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NP–COMPLETE PROBLEM: PARTITION INTO TRIANGLES

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1 Partition into Triangles

PARTITION INTO TRIANGLES (PT) is the following problem:

Instance: A graph $G = (V, E)$, with $|V| = 3q$ for some integer q.

Question: Can the vertices of G be partitioned into q disjoint sets $V_1, V_2, \ldots, V_q$, each containing exactly 3 vertices, such that each of these V_i is the node set of a triangle in G?

In this document we will prove the following result:

Theorem 1.1 *PARTITION INTO TRIANGLES is NP-Complete even when the input graph is 3-partite.*

It is easy to see that PARTITION INTO TRIANGLES (PT) is in NP, since a nondeterministic algorithm needs only to guess q disjoint triples of vertices in G and check in polynomial time that each of them induces a triangle in G.

To show that any problem in NP reduces to PT we need only to show how to reduce a known NP-complete problem to PT. We have chosen to build a reduction from 3DM.

3DM is the following problem:

Instance: Three sets X, Y and W, each containing q elements, and a set $M \subseteq X \times Y \times W$ of triples (x_i, y_i, w_i).

Question: There exists a perfect matching for M, that is, a subset M_1 of M such that each element of X, Y and W is contained in exactly one triple in M_1?

3DM has been shown to be NP-complete in [Karp72].

I have tried to reduce 3DM to PT in polynomial time, with the technique of local replacement. To do this, I searched a gadget to represent every triple in M. This gadget is a piece of graph with certain properties, such that the composition of these pieces is a graph that is a yes-instance of PT if and only if M contains a perfect matching. The gadget I have found is given in Figure 1.

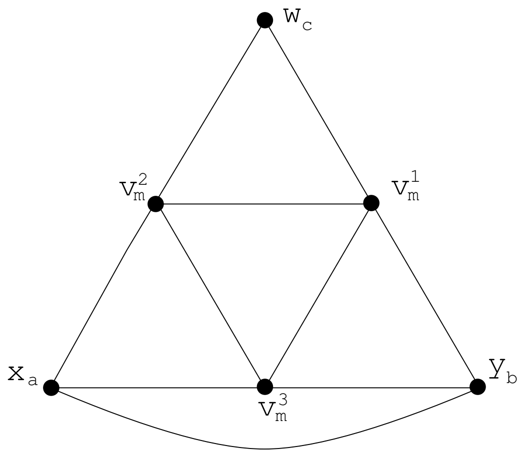

Figure 1: Gadget for 3DM $\propto$ PT.

Construction: Let $X = \{x_1, x_2, ..., x_q\}$, $Y = \{y_1, y_2, ..., y_q\}$, $W = \{w_1, w_2, ..., w_q\}$ and $M \subseteq X \times Y \times Z$ be the generic input instance of 3DM. Construct an initial set of vertices V, called the "public vertices", as the union of X, Y and W. For every triple $m = (x_a, y_b, w_c)$ in M, construct the graph $G_m = (V_m, E_m)$, depicted in Figure 1. Here V_m contains the three public vertices x_a, y_b and w_c plus three "private vertices" v_m^1, v_m^2 and v_m^3. The ten edges in E_m are $v_m^1 v_m^2$, $v_m^1 v_m^3$, $v_m^2 v_m^3$, $x_a v_m^3$, $x_a v_m^2$, $y_b v_m^3$, $y_b v_m^1$, $w_c v_m^1$, $w_c v_m^2$ and $x_a y_b$. The whole graph G_M is obtained from the union of all the gadgets G_m as follows: $G_M = (V_M, E_M) = (\bigcup_m V_m, \bigcup_m E_m)$.

(We call the vertices x_a, y_b and w_c public because they are the only nodes in V_m which are shared with other gadgets and which can be incident with further edges beyond those in E_m. The remaining three vertices are called private, because they cannot be connected to other gadgets by an edge in E_M.)

Observation 1.2 *Notice that, as a property of the construction, no edge in E_M has both end nodes in $X \cup W$ or in $Y \cup W$.*

Observation 1.3 *The graph G_M is 3-partite.*

Proof: Notice first that every gadget G_m is 3-partite: just consider the 3-partitioning $\{x_a, v_m^1\}$, $\{y_b, v_m^2\}$, $\{w_c, v_m^3\}$. Similarly, the whole graph G_M is 3-partite, just consider the three sets $X \cup \{v_m^1 : m \in M\}$, $Y \cup \{v_m^2 : m \in M\}$ and $W \cup \{v_m^3 : m \in M\}$. ∎

