Indecomposable r-graphs and some other counterexamples

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Abstract

An r-graph is any graph which can be obtained as a conic combination of its own 1-factors. An r-graph $G(V, E)$ is said *indecomposable* when its edge set E cannot be partitioned as $E = E_1 \cup E_2$ so that $G_i(V, E_i)$ is an r_i-graph for $i = 1, 2$ and for some r_1, r_2. We give an indecomposable r-graph for every integer $r \geq 4$. This answers a question raised in [11] and has interesting consequences for the Schrijver System of the T-cut polyhedron to be given in [9]. A graph in which every two 1-factors intersect is said *poorly matchable*. Every poorly matchable r-graph is indecomposable. We show that for every $r \geq 4$ "being indecomposable" does not imply "being poorly matchable". Next we give a poorly matchable r-graph for every $r \geq 4$. The paper provides counterexamples to some conjectures of Seymour [11].

Key words: r-graph, indecomposable, Petersen graph, Fulkerson Coloring.

1 Introduction

In this paper graphs may have parallel edges but contain no loop. The set of edges with precisely one endpoint in S is denoted by $\partial(S)$. To specify the graph, say G, we write $\partial_G(S)$. Let r be a positive integer. The notion of r-graph is due to Seymour [11]: an ***r*-*graph*** is a regular graph of valency r such that $|\partial(S)| \geq r$ for every set of nodes S with $|S|$ odd.
We rely on standard notation $d(S) = |\partial(S)|$ and $d_G(S) = |\partial_G(S)|$. Moreover $\partial(v) = \partial(\{v\})$ and $d(v) = d(\{v\})$. If G is an r-graph then G has an even number of nodes, since $d(V(G)) = 0$.
Given a graph G, a 1-*factor* of G is a spanning subgraph of G which is a 1-graph.
The celebrated Edmonds' matching polytope theorem [1] states that for every graph G the vertices of the following polytope are integral:

$$\begin{cases} x_e \geq 0 & \forall e \in E(G) \\ x(\partial(v)) = 1 & \forall v \in V(G) \\ x(\partial(S)) \geq 1 & \forall S \subseteq V(G) \text{ with } |S| \text{ odd} \end{cases} \tag{1}$$

Seymour [11] observed that Edmonds' theorem is equivalent to the following statement: a graph G is an r-graph if and only if G can be obtained as a conic combination of its own

1-factors. As a consequence, for every r-graph G and for every edge e of G, there exists a 1-factor of G containing e (see [11, 6]).

Let $G_1(V, E_1), \ldots, G_k(V, E_k)$ be graphs on a common node set V but with disjoint edge sets $E_1, \ldots, E_k$. We denote by $G_1 + \ldots + G_k$ the graph $G(V, E_1 \cup \ldots \cup E_k)$ and say that G is the *sum* of $G_1, \ldots, G_k$. Note that if $G_i(V, E_i)$ is an r_i-graph for $i = 1, \ldots, k$ then $G_1 + \ldots + G_k$ is an $(r_1 + \ldots + r_k)$-graph. For $k \in \mathbb{N}$, we denote by kG the graph obtained by summing up k copies of G (that is, replacing every edge of G by k parallel edges).

An *unslicable* r-graph is an r-graph which cannot be expressed as the sum of an $(r-1)$-graph and a 1-factor. An r-graph G which can be expressed as the sum of an r_1-graph and an r_2-graph is said *decomposable*. When no such decomposition exists G is said *indecomposable*. Finally, a graph in which every two 1-factors intersect is said *poorly matchable*.

In [11] Seymour raised the following question:

Question 1 *Does there exist a constant $\overline{r}$ such that every unslicable r-graph has $r < \overline{r}$?*

In the same paper, he proposed the following:

Conjecture 1.1 *The answer to Question 1 is positive and in fact we can take $\overline{r} = 4$.*

Conjecture 1.1 implies Conjecture 1.2 and makes Conjecture 1.3 imply Conjecture 1.4:

Conjecture 1.2 (Seymour [11]) *Every r-graph is $r+1$ edge colorable.*

A *Fulkerson coloring* of an r-graph G is a decomposition of $2G$ into 1-factors.

Conjecture 1.3 (Berge-Fulkerson) *Every 3-graph has a Fulkerson coloring.*

Conjecture 1.4 (Seymour [11]) *Every r-graph has a Fulkerson coloring.*

The author, while working on a bound for the size of the coefficients in the Schrijver System for the T-cut polyhedron (see [9]), became interested in the following question:

Question 2 *Does there exist a constant $\overline{r}$ such that every r-graph with $r \geq \overline{r}$ is decomposable?*

This paper presents a counterexample to Conjecture 1.1. In fact we settle Question 2 (and hence Question 1) in the negative by constructing for every r an indecomposable r-graph. More surprisingly, we exhibit for every r a poorly matchable r-graph.

2 Preliminary Observations

The Petersen graph is the 3-graph $\mathcal{P}$ shown in Figure 1. The six 1-factors of $\mathcal{P}$ are all equivalent under isomorphisms of $\mathcal{P}$. Let M be a 1-factor of $\mathcal{P}$. The essentially unique r-graph $\mathcal{P}(r) = \mathcal{P} + (r-3)M$, shown for $r = 4$ in Figure 1 and dating back to Meredith (see [7]), acts as a fundamental component in three of the constructions presented in this paper.

Every edge of $\mathcal{P}$ belongs to precisely two distinct 1-factors of $\mathcal{P}$. Conversely, every two of the six 1-factors of $\mathcal{P}$ have precisely one edge in common. This is expressed more formally by the following proposition:

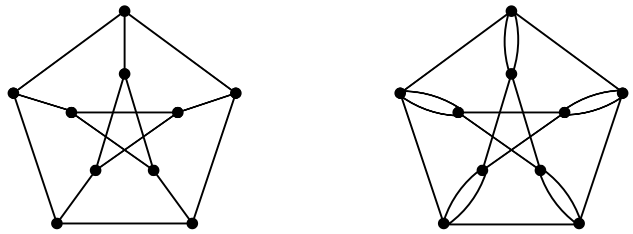

Figure 1: The Petersen Graph $\mathcal{P}$ and the Meredith Graph $\mathcal{P}(4)$.

