the indicated number of times.

LEFT: The Petersen graph. MIDDLE: A 1-factor. RIGHT: A bad weighting.

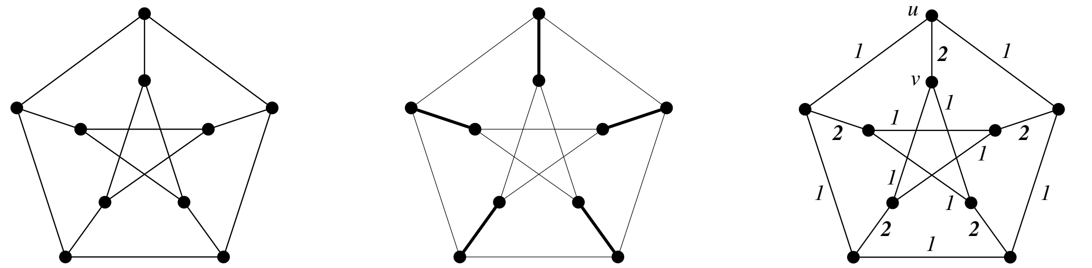

Figure 1: The Petersen graph does not have the $CPP = SCC$ property.

A weight function $w : E(G) \mapsto \{1, 2\}$ is called *Eulerian* if $\sum_{e \in \delta(S)} w_e$ is even for every cut $\delta(S)$ of G. Denote by $\mathcal{W}_G$ the set of all Eulerian weight functions for G. A $w \in \mathcal{W}_G$ is said to be *bad* when there exists no family $\mathcal{C}$ of cycles of G such that each edge e of G is in precisely w_e cycles of $\mathcal{C}$. When no $w \in \mathcal{W}_G$ is bad then G is said to have the *cycle cover property.* Note that the cycle cover property implies the $CPP = SCC$ property.

In Section 2, we give a counterexample to the following conjecture of Zhang [8, 9, 2, 10].

Conjecture 1 *The cycle cover property and the $CPP = SCC$ property are equivalent for 3-connected graphs.*

This will also be a counterexample to the stronger conjecture of Lai and Zhang stating that every 3-connected graph with the $CPP = SCC$ property has a nowhere-zero 4-flow. In Section 3, we derive infinitely many cyclically 4-connected counterexamples to both conjectures. Since the cycle cover property implies the $CPP = SCC$ property, the following conjecture of Jackson [6] would eventually come into play when one is willing to consider graphs with higher connectivity.

Conjecture 2 *The Petersen graph is the only cyclically 5-connected cubic graph without the cycle cover property.*

2 A first counterexample

In Figure 2, a first counterexample to Conjecture 1 is given.

LEFT: A counterexample. MIDDLE: $CPP = SCC$. RIGHT: A bad $w \in \mathcal{W}_{\mathcal{G}}$.

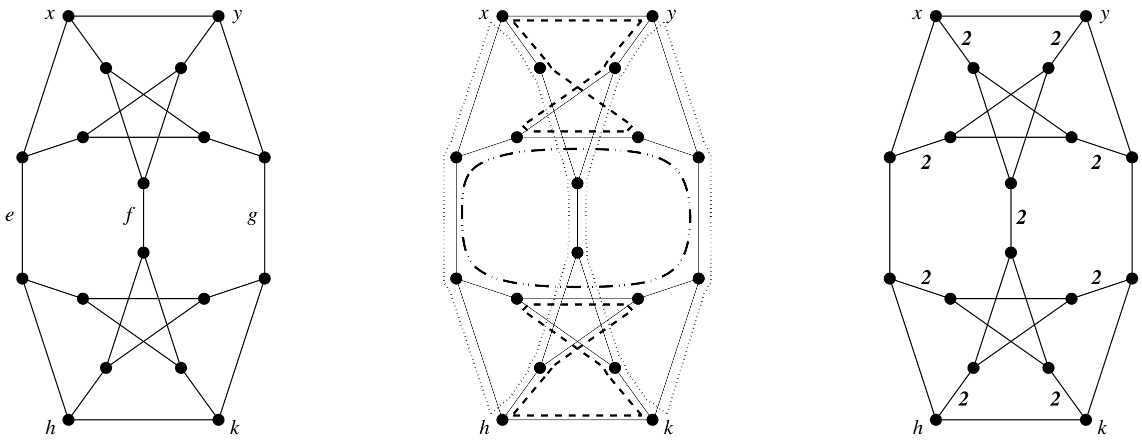

Figure 2: A graph $\mathcal{G}$ with the $CPP = SCC$ property but without the cycle cover property.

