Multidimensional wave propagation using the Waveguide Mesh

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1 Introduction

Wave propagation along multidimensional elastic media, as well as many other *evolution problems* can be simulated in the discrete time and space using *numerical schemes*. Inevitable differences between "real" waves and their numerical simulations are a consequence of the formulation of the problem in a discrete multidimensional domain. These differences are commonly interpreted in terms of *errors* caused by numerical approximations.

The structure of a numerical scheme depends on the nature of the medium. Provided the physical characteristics of a medium through the *partial differential equation* that describes it, there may exist several schemes suitable for simulating that equation, each scheme exhibiting a specific error and, likewise, requiring a certain amount of computational resources. The user is asked to choose between different modeling solutions, starting from considerations of precision and cost of the simulation. Moreover, changes in the characteristics of the medium (such as, for example, its *density* along the space) determine variations in the corresponding model parameters, and sometimes require the design of more complex models coupling several numerical schemes at the same time.

A critical aspect concerning any numerical scheme is *stability*. A numerical scheme can model a differential equation in the discrete time and space once the stability condition has been shown to hold, otherwise a simulation with that scheme is prone to *explosion* even after few computations. In practice simulations running close to the stability limit may explode as well, due to inevitable numerical errors introduced by the finite precision of the arithmetic in a computer.

The simplest wave propagation case springs out when the elastic medium is isotropic and its parameters are constant along the space. Considering for example pressure in air under ideal conditions, a pressure function $p(x, y, z, t)$ giving the value of pressure p at the coordinate point (x, y, z) during time t is governed by the following partial differential equation (also known as the *wave equation*):

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} \qquad (1)$$

This equation is easily modeled by means of Finite Difference Schemes (FDS). The properties of these schemes have been precisely assessed: in particular, the stability condition can be straightforwardly figured out. For this reason, FDS's are employed whenever possible, including cases when Eq. (1) is not strictly verified along the whole domain.

Eq. (1) allows a solution in closed form. The structure of this solution highlights the existence of *3d traveling waves* propagating along the medium at traveling speed c. This evidence suggests the use of the Waveguide Mesh (WM) to model wave propagation directly, instead of modeling the differential equation using FDS's.

The wave-based modeling approach determines strong differences in the nature of the simulations. Nevertheless, some fundamental similarities between waveguide and finite difference models also hold. In the following of this chapter WM's will be analyzed, together with their relationships with FDS's and with the real world.

2 Preamble: wave propagation along one dimension

As a preliminary task, we outline the differences existing between the finite difference and the waveguide approach in the one-dimensional case.

We consider a restriction of Eq. (1) in the case where only one direction of propagation is allowed—the corresponding physical system being the *ideal string* or, dually, the *ideal acoustic tube*[1].

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = \frac{\partial^2 p}{\partial x^2} \tag{2}$$

In that case, we can reformulate the differential equation over a discrete time and space domain by substituting each differential term with a correspondent *central difference*:

$$\left.\frac{\partial p}{\partial l}\right|_{l=l_0} \longleftrightarrow \frac{p(l_0+\frac{L}{2}) - p(l_0-\frac{L}{2})}{L} \tag{3}$$

Central differences approximate the partial derivative of p calculated over l_0 with the incremental factor calculated over the neighboring points $l_0 - \frac{L}{2}$ and $l_0 + \frac{L}{2}$. This approximation is as more precise as smaller the *step size* L is.

Using Eq. (3) twice both over the terms in x and t in Eq. (2) (with steps Δ and T, respectively), implicitly assuming the discrete domain to be the following set:

$$\{\ldots, -k\Delta, \ldots, -\Delta, 0, \Delta, \ldots, k\Delta, \ldots\} \times \{T, 2T, \ldots, nT, \ldots\} \tag{4}$$

we come up with the following discrete-time formulation of the wave equation in the one-dimensional case:

$$\begin{aligned} &\frac{1}{c^2}\frac{p[k\Delta,(n-1)T] - 2p[k\Delta,nT] + p[k\Delta,(n+1)T]}{T^2} \\ &= \frac{p[(k-1)\Delta,nT] - 2p[k\Delta,nT] + p[(k+1)\Delta,nT]}{\Delta^2} \end{aligned} \tag{5}$$

Setting $c = \Delta/T$, and writing $p[k\Delta, nT] = p_{k,n}$, we finally solve the evolution problem for the 1d wave equation using central differences:

$$p_{k,n+1} = p_{k+1,n} + p_{k-1,n} - p_{k,n-1} \tag{6}$$

[1] We will refer to the latter system provided that we continue to consider pressure waves.