Proposition 2.1 *Associate to every pair of* 1*-factors of* $\mathcal{P}$ *the edge they have in common. This is a one to one correspondence between edges and pairs of distinct* 1*-factors.*

Proof: The Petersen graph can be defined as follows (see [3]): the nodes of $\mathcal{P}$ are the pairs of elements in $\mathbb{N}_5 = \{1,2,3,4,5\}$, where two nodes $\{i,j\}$ and $\{h,k\}$ of $\mathcal{P}$ are adjacent if and only if they are disjoint. Thus $\mathcal{P}$ is a regular graph of valency $\binom{5-2}{2} = 3$ with $\binom{5}{2} = 10$ nodes and $\frac{10 \cdot 3}{2} = 15$ edges.

Let $e = \{i,j\}\{h,k\}$ be any edge of $\mathcal{P}$ and let x be the single element in $\mathbb{N}_5 \setminus \{i,j,h,k\}$. A 1-factor containing e can not contain any other edge with an endpoint in $\{i,j\}$ or $\{h,k\}$. Moreover, no 1-factor containing e can contain $\{i,h\}\{j,k\}$ or $\{i,k\}\{j,h\}$. Indeed, assume on the contrary and without lost of generality to have a 1-factor containing both e and $\{i,h\}\{j,k\}$. Then nodes $\{j,x\}$ and $\{h,x\}$ are both matched with node $\{i,k\}$, a contradiction.

We conclude that any 1-factor containing e has precisely 4 edges among the remaining 8 edges. These 8 edges form a circuit, whose nodes are listed in order here below:

$$\{j,k\},\ \{i,x\},\ \{j,h\},\ \{k,x\},\ \{i,h\},\ \{j,x\},\ \{i,k\},\ \{h,x\}$$

Therefore the 1-factors of $\mathcal{P}$ containing e are precisely two and have no edge in common other than e.

The number of distinct 1-factors of $\mathcal{P}$ is therefore $\frac{2|E(\mathcal{P})|}{5} = 6$. The function which associates to each edge e of $\mathcal{P}$ the pair of 1-factors containing e is injective, for we said that the two 1-factors have no edge in common other than e. Since $|E(\mathcal{P})| = 15 = \binom{6}{2}$ is the number of pairs of 1-factors, the function is also surjective and bijective. $\square$

An immediate consequence of Property 2.1 is the following:

Lemma 2.2 *Let* M_1, M_2 *be two edge-disjoint* 1*-factors of* $\mathcal{P}(r) = \mathcal{P} + (r-3)M$*. Then either* $M_1 = M$ *or* $M_2 = M$*.*

The following lemma is involved in a first construction of indecomposable r-graphs:

Lemma 2.3 *Assume* $\mathcal{P}(r) = G_1 + G_2$ *where, for* $i = 1,2$, G_i *is an* r_i*-graph. Then there exist* k_1, k_2 *such that* $G_1 = \mathcal{P} + k_1 M$ *and* $G_2 = k_2 M$ *or viceversa.*

Proof: It suffices to show that $\mathcal{P} \setminus M$ is contained in either G_1 or G_2. Assume the contrary and let e_i $(i = 1,2)$ be an edge of G_i contained in $\mathcal{P} \setminus M$. Let M_i be a 1-factor of G_i containing

e_i. Thus M_1 and M_2 are two edge-disjoint 1-factors of $\mathcal{P}(r)$ contradicting Lemma 2.2. $\square$

A *tight cut* in an r-graph is an edge set of the form $\partial(S)$ where S is a set of nodes of odd cardinality and $d(S) = r$. The following proposition plays a central role in proving that the graphs to be constructed in the next section are indecomposable:

Proposition 2.4 *Let $\partial(S)$ be a tight cut in an r-graph G. Then the graph G^* obtained from G by identifying all nodes in S is an r-graph.*
Assume $G = G_1 + \ldots + G_h$ where G_i is an r_i-graph ($i = 1, \ldots, h$). Note that $d_{G_i}(S) = r_i$ ($i = 1, \ldots, h$). Let $G_1^*, \ldots, G_h^*$ be the graphs obtained from $G_1, \ldots, G_h$ by identifying all nodes in S. As above G_i^* is an r_i-graph ($i = 1, \ldots, h$). Moreover $G^* = G_1^* + \ldots + G_h^*$.

Lemma 2.5 *Let G be a graph and $S \subseteq V(G)$ with $|S|$ odd. Let G_S and $G_{\overline{S}}$ be the graphs obtained from G by identifying all nodes in S and in $\overline{S} = V(G) \setminus S$ respectively. If G_S and $G_{\overline{S}}$ are both r-graphs then G is an r-graph.*

Proof: Obviously G is r-regular. Assume $d_G(X) < r$ and $|X|$ odd. Exchanging S and $\overline{S}$ if necessary, we can assume that $|X \cap S|$ and $|X \cup S|$ are odd. By submodularity of d_G, $d_G(X \cap S) + d_G(X \cup S) \leq d_G(X) + d_G(S) < r + r$. So, either $d_{G_{\overline{S}}}(X \cap S) = d_G(X \cap S) < r$ or $d_{G_S}(X \cup S) = d_G(X \cup S) < r$ contrary to the assumption that both G_S and $G_{\overline{S}}$ are r-graphs. $\square$

3 An infinite family of counterexamples

Let r be any integer with $r \geq 4$. In this section we construct an unslicable r-graph $U(r)$.

By Lemma 2.3, the only way to decompose the r-graph $\mathcal{P}(r) = \mathcal{P} + (r-3)M$ into an $(r-1)$-graph and a 1-factor is $\mathcal{P}(r) = \mathcal{P}(r-1) + M$. Let $e = uv$ be any edge of $\mathcal{P}$ which is not in M (all such edges are equivalent by symmetry). Take r distinct copies $C_1, \ldots, C_r$ of $\mathcal{P}(r) \setminus e$. For $i = 1, \ldots, r$, copy C_i contains two nodes of degree $r-1$, namely u_i and v_i. Let x and y be two nodes not belonging to $V(C_i)$ for any i. The r-graph $U(r)$ is obtained from the components $C_1, \ldots, C_r$ and the nodes x, y by adding all edges xu_i and yv_i for $i = 1, \ldots, r$.