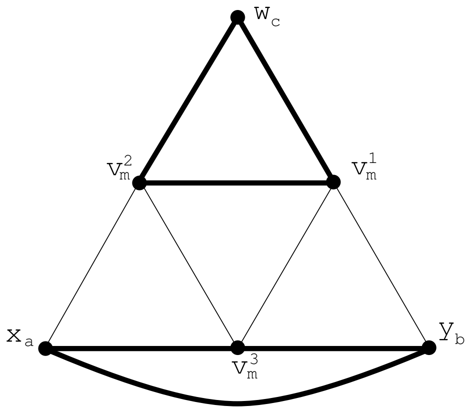

Figure 2: *Complete state.*

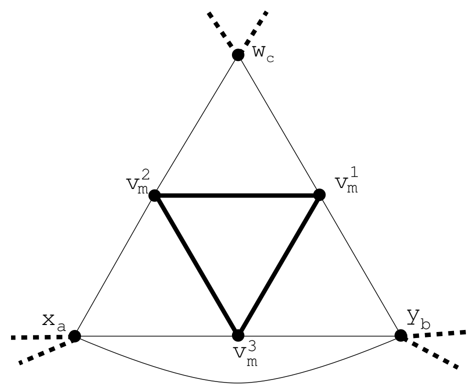

Figure 3: *Shared state.*

Properties of the gadget: Every single gadget G_m, taken in isolation, can be partitioned into the two triangles with the vertices $x_a y_b v_m^3$ and $w_c v_m^1 v_m^2$ as in Figure 2. I call this the *complete state*. A second possibility to cover the nodes of G_m by a packing of triangles in G_M is given by taking the triangle $v_m^3 v_m^1 v_m^2$ to cover the private nodes plus other triangles of G_M to cover the public nodes. This is called the *shared state* (Figure 3).

Claim 1.4 *Let m be any triple in M and let G_m be the corresponding gadget in G_M. The only possible triangle of G_M, that covers precisely one private vertex of G_m, is the triangle $x_a y_b v_m^3$.*

Proof: A triangle with only one private vertex v requires two public vertices that are adjacent and connected to v. By Observation 1.2, $x_a y_b$ is the *only* edge in G_M between public vertices of the gadget G_m. The claim follows. ■

Lemma 1.5 *Given any partition of the graph G_M into triangles, each gadget G_m must be either in the complete state or in the shared state.*

Proof: Consider a partition $\mathcal{P}$ of G_M into triangles and a generic gadget G_m. Assume G_m is not in the shared state. This implies that the three private vertices are not in the same triangle in $\mathcal{P}$. The only triangle of G_m that takes precisely one private vertex is the triangle $x_a y_b v_m^3$ (Claim 1.4). However, if $x_a y_b v_m^3$ is in $\mathcal{P}$, the gadget has to be in the *complete state*, since the remaining private vertices have no other possibility than to be covered by triangle $w_c v_m^1 v_m^2$. ■

Lemma 1.6 *The Graph G_M is a yes-instance of PT if and only if the 3DM-instance M has a matching.*

Proof: Assume that M has a perfect matching $M_1 \subseteq M$. Every time a triple of M is in M_1, set the corresponding gadget to the complete state. Otherwise set it in the shared state (the remaining vertices will be taken by other gadgets whose triple is in M_1). Notice that all private vertices are taken precisely once. Furthermore, the public vertices are also taken precisely once since, by definition of 3DM, all the triples in M_1 are disjoint and cover all elements in X, Y and W.

Conversely, assume that the graph G_M is partitionable into triangles. Let $\mathcal{P}$ be such a partition. As a consequence of Observation 1.2, no new triangles are introduced connecting gadgets to a graph G_M. Therefore every triangle in $\mathcal{P}$ belongs to a gadget G_m of G_M that is either in the *complete* or in the *shared state* (Lemma 1.5). Notice that all public vertices of G_M have to be covered by triangles in $\mathcal{P}$ that belong to gadgets in the *complete state*, because the *shared state* does not cover any public vertex. Call M_1 the set of all triples corresponding to the gadgets in the *complete state*. So M_1 covers all elements in X, Y and W (as a property of the construction these elements correspond to the public vertices of the gadgets).

Since the triangles in $\mathcal{P}$ are vertex-disjoint, no gadget in the *complete state* shares a vertex with other gadgets in the *complete state*. Therefore the triples of M_1 are also disjoint and so M_1 is a perfect matching for M. ■

Lemma 1.7 *The reduction needs only polynomial time in dependence to the size of the input instance of 3DM.*

Proof: For every triple in input we need to create one gadget. So the reduction requires only linear time and logarithmic space. ■

References

[Karp72] Karp, R.M. "Reducibility among combinatorial problems" pages 85–103 in *Complexity of Computer Computations*, Plenum Press, New York, 1972.