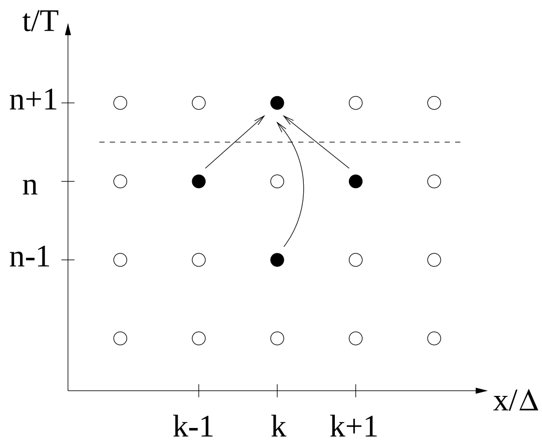

Figure 1: 1d wave equation: evolution problem using central differences.

Once the function p is known over all the spatial grid during time steps $n-1$ and n, then the spatial grid during time steps $n+1$ can be calculated. Also notice from Fig. 1 that computations evolve along two independent (interleaved) time-space grids, i.e., the computations of $p_{k,n+1}$ when k are even and when k are odd evolve independently one from the other.

We now show that an equivalent solution is achieved following a wave-based approach. Eq. (2) allows a solution in closed form, formed by the superposition of two functions representing waves traveling at speed c into opposite directions along the 1d medium:

$$p(x,t) = p_+(x-ct) + p_-(x+ct) \tag{7}$$

Discretizing this solution using the same space and time steps as before

$$p_{k,n} = p_+[k\Delta - cnT] + p_-[k\Delta + cnT]\ ,\ c = \Delta/T \tag{8}$$

and noticing that

$$\begin{aligned} p_-[(k+1)\Delta + cnT] &= p_-[k\Delta + c(n+1)T] \\ p_+[(k-1)\Delta - cnT] &= p_+[k\Delta - c(n+1)T] \end{aligned}$$

it can be easily verified by direct substitution that Eq. (8) satisfies condition (6). Hence, the finite difference and the wave based approaches are equivalent.

Eq. (8) is immediately implemented by the digital waveguide. We then conclude that *the evolution problem for the 1d wave equation using central differences can be solved more efficiently at the level of the solution of that equation.* In fact, some advantages of the wave approach are worth noting:

- the pressure function is computed just summing the values coming from two delay lines, instead of evolving a FDS;
- DW's form a system that is inherently stable (in fact their internal state does not change over the time);
- the waveguide approach leads to structures that allow an immediate physical interpretation of the process. This property turns out to be very useful when modeling more complex elastic systems.

2.1 Error of the WG scheme

The waveguide approach also allows an immediate analysis of the error coming from the scheme. Once the wave signals are sampled in the space grid so as not to create (spatial) aliasing, i.e., if the space step observes the non-aliasing conditions

$$\Delta < \frac{1}{2}\frac{1}{\max\{\xi_-, \xi_+\}} \tag{9}$$

where ξ_- and ξ_+ are the bandwidths of signals p_- and p_+, respectively, then the scheme is error-free. Moreover, as seen before the waveguide approach leads to an inherently stable scheme.

In other words, *aliasing is the only error generated by the scheme, and this error can be avoided in the case of a band-limited signal.*

3 Wave propagation along two dimensions: ideal membranes

Wave propagation along two dimensions involves *mechanical* rather than fluid-dynamical systems, such as membranes. For this reason, an adaptation of Eq. (1) to a 2d domain is meaningless in almost all cases, since it governs a pressure function. For simplicity, we will continue to deal with that equation, thinking of p as a mechanical quantity such as force or velocity over a membrane:

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} \tag{10}$$

Following the same outlining as done for the 1d case, we use central differences to develop a discrete formulation of Eq. (10)

$$\begin{gathered}\frac{1}{c^2}\frac{p[k\Delta_x, h\Delta_y, (n-1)T] - 2p[k\Delta_x, h\Delta_y, nT] + p[k\Delta_x, h\Delta_y, (n+1)T]}{T^2} = \\ \frac{p[(k-1)\Delta_x, h\Delta_y, nT] - 2p[k\Delta_x, h\Delta_y, nT] + p[(k+1)\Delta_x, h\Delta_y, nT]}{\Delta_x^2} \\ + \frac{p[k\Delta_x, (h-1)\Delta_y, nT] - 2p[k\Delta_x, h\Delta_y, nT] + p[k\Delta_x, (h+1)\Delta_y, nT]}{\Delta_y^2}\end{gathered} \tag{11}$$

that is defined over the following 2d-space and time domain:

$$\begin{aligned}&\{\ldots, -k\Delta_x, \ldots, -\Delta_x, 0, \Delta_x, \ldots, k\Delta_x, \ldots\} \times \\ &\{\ldots, -h\Delta_y, \ldots, -\Delta_y, 0, \Delta_y, \ldots, h\Delta_y, \ldots\} \times \\ &\{T, 2T, \ldots, nT, \ldots\}\end{aligned} \tag{12}$$