When $G = G_1 + G_2$ we say that G_2 is the *complement* of G_1 in G.

Claim 3.1 *The r-graph $U(r)$ is unslicable.*

Proof: Any 1-factor F of $U(r)$ contains an edge incident with x. Assume without lost of generality that $xu_1 \in F$. By parity reasons $yv_1 \in F$. Therefore $F \cap E(C_1) + u_1v_1$ is a 1-factor of $C_1 + u_1v_1$. Moreover the complement of $F \cap E(C_1) + u_1v_1$ in $C_1 + u_1v_1$ is an $(r-1)$-graph. Apply Lemma 2.3. $\square$

Evidently "being indecomposable" is a stronger property than "being unslicable". Since every 2-graph is decomposable the two properties are equivalent for $r < 6$. To prove them to be distinct for every $r \geq 6$, we show that the unslicable r-graph $U(r)$ is decomposable whenever $r \geq 6$. Indeed $U(r) = G_1(r) + G_2(r)$, where $G_1(r)$ is the 3-graph collecting a copy of $\mathcal{P}$ from components C_1, C_2 and C_3 and a $3M$ from every other component. Also $G_2(r)$, which results as the complement of $G_1(r)$ in $U(r)$, is an $(r-3)$-graph.

4 The indecomposable r-graph $G(r)$

Let r be any integer with $r \geq 4$. In this section we construct an indecomposable r-graph $G(r)$.

Let z be any node of $\mathcal{P}(r) = \mathcal{P} + (r-3)M$ (all nodes are equivalent under isomorphism). Let x, a, b be the neighbors of z, where $xz \in M$. We indicate with $< a, x, b >^{(r)}$ the graph obtained from $\mathcal{P}(r)$ by removing node z. Symbolic representation of $< a, x, b >^{(r)} = \mathcal{P}(r) \setminus z$ is indicated in Figure 2.

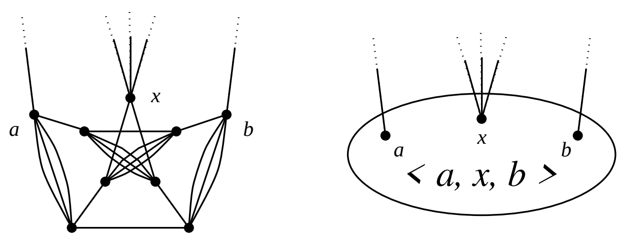

Figure 2: $\mathcal{P}(r) \setminus z = < a, x, b >^{(r)}$.

Let $< a_1,\, x_1,\, b_1 >_1^{(r)}, \ldots, < a_r,\, x_r,\, b_r >_r^{(r)}$ be distinct copies of $< a, x, b >^{(r)}$. Define C to be the set of edges $\{b_i a_{i+1} \ : \ i = 1, \ldots, r-1\} \cup \{b_r a_1\}$. Let $\{v_1, ..., v_{r-2}\}$ be a set of nodes disjoint from all the $V(< a_i,\, x_i,\, b_i >_i^{(r)})$. For $i = 1, \ldots, r-2$ define E_i as the set of edges $\{v_i x_j \ : \ j = 1, \ldots, r\}$. The graph $G(r)$ is obtained from the components $< a_1,\, x_1,\, b_1 >_1^{(r)}, \ldots, < a_r,\, x_r,\, b_r >_r^{(r)}, v_1, ..., v_{r-2}$ by adding all the edges in $C \cup E_1 \cup E_2 \cup \ldots \cup E_{r-2}$.
For example $G(4)$ and $G(5)$ are shown in Figure 3.

By Lemma 2.5 $G(r)$ is an r-graph.

Claim 4.1 *For every integer $r \geq 4$, $G(r)$ is indecomposable.*

Proof: Assume $G(r) = G_1 + G_2$ with G_1 r_1-graph and G_2 r_2-graph. By Lemma 2.3 and Proposition 2.4 either $C \subseteq G_1$ or $C \subseteq G_2$. Assume without lost of generality that $C \subseteq G_1$. Proposition 2.4 implies:

$$\begin{aligned} |E(G_2) \cap (E_1 \cup \ldots \cup E_{r-2})| &= \textstyle\sum_{i=1}^{r} d_{G_2 \setminus C}(V(< a_i,\, x_i,\, b_i >_i^{(r)})) = \\ &= \textstyle\sum_{i=1}^{r} d_{G_2}(V(< a_i,\, x_i,\, b_i >_i^{(r)})) = r r_2 \end{aligned} \tag{2}$$

However $|E(G_2) \cap E_i| = d_{G_2}(v_i) = r_2$ $(i = 1, \ldots, r-2)$ implies $|E(G_2) \cap (E_1 \cup \ldots \cup E_{r-2})| = (r-2) r_2$, in contradiction with (2). We conclude that $G(r)$ is indecomposable. $\square$

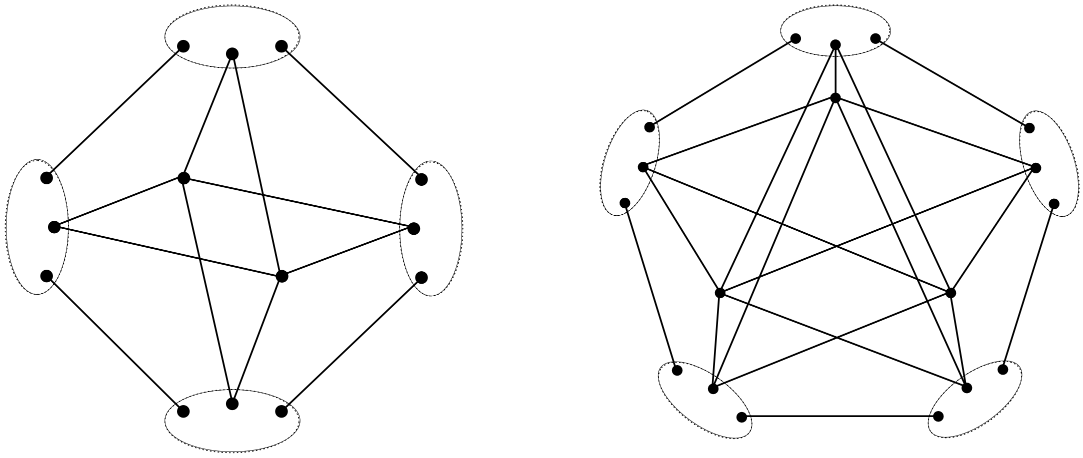

Figure 3: Graphs $G(4)$ and $G(5)$.

