# *A note on range-restricted circuit covers*

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**Abstract**

Let $Cone(G)$, $Int.Cone(G)$ and $Lat(G)$ be the cone, the integer cone and the lattice of the incidence vectors of the circuits of graph $G$. A *good range* is a set $\mathbb{K} \subseteq \mathbb{N}$ such that $Cone(G) \cap Lat(G) \cap \mathbb{K}^E \subseteq Int.Cone(G)$ for every graph $G(V, E)$. We give a counterexample to a conjecture of Goddyn [1] stating that $\mathbb{N} \setminus \{1\}$ is a good range.

**Key words**: range-restricted circuit covers, Petersen graph, Hilbert basis.

## 1 Introduction

We denote by $\mathbb{N}$ the set of natural numbers and by $\mathbb{R}_+$ the set of non-negative reals. Let $Cone(G)$, $Int.Cone(G)$ and $Lat(G)$ be the cone, the integer cone and the lattice of the incidence vectors of the circuits of graph $G$. Obviously, $Int.Cone(G) \subseteq Cone(G) \cap Lat(G)$. Seymour in [2] proved that graphic matroids have the sum of circuits property: $p \in \mathbb{R}_+^E(G)$ is in $Cone(G)$ if and only if $p$ is *balanced*, i.e. $p(e) \leq p(B \setminus \{e\})$ for all $e \in B$ for all bonds $B$ of $G$. Moreover, if $p$ is in $Cone(G) \cap \mathbb{N}^E(G)$, then $p$ is in $Lat(G)$ whenever $p$ is *eulerian*, i.e. $p(B)$ is even for every bond $B$ of $G$. (See [1] for a surway).

A *range* is any set of natural numbers. Range $\mathbb{K}$ is *good* if $Cone(G) \cap Lat(G) \cap \mathbb{K}^E \subseteq Int.Cone(G)$ for every graph $G(V, E)$ and is *bad* otherwise. The study of good ranges could possibly give some insight towards the following:

**Conjecture 1.1 (circuit double cover)** *The set $\{2\}$ is a good range.*

For example, Goddyn [1] showed the equivalence between Conjecture 1.1 and Conjecture 1.2.

**Conjecture 1.2** *The set $\{2, 3\}$ is a good range.*

In the same paper, Goddyn proposed the following:

**Conjecture 1.3 (Goddyn (1991))** *The set of the integers bigger than one is a good range.*

We refer to [1] for more motivation in range-restricted circuit covers. In the next section we give a counterexample to Conjecture 1.3.

## 2 Some bad ranges

We denote by $\mathcal{P}$ the Petersen graph. Any two of the six 1-factors of $\mathcal{P}$ have precisely one edge in common. Thus, when $M$ is any 1-factor of $\mathcal{P}$, then we have the following:

**Property 2.1** *Every $M$-alternating circuit in $\mathcal{P}$ has length* 8.

The falsity of Conjecture 1.3 is a consequence of the following:

**Claim 2.2** *Let $k$ be any odd natural number. Then $\{k, 2k\}$ is a bad range.*

*Proof:* The vector $p \in \{k, 2k\}^{E(\mathcal{P})}$ defined by $p(e) = 2k$ if and only if $e \in M$ is balanced and eulerian. Hence $p \in Cone(\mathcal{P}) \cap Lat(\mathcal{P})$. Assume $p \in Int.Cone(\mathcal{P})$ and let $\mathcal{C}$ be a family of circuits of $\mathcal{P}$ such that every edge $e$ of $\mathcal{P}$ is in precisely $p(e)$ members of $\mathcal{C}$. For any circuit $C \in \mathcal{C}$, $p - \chi_C$ is balanced since $p - \chi_C \in Int.Cone(\mathcal{P}) \subset Cone(\mathcal{P})$. We conclude that every circuit $C \in \mathcal{C}$ is $M$-alternating. However $\sum_{e \in E(\mathcal{P})} p(e) = 20k$ is not divisible by 8 since $k$ is odd. This contradicts Property 2.1. □

The three crucial properties of vector $p$ in the proof of Claim 2.2 are the following:

1) $p \in Cone(\mathcal{P}) \cap Lat(\mathcal{P})$,

2) $p(uv) = p(\delta(u) \setminus \{uv\})$ for every edge $uv \in M$,

3) $\sum_{e \in E(\mathcal{P})} p(e)$ is not divisible by 8.