Graph $\mathcal{G}$, given in Figure 2 on the left, is indeed 3-connected. Let $\mathcal{C}$ be the family of cycles shown in Figure 2 in the middle. Every edge of $\mathcal{G}$ belongs to either 1 or 2 of the cycles in $\mathcal{C}$. Moreover the edges of $\mathcal{G}$ belonging to 2 cycles in C give a 1-factor of $\mathcal{G}$ and hence a minimum $V_o(\mathcal{G})$-join of $\mathcal{G}$. Hence $\mathcal{G}$ has the $CPP = SCC$ property. Consider now the weighting w indicated in Figure 2 on the right. Note that $w \in \mathcal{W}_{\mathcal{G}}$. We will show that w is bad, hence $\mathcal{G}$ does not have the cycle cover property. Assume on the contrary that there exists a family of cycles $\mathcal{C}$ such that every edge e is in precisely w_e cycles of $\mathcal{C}$. Let e, f, g be the three edges of $\mathcal{G}$ indicated in Figure 2 on the left. Let C_1 and C_2 be the two cycles of $\mathcal{C}$ containing f. We can assume w.l.o.g. that e belongs to C_1 and g belongs to C_2. Let $\mathcal{G}_A$ and $\mathcal{G}_B$ be the two connected components of $\mathcal{G} \setminus \{e, f, g\}$. Now $\mathcal{C} \setminus \{C_1, C_2\}$ can be partitioned into $\mathcal{C}_A$ and $\mathcal{C}_B$, where $\mathcal{C}_A$ is the set of those cycles in $\mathcal{C}$ which are cycles of $\mathcal{G}_A$ and $\mathcal{C}_B$ is the set of those cycles in $\mathcal{C}$ which are cycles of $\mathcal{G}_B$. Consider the Petersen graph $\mathcal{P}$ obtained from $\mathcal{G}$ by identifying all nodes in $V(\mathcal{G}_B)$ into a single node. Here $\mathcal{C}_A \cup \{C_1 \setminus E(\mathcal{G}_B), C_2 \setminus E(\mathcal{G}_B)\}$ would be a cycle cover of $\mathcal{P}$ contradicting the fact that the edge weighting shown in Figure 1 on the right is bad for $\mathcal{P}$.

3 Infinitely many cyclically 4-connected counterexamples

Although the original conjectures were about 3-connected graphs, it is now pertinent to investigate what happens for higher connectivity values. In this section, we show that infinitely many cyclically 4-connected counterexamples exist. To do so, we consider an operation that merges two cubic graphs, endowed by Eulerian 1, 2-weightings, into a single cubic graph, endowed by a corresponding Eulerian 1, 2-weighting. This operation is called *dot product*, since it is a natural extension of the celebrated operation introduced by Isaacs in [5] to generate new snarks by combining old ones.

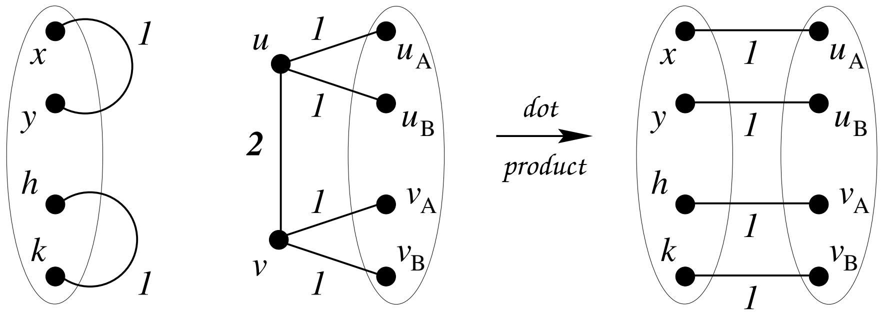

Figure 3: The dot product $(G, w) = (G_1, w_1) \cdot (G_2, w_2)$.

We are given two pairs (G_1, w_1) and (G_2, w_2), with G_i cubic and $w_i \in \mathcal{W}_{G_i}$, for $i = 1, 2$. Let hk and xy be two edges of G_1 and assume $w_1(hk) = w_1(xy) = 1$. Let uv, uu_A, uu_B, vv_A, vv_B, be edges of G_2 and assume $w_2(uv) = 2$, whereas $w_2(uu_A) = w_2(uu_B) = w_2(vv_A) = w_2(vv_B) = 1$. Then the *dot product* $(G_1, w_1) \cdot (G_2, w_2)$ is the pair (G, w) obtained from (G_1, w_1) and (G_2, w_2) by removing nodes u and v and removing edges hk, xy, uv, uu_A, uu_B, vv_A, and vv_B and adding edges u_Ax, u_By, v_Ah and v_Bk with $w(u_Ax) = w(u_By) = w(v_Ah) = w(v_Bk) = 1$. Every other edge e of G either belongs to G_1 or to G_2 and we set $w(e) = w_1(e)$ or $w(e) = w_2(e)$, accordingly. The operation is shown in Figure 3 and had been introduced by Jackson in [6] for the special case when the edges of weight 2 form a 1-factor. In [6], the following lemma had also been given.