5 More indecomposable r-graphs

Let G_1, G_2 be two node-disjoint r-graphs. Choose $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$. Let G be any r-regular graph obtained from G_1 and G_2 by first removing nodes v_1, v_2 and then adding some new edges with one endpoint in $V(G_1) \setminus \{v_1\}$ and the other in $V(G_2) \setminus \{v_2\}$. We say that G has been obtained by *splicing* G_1 and G_2 (at v_1, v_2). Proposition 2.4 and Lemma 2.5 imply the following:

Lemma 5.1 *If G has been obtained by splicing two r-graphs G_1 and G_2 then G is an r-graph. Moreover if G_1 is indecomposable then G is indecomposable.*

Hence we have an infinite number of indecomposable r-graphs for any given integer $r \geq 4$. Let K_n be the complete graph on n nodes. When r is odd then $S_r = K_{r+1}$ is a *simple* (no parallel edges) r-graph. When r is even then let M be any matching of K_{r+1} with $|M| = \frac{r}{2}$. Let S_r be the graph obtained from K_{r+1} by first subdividing every edge in M into two edges and next identifying all nodes of degree two so introduced. Again S_r is a simple r-graph. To obtain a simple indecomposable r-graph start from any indecomposable r-graph and while some parallel edges are incident with a node x splice at x with some simple r-graph like S_r.

The smallest indecomposable r-graphs (for $r = 4, 5, 6$), we were able to construct, are given in Figure 4. The first graph in Figure 4 is in fact the smallest possible counterexample to Conjecture 1.1 (see [8]).

6 Poorly matchable r-graphs: a recursive construction

An r-graph G is said *poorly matchable* if G does not contain two disjoint 1-factors.
Since every r-graph has a 1-factor, every poorly matchable r-graph is indecomposable. Thus "being poorly matchable" is a stronger property than "being indecomposable". For $r = 3$ the two properties are equivalent since the presence of two disjoint 1-factors implies 3 edge

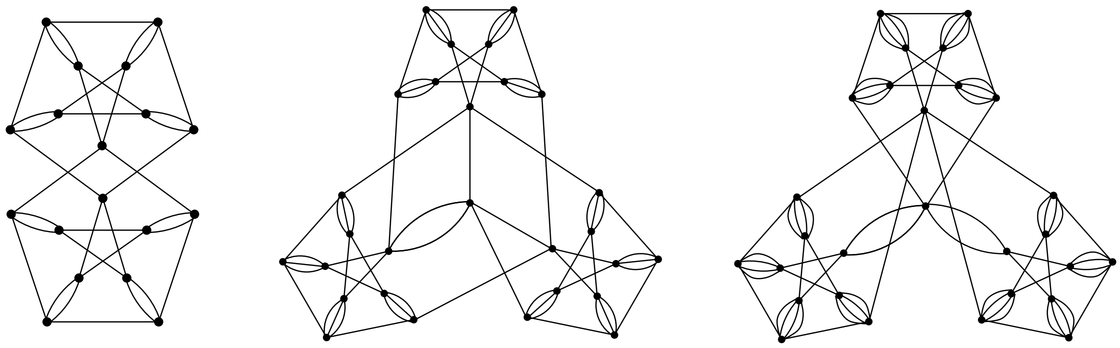

Figure 4: Small indecomposable r-graphs.

colorability. However the two properties are distinct for every $r \geq 4$. This is proven in Subsection 6.1 by showing that, for all $r \geq 4$, the indecomposable r-graph $G(r)$ from Section 4 has two edge-disjoint 1-factors.

Subsection 6.2 gives a poorly matchable r-graph G^r for every integer $r \geq 4$. The construction we propose is however recursive and the size of G^r is probably exponential in r. (Whereas for $G(r)$ we have $|V(G(r))| = (10-1)r + (r-2) = 10r - 2$, which is linear in r).

The following three statements are equivalent for an r-graph G: (i) G is poorly matchable. (ii) G does not contain a spanning 2-graph. (iii) G does not contain two disjoint spanning r-graphs.

Therefore the existence of a poorly matchable r-graph for every integer $r \geq 3$ has the following consequence:

Proposition 6.1 *There exists no constant K such that every r-graph with $r > K$ can be expressed as the sum of a K-regular graph and an $(r-K)$-graph.*

Proof: For any given $K \in \mathbb{N}$, consider a poorly matchable $(K+2)$-graph. □

We propose the following conjecture:

Conjecture 6.2 *Every $3r$-graph is the sum of r 3-regular graphs.*

6.1 Two edge-disjoint 1-factors in $G(r)$

This subsection gives two edge-disjoint 1-factors in $G(r)$: $M_1(r)$ and $M_2(r)$.
To specify $M_1(r)$ and $M_2(r)$ we rely on the description of $G(r)$ in Section 4.

- $M_1(r) \cap C = \{b_{r-1}a_r\}$ whereas $M_2(r) \cap C = \{b_1 a_2\}$.
- For $1 \leq i \leq r-2$ $\partial_{M_1(r)}(v_i) = v_i x_i$ whereas $\partial_{M_2(r)}(v_i) = v_i x_{i+2}$.
- For $3 \leq i \leq r-2$ and $j = 1, 2$ $M_j(r) \cap E(< a_i, x_i, b_i >_i^{(r)})$ is a copy of M with z removed. (Remember $< a, x, b >^{(r)}$ is $\mathcal{P}(r) = \mathcal{P} + (r-3)M$ with a node z removed).

