

# A short proof of König's matching theorem

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## Abstract

We give a short proof of the following basic fact in matching theory: in a bipartite graph the maximum size of a matching equals the minimum size of a node cover.

**Key words:** bipartite graphs, maximum matching, minimum node cover.

A *matching* of a graph  $G(V, E)$  is a subset  $M$  of  $E$  such that every node of  $G$  is incident with at most one edge in  $M$ . A *cover* of  $G$  is a set of nodes  $W$  such that  $G \setminus W$  has no edges. Denote by  $\nu(G)$  the maximum cardinality of a matching of  $G$  and by  $\tau(G)$  the minimum cardinality of a cover of  $G$ . Clearly,  $\nu(G) \leq \tau(G)$ .

We give a short proof of the following basic fact [1] in matching theory.

**Theorem.** *Let  $G$  be a bipartite graph. Then  $\nu(G) = \tau(G)$*

*Proof:* Let  $G$  be a minimal counterexample. Then  $G$  is connected, is not a circuit, nor a path. So,  $G$  has a node of degree at least 3. Let  $u$  be such a node and  $v$  one of its neighbors. If  $\nu(G \setminus v) < \nu(G)$ , then, by minimality,  $G \setminus v$  has a cover  $W'$  with  $|W'| < \nu(G)$ . Hence,  $W' \cup \{v\}$  is a cover of  $G$  with cardinality  $\nu(G)$  at most. Assume, therefore, there exists a maximum matching  $M$  of  $G$  having no edge incident at  $v$ . Let  $f$  be an edge of  $G \setminus M$  incident at  $u$  but not at  $v$ . Let  $W'$  be a cover of  $G \setminus f$  with  $|W'| = \nu(G)$ . Since no edge of  $M$  is incident at  $v$ , it follows that  $W'$  does not contain  $v$ . So  $W'$  contains  $u$  and is a cover of  $G$ .  $\square$

The same proof easily extends to Egerváry's generalization [2] of König's result to graphs with nonnegative weights on the edges.

## References

- [1] D. König, Graphs and matrices, Mat Fiz Lapok 38 (1931), 116–119 (in Hungarian).
- [2] E. Egerváry, On combinatorial properties of matrices, Mat Lapok 38 (1931), 16–28 (in Hungarian).

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