Setting $c = \sqrt{1/2}(\Delta/T)$ in the special case where the space steps are equal along the x and y directions, i.e., $\Delta_x = \Delta_y = \Delta$, and operating as in the previous Chapter, we finally find out the FDS solving the 2d wave equation:

$$p_{k,h,n+1} = \frac{1}{2}(p_{k+1,h,n} + p_{k-1,h,n} + p_{k,h+1,n} + p_{k,h-1,n}) - p_{k,h,n-1} \tag{13}$$

where it is $p_{k,h,n} = p[k\Delta, h\Delta, nT]$. We have thus solved the evolution problem for the 2d wave equation using central differences. The corresponding procedure

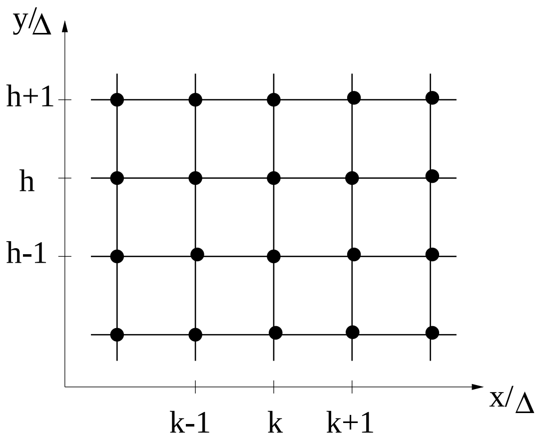

Figure 2: Square Waveguide Mesh (SWM).

generalizes to the x and y directions all the considerations made at the previous Chapter, already highlighted in Fig. 1. We remark that now the numerical scheme sets the propagation speed to $c = \sqrt{1/2}(\Delta/T)$.

The solution of Eq. (10) is formed by the superposition of an infinite number of functions representing waves traveling at speed c along all directions. Again, this suggests to treat the propagation problem modeling the solution rather than the equation.

N ideal vibrating strings joint together through a lossless junction can be modeled by N DW's connected via a Lossless Scattering Junction. Such a junction puts in relationship the N signals $p_{i-}[k\Delta_x, h\Delta_y, nT]$ coming out from the junction to the N signals $p_{i+}[k\Delta_x, h\Delta_y, nT]$ going to the junction via the instantaneous following relations:

$$p_{i-}[k\Delta_x, h\Delta_y, nT] = p[k\Delta_x, h\Delta_y, nT] - p_{i+}[k\Delta_x, h\Delta_y, nT] \quad (14)$$

$$p[k\Delta_x, h\Delta_y, nT] = \frac{1}{2N} \sum_{l=1}^{N} p_{l+}[k\Delta_x, h\Delta_y, nT] \quad (15)$$

where $p[k\Delta_x, h\Delta_y, nT]$ is the value of the signal at the junction, located at position $(k\Delta_x, h\Delta_y)$. Lossless Scattering Junctions thus allow to connect DW's in order to form various network designs (or *meshes*).

We can in particular design a network (called *square waveguide mesh*, briefly SWM) that solves the evolution problem for the 2d wave equation exactly as the FDS seen before. For this purpose we build a rectangular grid of DW's one unit long, each one connected through a Lossless Scattering Junction like in Fig. 2, where dots are junctions and lines are DW's. Adjacent junctions are separated by a distance $\Delta_x = \Delta_y = \Delta$.

Eqs. (14) and (15) can be written at time steps $nT - T$ and $nT + T$ over grid point $(k\Delta, h\Delta)$, and at time step nT over grid points $(k\Delta + \Delta, h\Delta)$, $(k\Delta - \Delta, h\Delta)$, $(k\Delta, h\Delta + \Delta)$, $(k\Delta, h\Delta - \Delta)$, then rearranged (with some patience) to figure out values for p that relate each other such as in Eq. (13). This proves that the two schemes are equivalent.