It remains to determine $M_1(r)$ and $M_2(r)$ on $E(<a_r,\, x_r,\, b_r>_r^{(r)}) \cup E(<a_{r-1},\, x_{r-1},\, b_{r-1}>_{r-1}^{(r)})$ and $E(<a_1,\, x_1,\, b_1>_1^{(r)}) \cup E(<a_2,\, x_2,\, b_2>_2^{(r)})$: both of them are described by Figure 5.

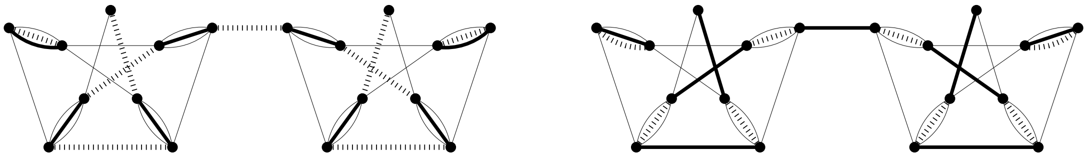

Figure 5: $M_1(r)$ and $M_2(r)$ on the first and last two $<a, x, b>^{(r)}$ components.

As an example Figure 6 shows $M_1(r)$ and $M_2(r)$ in $G(r)$ for $r = 4, 5$.

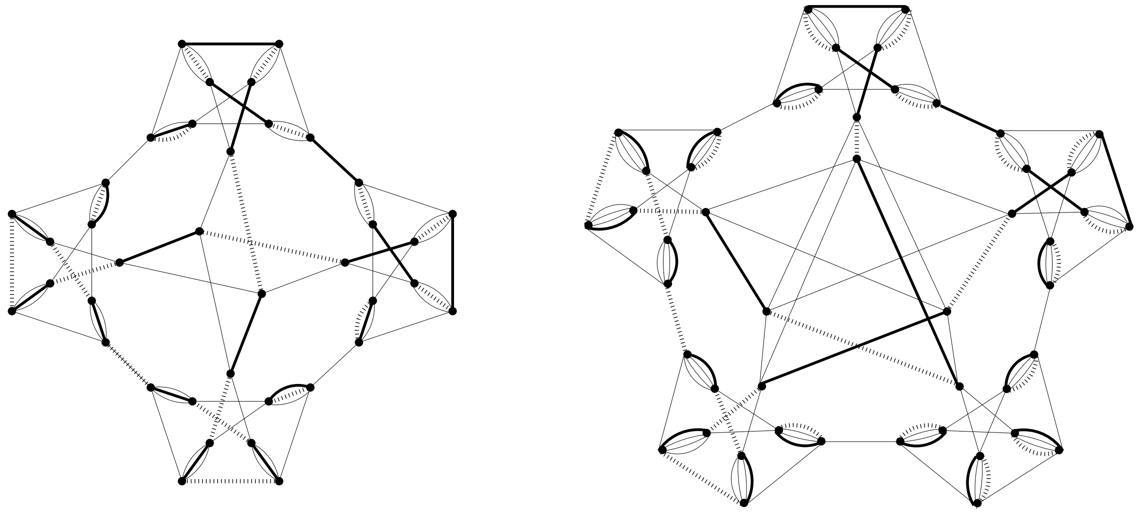

Figure 6: M_1 and M_2 in $G(4)$ and $G(5)$.

6.2 Constructing the poorly matchable r-graph G^r

The Petersen graph $\mathcal{P}$ is an indecomposable 3-graph and hence a poorly matchable 3-graph. Thus let $G^3 = \mathcal{P}$. This subsection shows how to construct a poorly matchable r-graph G^r from a poorly matchable $(r-1)$-graph G^{r-1} whenever $r \geq 4$.

Let G be an r-graph and let $e, f = uv$ be two parallel edges of G. Take a copy of $\mathcal{P}(r)$ node-disjoint from G and choose a node z in $\mathcal{P}(r)$. Let x be the node of $\mathcal{P}(r)$ joined to z by $r-2$ edges and let a, b be the other two nodes of $\mathcal{P}(r)$ adjacent to z. Remove node v from G and node z from $\mathcal{P}(r)$. Next, add edges ua and ub. Finally, add a set of edges with one endpoint in $V(\mathcal{P}(r)) \setminus \{z\}$ and the other in $V(G) \setminus \{v\}$ as to obtain an r-regular graph G^*. We say that G^* is obtained from G by *$\mathcal{P}$-splicing at v distinguishing e and f*.
Note that $\mathcal{P}$-splicing is a particular instance of the splice operation defined in Section 5. Hence, by Lemma 5.1, G^* is an r-graph. Moreover we have the following:

Lemma 6.3 *If G^* has two edge-disjoint 1-factors then G has two edge-disjoint 1-factors M_1 and M_2 such that $\{e,f\} \not\subseteq M_1 \cup M_2$.*

Proof: Let M_1^*, M_2^* be two edge-disjoint 1-factors of G^*. Let $K = \partial_{G^*}(V(\mathcal{P}(r)) \setminus \{z\})$ denote the set of edges which have been added by $\mathcal{P}$-splicing. Since $|V(\mathcal{P}(r)) \setminus \{z\}| = 9$ is odd then $|M_1^* \cap K|$ and $|M_2^* \cap K|$ are both odd. Hence $|M_1^* \cap K|, |M_2^* \cap K| \geq 1$. In fact $|M_1^* \cap K|, |M_2^* \cap K| = 1$ since all edges in K are incident either with x or with u. Therefore, after identifying all nodes in $V(\mathcal{P}(r)) \setminus \{z\}$, M_1^* and M_2^* become two edge-disjoint 1-factors M_1 and M_2 of G.
If $\{e,f\} \subseteq M_1 \cup M_2$ then $\{e,f\} \subseteq M_1^* \cup M_2^*$ and after identifying in G^* all nodes in $V(G) \setminus \{v\}$, M_1^* and M_2^* become two edge-disjoint 1-factors of $\mathcal{P}(r)$ contradicting Lemma 2.2. □