**Observation 2.3** *Let $p \in \mathbb{N}^{E(\mathcal{P})}$ be a vector satisfying 1) 2) and 3). Let $\mathcal{C}$ be a family of $M$-alternating circuits in $\mathcal{P}$. Then $p + \sum_{C \in \mathcal{C}} \chi_C$ satisfies 1) 2) and 3).*

Observation 2.3 allows to obtain further sufficient conditions for a range to be bad:

**Claim 2.4** *Let $k$ be any odd natural number. Then $\{k, k+1, k+2, 2k+1, 2k+2\}$ is bad.*

*Proof:* In $\mathcal{P}$ there exist $M$-alternating circuits $C_1, C_2$ such that $[\chi_{C_1} + \chi_{C_2}](e) \in \{0, 1, 2\}$ if $e \notin M$ and $[\chi_{C_1} + \chi_{C_2}](e) \in \{1, 2\}$ if $e \in M$. □

Analogously we can obtain the following:

**Claim 2.5** *Let $k$ be any odd natural number. Then $\{k, k+1, k+2, 2k+2, 2k+3\}$ is bad.*

**Claim 2.6** *Let $k$ be any odd natural number. Then $\{k+1, k+2, 2k+3, 2k+4\}$ is bad.*

More general sufficient conditions for a range to be bad are obtained by combining the above observations with the construction of Seymour described in [1] and used there to prove *Proposition 4.5*. For example Claim 2.2 becomes the following:

**Claim 2.7** *Let $\mathbb{K} = \{\overline{k}, k_0, k_1 \ldots, k_n\}$ with $k_0$ odd. Assume there exist non-negative integers $\lambda_0, \lambda_1, \ldots, \lambda_n$ such that $\overline{k} - \sum_{i=0}^{n} \lambda_i k_i = 2k_0$. Then $\mathbb{K}$ is bad.*

## 3 A conjecture of Seymour

The following conjecture was posed in [1]:

**Conjecture 3.1 (Goddyn [1])** *The range* $\{k, k+2\}$ *is good for every natural* $k \geq 2$.

In the same paper, Goddyn said the above conjecture to have "a very different flavor between odd and even values of $k$". Let $2\mathbb{N}$ be the set of even naturals. The following lemma generalizes a result introduced in [1].

**Lemma 3.2** *Let* $\mathbb{K} \subseteq 2\mathbb{N}$ *be a good range. Denote by* $\max(\mathbb{K})$ *the biggest integer in* $\mathbb{K}$. *Let* $y$ *be any odd positive integer. Then range* $\overline{\mathbb{K}} = \mathbb{K} \cup \{k + y\ : k \in \mathbb{K}, k \geq \frac{\max(\mathbb{K})}{2}\}$ *is good.*

*Proof:* Let $G(V, E)$ be any graph. Let $p$ be any vector in $Cone(G) \cap Lat(G) \cap \overline{\mathbb{K}}^E$. Let $F \subseteq E$ be the set of those edges $f$ such that $p(f) \in \overline{\mathbb{K}} \setminus \mathbb{K}$. Equivalently, $F$ is the set of those edges $f$ such that $p(f)$ is odd. Since $p \in Lat(G)$ then $F$ is an Eulerian subgraph, hence a disjoint union of circuits $C_1, \ldots, C_n$. Consider $p' = p - \sum_{i=1}^{n} y\chi_{C_i}$. Obviously $p' \in Lat(G) \cap \mathbb{K}^E$. We claim that $p' \in Cone(G)$, hence the proof is complete. Assume not, then there exists a bond $B = \delta_G(S)$ and an edge $b \in B$ such that $p'(b) > p'(B \setminus \{b\})$. However $p(b) \leq p(B \setminus \{b\})$ and so $|(B \setminus \{b\}) \cap F| \geq 2$ since $|B \cap F|$ is even. But then $p'(B \setminus \{b\}) \geq 2\frac{\max(\mathbb{K})}{2}$ in contradiction with $p'(b) > p'(B \setminus \{b\})$ since $p'(b) \leq \max(\mathbb{K})$. □

By Lemma 3.2, if Conjecture 3.1 is true for $k$ even, then Conjecture 3.1 is true for $k$ odd. In particular, Conjecture 1.1 turns out to be a special case of the following relevant conjecture:

**Conjecture 3.3 (Seymour (1981))** $2\mathbb{N}$, *the set of even natural numbers, is a good range.*

In fact Seymour posed Conjecture 3.3 in the more general setting of matroids with the sum of circuits property (see *Conjecture 16.6* in [3]) and obtained a quite natural and appealing formulation.

Assume Conjecture 3.3 to be true. Then many ranges are good in virtue of Lemma 3.2. On the other side several ranges were shown to be bad in the previous section. Right now the gap of limbo ranges is not empty. For example we pose the following:

**Question 1** *Is* $\{2, 3, 4, 5, 7\}$ *a good range?*

## References


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