Even if scattering operations are needed to compute this scheme, the advantages outlined in the 1d case for the waveguide approach remain valid also

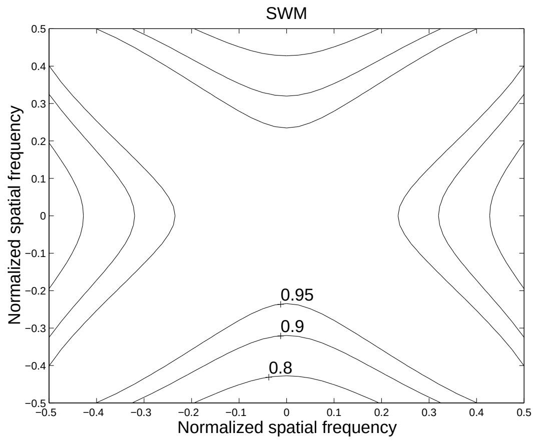

Figure 3: Dispersion error ratio in the SWM as a function of normalized spatial frequencies.

for the 2d case. More in general, propagation schemes dealing (when possible) with waves instead of normal signals exhibit more stable behavior and straightforward physical interpretation.

3.1 Error in the SWM

As long as we move to multidimensional domains, discrete models realized using DW's cannot track all the directions of propagation followed by real waves: For instance, the SWM only models four directions. This further approximation translates into a new kind of error, called *dispersion*. About this phenomenon, a *Von Neumann analysis* conducted over the scheme shows that the speed of propagation associated to each single frequency component of the signal p traveling along the mesh is no longer equal to $c = \sqrt{1/2}(\Delta/T)$. In fact, it decreases of an amount that depends on the (*spatial*) frequency of that signal component.

Fig. 3 shows a plot of the dispersion error ratio versus spatial frequencies, the value 1 meaning that a frequency component propagates at nominal speed c, otherwise this speed being lower. Spatial frequencies have been normalized to the value 0.5. It can be noticed that there is no dispersion at very low frequency, and for the components propagating along diagonal directions (whose dispersion can be read, indeed, along the diagonals of the plot in Fig. 3). This suggests that the most dispersed components are high-frequency sinusoidal waves traveling along directions parallel to the WG's. For this reason, dispersion can be arbitrarily reduced increasing the junction density until the error reaches the desired tolerance.

As in the 1d case, the SWM is inherently stable but exhibits aliasing error. Like dispersion, aliasing can be reduced to the desired value by increasing the junction density, and possibly canceled if the signal we want to study is (spatially) band-limited. Note that the correspondent number of computations in the SWM increases with the junction density such as $O(n^3)$: an increase of density can lead to serious computational problems.

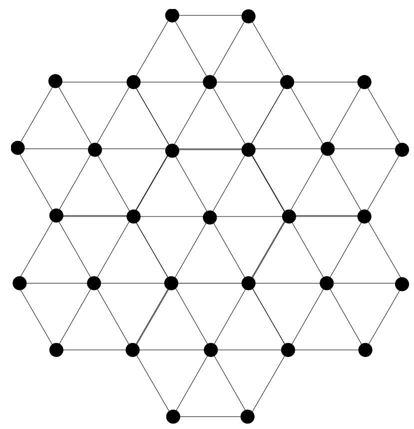

Figure 4: Triangular Waveguide Mesh (TWM).

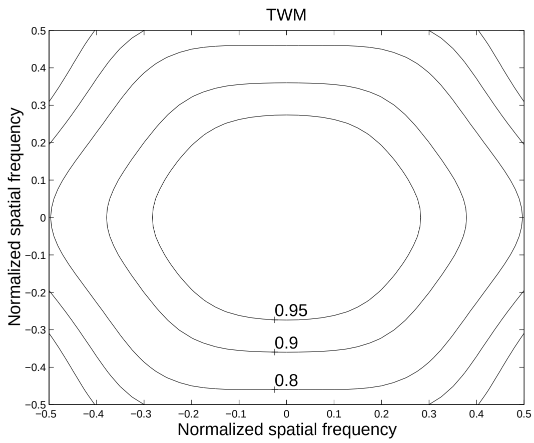

Figure 5: Dispersion error ratio in the TWM as a function of normalized spatial frequencies.

3.2 Reducing dispersion: the TWM

As long as dispersion is concerned, we can try to compensate it by increasing the number of direction of propagation in the waveguide model. The most effective method found so far consists in changing the geometry of the waveguide network, creating the mesh depicted in Fig. 4.

The Triangular Waveguide Mesh (TWM) has been shown to preserve the characteristics of stability and the correspondence with FDS's. As expected, it smooths the direction-dependent behavior of the dispersion function, now plotted in Fig. 5. Hence, in the TWM dispersion becomes to a large extent a function of the absolute frequency of the signal component. Notice that the absolute spatial frequency is directly related to temporal frequency: spatial components of the signal traveling along the mesh having the same absolute frequency will result in components having the same temporal frequency when they are picked up from a point of the mesh and sent to audio reproduction.