We are now ready for the recursive construction: Let G^{r-1} be a poorly matchable $(r-1)$-graph. Let M be a 1-factor of G^{r-1}. Then $H^r = G^{r-1} + M$ is an r-graph. Let $\overline{M}$ be the set of those edges of G^{r-1} which have multiplicity 1 in G^{r-1} and 2 in H^r. (M will stand for the edges in $H^r \setminus G^{r-1}$). Let $V_{\overline{M}}$ be a node cover for $\overline{M}$ with $|V_{\overline{M}}| = |\overline{M}|$. Obtain G^r from H^r by $\mathcal{P}$-splicing at every node $\overline{v} \in V_{\overline{M}}$ distinguishing the unique edge in $\partial_M(\overline{v})$ and the unique edge in $\partial_{\overline{M}}(\overline{v})$. If G^r has two edge-disjoint 1-factors then, by Lemma 6.3, H^r has two edge-disjoint 1-factors M_1 and M_2 with $(M_1 \cup M_2) \cap (M \cup \overline{M})$ having no parallel edges. But then, by eventually substituting the edges in M with those in $\overline{M}$ having the same endpoints, M_1 and M_2 are two edge-disjoint 1-factors of G^{r-1}. We conclude that G^r is a poorly matchable r-graph.

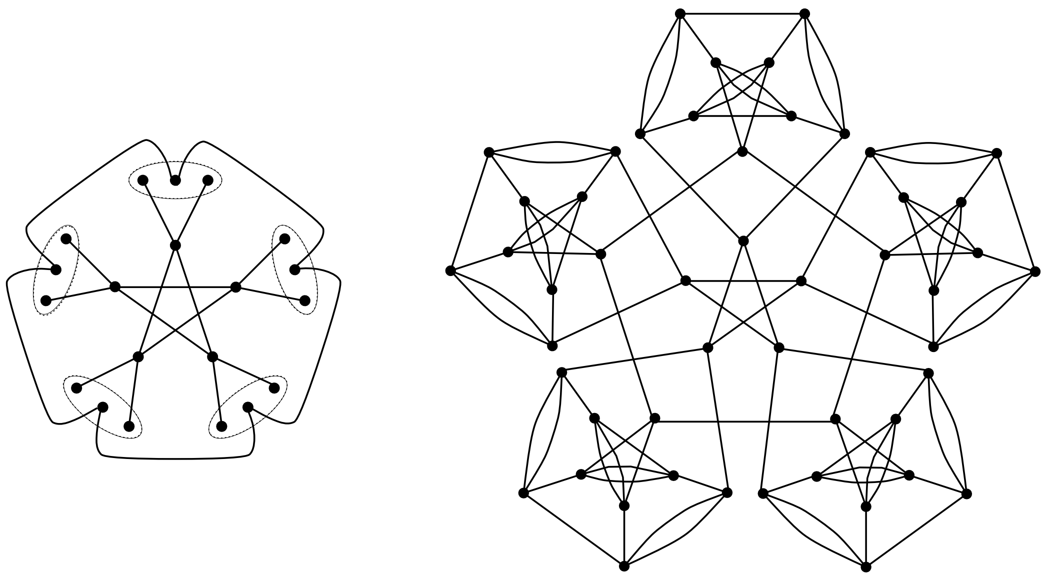

Figure 7: The poorly matchable 4-graph G^4.

The construction proposed is not deterministic. Non-isomorphic r-graphs can in fact be obtained starting from a same $(r-1)$-graph. From $\mathcal{P}$ however a sole graph G^4 can be derived. Graph G^4 has 50 nodes. We have found no poorly matchable 4-graph on less than 50 nodes.

7 Avoiding tight cuts

All the unslicable, indecomposable or poorly matchable r-graphs seen until now contain some tight cuts. This section gives a poorly matchable 4-graph without tight cuts as counterexample to the following conjectures:

Conjecture 7.1 *Every unslicable r-graph with $r \geq 4$ has a tight cut.*

Conjecture 7.2 *Every indecomposable r-graph with $r \geq 4$ has a tight cut.*

Conjecture 7.1 is still strong enough to imply Conjecture 1.2. Conjecture 7.2 is still strong enough to make Conjecture 1.3 imply Conjecture 1.4.

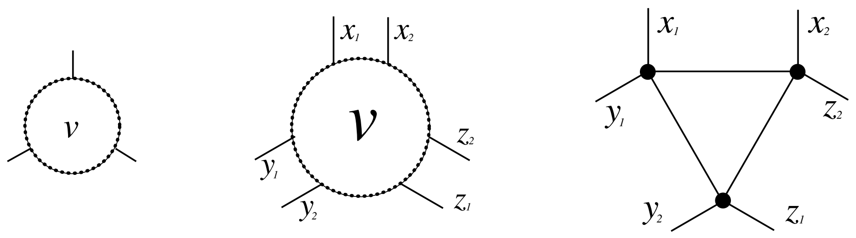

Figure 8: The node gadget.

We shall use a technique with some similarities to *superposition*. Superposition is a method for constructing snarks introduced in [4, 5] as a practical and effective mean for capturing and exploiting "global type conditions" as suggested in [2].

The idea is to take a poorly matchable 3-graph, like $\mathcal{P}$, as skeleton. Next, every node in the skeleton is replaced by a distinct copy of the "node gadget" shown in Figure 8 and every edge in the skeleton is replaced by a distinct copy of the "edge gadget" shown in Figure 9.

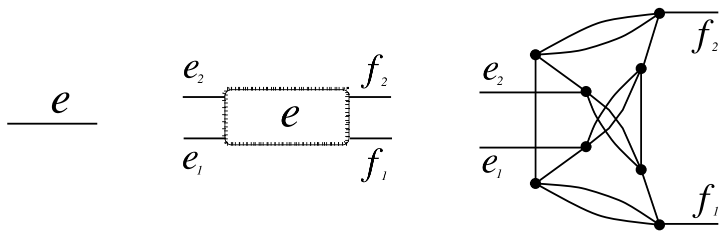

Figure 9: The edge gadget.