Lemma 3 *If $w_1 \in \mathcal{W}_{G_1}$ and $w_2 \in \mathcal{W}_{G_2}$ are bad, and $(G, w) = (G_1, w_1) \cdot (G_2, w_2)$, then w is bad for G.*

Proof: Assume w is not bad for G. Let $\mathcal{C}$ be a family of cycles of G such that each edge e of G is in precisely w_e cycles of $\mathcal{C}$. Let C be the unique cycle in $\mathcal{C}$ containing edge u_Ax. If C contains also edge u_By then we have a contradiction with the fact that w_1 was bad for G_1. Otherwise we have a contradiction with the fact that w_2 was bad for G_2. $\square$

Let G be a cubic graph with the $CPP = SCC$ property but without the cycle cover property. If G is 3-connected, then G is bridgeless and hence, by Petersen's theorem, G has a 1-factor. Therefore, when $\mathcal{C}$ is a shortest cycle cover of G, and since G has the $CPP = SCC$ property, then the edges of G which are contained in two cycles of $\mathcal{C}$ form a 1-factor of G, denoted by $F_G(\mathcal{C})$. Let hk and xy be any two edges of G. Graph G is called an hk, xy-counterexample if there exists a shortest cycle cover $\bar{\mathcal{C}}$ of G with $hk, xy \notin F_G(\bar{\mathcal{C}})$ and a bad $\bar{w}_G \in W_G$ with $\bar{w}(hk) = \bar{w}(xy) = 1$. Note that the graph $\mathcal{G}$ given in Figure 2 is an hk, xy-counterexample. Denote by $\bar{w}_{\mathcal{P}}$ the bad weighting of $\mathcal{P}$ given in Figure 1 on the right. When G is an hk, xy-counterexample, then in the dot product $(H, w_H) = (G, \bar{w}_G) \cdot (\mathcal{P}, \bar{w}_{\mathcal{P}})$, graph H has the $CPP = SCC$ property, as shown in Figure 4. Moreover, by Lemma 3, w_H is a bad weighting for H. Hence, H too is a cubic graph with the $CPP = SCC$ property but without the cycle cover property. Moreover many choices for hk and xy are possible in H so that H is actually an hk, xy-counterexample. (One such choice is indicated in Figure 4). This means that the above operation can be repeated indefinitely many times, and in several ways.

For the graph $\mathcal{G}$, the choice of hk and xy indicated in Figure 2 was particularly fortunate: under this choice, the graph $\mathcal{H} = \mathcal{G} \cdot \mathcal{P}$, also displayed in Figure 4, is cyclically 4-connected. Finally, the property of being cyclically 4-connected is maintained when further dot product operations are performed.

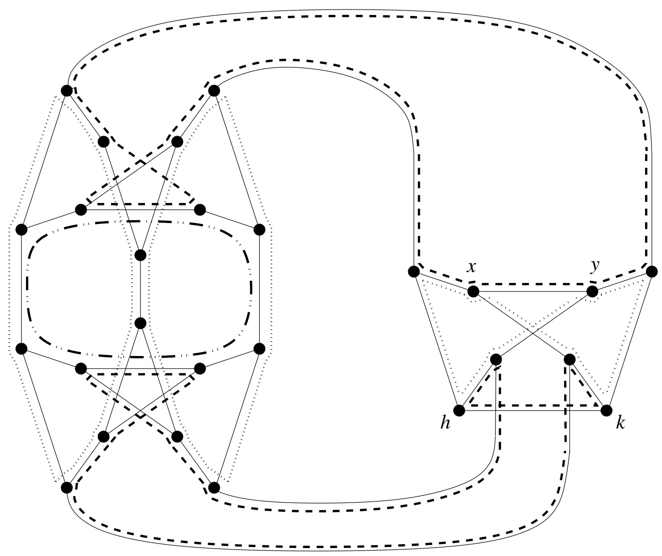

Figure 4: A cyclically 4-connected graph with the $CPP = SCC$ but without the cycle cover property.

Acknowledgments

I thank Professor Zhang for the interest shown. He also motivated me in obtaining counterexamples for higher connectivity values.

References

[1] Conforti, M., Rizzi, R.: Shortest Paths in Conservative Graphs. Discrete Mathematics **226**, 143-153 (2001)

[2] Dean, N.: Open Problems. Graph Structure Theory, AMS, Providence, Rhode Island, 677-688 (1993)

[3] Edmonds, J., Johnson, E.: Matching, Euler tours and the Chinese postman problem. Mathematical Programming **5**, 88–124 (1973)

[4] Guan, M.G.: Graphic programming using odd or even points. Chinese Mathematics **1**, 273–277 (1962)

[5] Isaacs, M.A.: Infinite families of snarks. *J.*, (Y. Alavi, G. Chartrand, O. R. Oellermann, A. J. Shwenk, Eds.) 493–524, Wiley, New York, 1991

[6] Jackson, B.: On Circuit Covers, Circuit Decompositions and Euler Tours of Graphs, *Surveys in Combinatorics 1993*, LMS Lecture Notes Series 187, (ed. K. Walker) CUP 1993, 191–210.

[7] Sebő, A.: Finding the $t-$join structure of graphs. Mathematical Programming **36**, 123–134 (1986)

[8] Zhang, C.Q.: Minimum cycle coverings and integer flows. Journal of Graph Theory **14 (5)**, 537–546 (1990)

[9] Zhang, C.Q.: Cycle covers and cycle decompositions of graphs. Annals Discrete Mathematics **55**, 183-190 (1993)

[10] Zhang, C.Q.: Integer Flows and Cycle Covers of Graphs. Marcel Dekker Inc. New York, (1997) ISBN: 0-8247-9790-6.