Aliasing error and, likewise, dispersion can be reduced in the TWM by in-

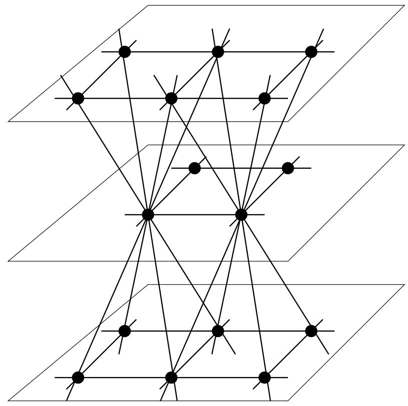

Figure 6: 3d Triangular Waveguide Mesh (3DTWM). All DW are one unit long.

creasing the junction density. Once again the computational complexity increases with the junction density such as $O(n^3)$.

4 Wave propagation along three dimensions: modeling of resonators and small enclosures

Numerical schemes devoted to model the 3d wave equation (1) are very important in the study of models of ideal resonators such as small enclosures, musical instrument bodies and resonating objects in general. These models allow a careful resonance analysis, although their computational requirements usually prevent their use in simulations of large resonators such as wide enclosures and musical halls. These applications require different approaches.

All the mathematical considerations made for the 2d case can be extended to the third dimension, once Eqs. (10), (11), (12) and (13) are completed with components accounting for the z direction. In particular, Eq. (10) gets back to the original formulation expressed by (1).

The 3d Square Waveguide Mesh (3DSWM) is obtained by connecting an orthogonal grid of Lossless Scattering Junctions in the 3d space by means of Eqs. (14) and (15), where adjacent grid points are separated by a distance Δ. It can be shown that it equals an FDS solving the evolution problem for the 3d wave equation using central differences, where waves travel at a nominal speed equal to $c = \sqrt{1/3}(\Delta/T)$.

Dispersion error is now defined on the (3d) domain of the spatial frequencies. Orthogonal projections of its plot calculated along diagonal axes equal the plot shown in Fig. 3. Again, dispersion is maximum along the main axes, confirming that it affects in particular high-frequency spatial components traveling along directions that are parallel to the main axes. An increase in the junction density lowers both dispersion and aliasing error, although the computational complexity grows up with density such as $O(n^4)$.

Also in the 3d case dispersion can be made less direction-dependent by changing the mesh geometry. We show in Fig. 6 the 3d counterpart of the TWM, called 3DTWM. Note that the axis orthogonal to the planes depicted in the figure has been stretched to make the sketch more clear: all the DW's are in fact one unit long. Also in this case, orthogonal projections of the 3DTWM dispersion plot equal the characteristic shown in Fig. 5.

5 Modeling boundaries

As long as we want to model a resonating structure we have to limit the size of the waveguide model. This is true whatever the number of dimensions are, either one (e.g., finite-length ideal string), two (e.g., finite-area ideal membrane) or three (e.g., finite-volume ideal cavity).

Wave theory states that resonances occur on a system governed by Eq. (1) (or its reductions (2) and (10)) once the wave equation is completed by proper *boundary conditions*. These conditions depend on the reflection properties of the boundary. In the ideal case of perfect reflection, the solution of the bounded problem shows that *waves are reflected by the boundary without energy loss and phase distortion at any frequency.*

Once again the waveguide approach turns out to be useful, in fact it allows to deal with the ideal solution of the reflection problem quite easily. Since wave signals are processed, perfect reflection at the boundary can be modeled by instantaneous lossless loopbacks. For example, hypothesizing a 3DSWM model of a parallelepiped having volume equal to $N_x\Delta \times N_y\Delta \times N_z\Delta$, the plane located at $z = N_z\Delta$ reflects signals according to the following equation:

$$p_-[k\Delta, h\Delta, N_z\Delta, nT] = \pm p_+[k\Delta, h\Delta, N_z\Delta, nT]\ ,\ 0 < k < N_x\ ,\ 0 < h < N_y \tag{16}$$

where p_+ are incident wave signals, and p_- reflected wave signals.

The sign of the reflection coefficient in Eq. (16) depends on the type of reflection applied to the incident waves. 3d waveguide models typically deal with pressure signal reflected by walls: in that case the coefficient is equal to $+1$. 2d waveguide models typically process velocity waves traveling along membranes: in that case reflection at the membrane edge is modeled by a coefficient equal to -1.

As expected, the reflection coefficient affects the resonances generated by the model. In particular, the positive sign allows the mode at zero frequency (also known as *Helmholtz's mode*) to be excited.