The skeleton acts like a map, telling how edge and node gadgets are mutually connected. The resulting graph G_4 is shown in Figure 10.

One can check that G_4 is a 4-graph without tight cuts. Let M_1 and M_2 be two edge-disjoint 1-factors of G_4. Let $\phi : E(G_4) \mapsto \{(0,0),(0,1),(1,0)\}$ be defined as follows:

$$\phi(e) = \begin{cases} (1,0) & \text{if } e \in M_1 \\ (0,1) & \text{if } e \in M_2 \\ (0,0) & \text{if } e \notin M_1 \cup M_2 \end{cases}$$

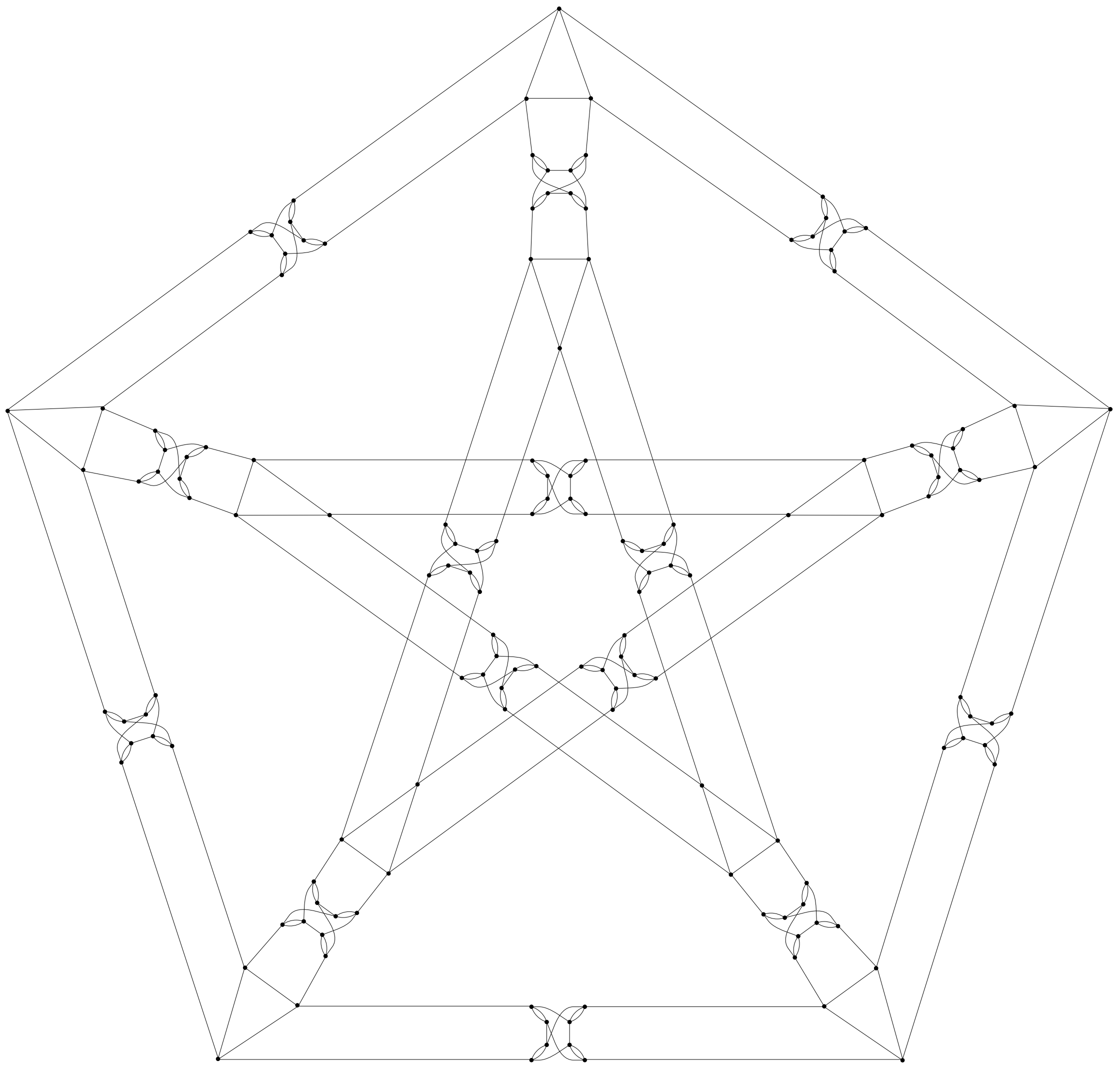

Figure 10: A poorly matchable 4-graph without tight cuts.

When $F \subseteq E(G_4)$ we define $\phi(F) = \sum_{e \in F} \phi(e)$, where the sum is componentwise and modulo 2. Then ϕ satisfies the following conditions:

EVEN SET: Let S be an even set of nodes. Then $\phi(\partial(S)) = (0,0)$.

ODD SET: Let S be an odd set of nodes. Then $\phi(\partial(S)) = (1,1)$.

EDGE GADGET: $\phi(\{e_1, e_2\}) = \phi(\{f_1, f_2\}) \neq (1,1)$.

Proof: Edge gadgets contain an even number of nodes. Hence $\phi(\{e_1, e_2\}) = \phi(\{f_1, f_2\})$ by the Even Set Condition. Moreover $\phi(\{e_1, e_2\}) = \phi(\{f_1, f_2\}) \neq (1,1)$ by Lemma 2.2. □

NODE GADGET: $\{\phi(\{x_1, x_2\}), \phi(\{y_1, y_2\}), \phi(\{z_1, z_2\})\} = \{(0,0),(0,1),(1,0)\}$.

Proof: By the Odd Set Condition $\phi(\{x_1, x_2\}) + \phi(\{y_1, y_2\}) + \phi(\{z_1, z_2\}) = (1,1)$. By the Edge Gadget Condition $\phi(\{x_1, x_2\}), \phi(\{y_1, y_2\}), \phi(\{z_1, z_2\}) \neq (1,1)$. Thus $(1,0),(0,1) \in \{\phi(\{x_1, x_2\}), \phi(\{y_1, y_2\}), \phi(\{z_1, z_2\})\}$.

But then $\{\phi(\{x_1, x_2\}), \phi(\{y_1, y_2\}), \phi(\{z_1, z_2\})\} = \{(0,0),(0,1),(1,0)\}$. □

Let e be any edge of $\mathcal{P}$. Let e_1, e_2 be two edges of G_4 entering the edge gadget relative to e on a same side. Define $\phi'(e) = \phi(\{e_1, e_2\})$. By the Edge Gadget Condition ϕ' is well defined. By the Edge Gadget and Node Gadget Condition ϕ' is a coloring of the edges of $\mathcal{P}$ by colors $(0,0)$, $(0,1)$ and $(1,0)$. Since $\mathcal{P}$ is not 3 edge colorable, G_4 is poorly matchable.

We remind that a *Fulkerson coloring* of an r-graph G is a decomposition of $2G$ into 1-factors.

Observation 7.3 *Let H be an indecomposable 3-graph and let H_4 be the poorly matchable 4-graph obtained from H as skeleton graph through the above described construction with Node and Edge Gadgets as in Figures 8 and 9. From a Fulkerson coloring for H one can derive a Fulkerson coloring for H_4 as shown in Figure 11.*

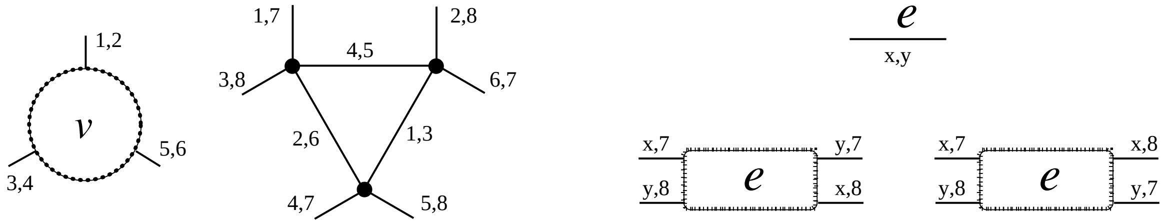

Figure 11: Deriving a Fulkerson coloring for H_4 from one for H.

8 Poorly joinable r-graphs: a positive result

Given a graph G, a *join* of G is a set of edges $J \subseteq E(G)$ such that an odd number of edges in J is incident with each node in $V(G)$. An r-graph G is *poorly joinable* if every two joins of G intersect. By definition, "poorly joinable" $\Rightarrow$ "poorly matchable". For a 3-graph the two properties are equivalent. In an early attempt of extending the construction proposed in the previous section and obtain poorly matchable r-graphs without tight cuts for $r > 4$ we found ourselves looking for poorly joinable r-graphs with $r > 3$. This approach ended in the following proposition.

Proposition 8.1 *There exists no poorly joinable r-graph for $r > 3$.*

Proof: Let G be an r-graph with $r > 3$. If r is even then let M be any 1-factor of G and observe that M and $G \setminus M$ are two disjoint joins of G. So r is odd and we want to prove that G is the sum of three disjoint joins of G. We can assume that G is 4-edge connected since 2-edge cuts give rise to an easy decomposition of the problem. By [13], G contains two disjoint spanning trees. So G contains two disjoint joins. $\square$

In our opinion, the next thing which makes sense trying to pack into r-graphs are joins. To stress this belief we pose the following:

Question 3 *Which functions $f(r)$ are there such that every r-graph with $r \geq \overline{r}$ admits $f(\overline{r})$ disjoint joins? Could $f(r) = \lfloor \frac{r}{2} \rfloor$ possibly work? What about $f(r) = r - 2$?*

The above question becomes even more relevant in view of its extension to grafts by arguments as given in [8].

9 Open Problems

The Petersen graph seems quite unavoidable in all our counterexamples. This suggests to generalize Tutte's conjecture as follows:

Conjecture 9.1 *Every indecomposable r-graph has a Petersen minor.*

Sebő pointed out that Conjecture 9.1 is equivalent to the following conjecture of Lovász:

Conjecture 9.2 *The 1-factors of a graph with no Petersen minor form a Hilbert basis.*

The following questions are left open:

Question 4 *Does there exist a constant $\overline{r}$ such that every unslicable r-graph with $r \geq \overline{r}$ contains some tight cuts?*

Question 5 *Does there exist a constant $\overline{r}$ such that every indecomposable r-graph with $r \geq \overline{r}$ contains some tight cuts?*

We propose the following:

Conjecture 9.3 *The answer to Question 4 is positive and in fact we can take $\overline{r} = 5$.*

In [11], Seymour mentioned to have proven Conjecture 1.2 for $r \leq 6$. In [12], Seymour gave a second proof that Conjecture 1.2 holds for $r \leq 6$. In fact, as a consequence of the approximation algorithm to edge color multigraphs described in [10], Conjecture 1.2 holds for $r \leq 12$. Therefore a positive answer to Question 5 with $\overline{r} \leq 13$ would imply Conjecture 1.2.

As far as we know the following is still open:

Conjecture 9.4 *Every planar r-graph is decomposable (and hence is r edge colorable).*

Conjecture 9.5 *The 1-factors of a planar graph form a Hilbert basis.*

Finally, we insist on a conjecture introduced in Section 7:

Conjecture 9.6 *Every r-graph contains $r - 2$ disjoint joins.*

10 Acknowledgments

We thank Michele Conforti for suggesting the problem and Ajai Kapoor for his assistance in simplifying the proposed family of indecomposable r-graphs. Bill Jackson suggested how to prove that there exists no poorly joinable r-graph for $r > 3$. Poul Seymour, besides posing the problem in [11], contributed with a second original and unpublished proof in [12].

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