Worksheet 4: Structural induction – Sample Answers

1. We must prove that, for any expression E, if $E \Downarrow \mathtt{n}$ and $E \Downarrow \mathtt{n}'$ then $n = n'$. We are asked to do this by induction on the structure of the expression E.

   Base Case: The base case is the case where E is a numeral. The only rule of the big-step semantics that lets us infer that

   $$\text{numeral} \Downarrow \text{something}$$

   is the axiom, so if $E \Downarrow \mathtt{n}$ then E is $\mathtt{n}$, and if also $E \Downarrow \mathtt{n}'$ then it must be that $\mathtt{n}$ and $\mathtt{n}'$ are the same numeral, that is, $n = n'$.

   Inductive Step: The inductive step is the case where E is $(E_1 + E_2)$. Again, by inspection of the rules, $E \Downarrow \mathtt{n}$ and $E \Downarrow \mathtt{n}'$ can only have been derived by using the rule for $+$, so we have derivations of the form

   $$\frac{E_1 \Downarrow \mathtt{n}_1 \quad E_2 \Downarrow \mathtt{n}_2}{(E_1 + E_2) \Downarrow \mathtt{n}}$$

   and

   $$\frac{E_1 \Downarrow \mathtt{n}_1' \quad E_2 \Downarrow \mathtt{n}_2'}{(E_1 + E_2) \Downarrow \mathtt{n}'}$$

   where $n = n_1 + n_2$ and $n' = n_1' + n_2'$. Since we have both $E_1 \Downarrow \mathtt{n}_1$ and $E_1 \Downarrow \mathtt{n}_1'$, and since E_1 is a *subexpression* of E, we can apply the inductive hypothesis to conclude that $n_1 = n_1'$. Applying the inductive hypothesis to E_2 yields that $n_2 = n_2'$, so it follows that $n = n'$, as required.

2. The function plusses is easy to define:

   $$\begin{aligned} \mathsf{plusses}(\mathtt{n}) &= 0 \\ \mathsf{plusses}((E_1 + E_2)) &= \mathsf{plusses}(E_1) + \mathsf{plusses}(E_2) + 1. \end{aligned}$$

   As usual, in the case for the compound expression $(E_1 + E_2)$, we are allowed to make use of $\mathsf{plusses}(E_1)$ and $\mathsf{plusses}(E_2)$, since E_1 and E_2 are subexpressions of the expression we're interested in.

3. The function nums is only a minor variation:

   $$\begin{aligned} \mathsf{nums}(\mathtt{n}) &= 1 \\ \mathsf{nums}((E_1 + E_2)) &= \mathsf{nums}(E_1) + \mathsf{nums}(E_2). \end{aligned}$$

   Let $P(E)$ be the property $\mathsf{plusses}(E) < \mathsf{nums}(E)$. We will show by structural induction on E that $P(E)$ is true for every expression E.

   Base case: Here E is a numeral, say $\mathtt{n}$, and we need only look up the definitions of the two functions:

   $$\mathsf{plusses}(\mathtt{n}) = 0 \quad < \quad 1 = \mathsf{nums}(\mathtt{n})$$

   Inductive Step: Here E has the form $E_1 + E_2$ and we may assume that the statement P is true of E_1 and E_2. So we may assume

   $$\begin{aligned} \mathsf{plusses}(E_1) &< \mathsf{nums}(E_1) \\ \mathsf{plusses}(E_2) &< \mathsf{nums}(E_2) \end{aligned}$$

   We refer to these assumptions as IH.

Now we look up the definition of the functions applied to E and $P(E)$ follows by simple calculation:

$$\begin{aligned} \mathsf{plusses}(E) &= \mathsf{plusses}(E_1) + \mathsf{plusses}(E_2) + 1 & \text{(definition)} \\ &< \mathsf{nums}(E_1) + \mathsf{nums}(E_2) & \text{(IH)} \end{aligned}$$

Question: Can you justify the last step ?

By structural induction we may now conclude the $P(E)$ is true for every expression E.

4. Let $P(n)$ be the property

$$E \to^n E' \textit{ implies } E + F \to^n E' + F$$

We show, by *mathematical induction*, that $P(n)$ is true for every natural number n.

Base case: We have to show $P(0)$ is true; this case is pretty trivial. Note that $E \to^0 G$ is only true when G is equal to E, since $E \to^0 G$ means that E evaluates to G in zero steps, that is in no steps. Suppose $E \to^0 E'$. This mean E' *is* E. Which means of course that $E + F$ is $E' + F$, that is $E + F \to^n E' + F$.
So, rather vacuously, $P(0)$ is true.

Inductive step: Here we can assume $P(k)$ is true; we call this IH, the induction hypothesis. Using this we have show $P(k+1)$ is true; that is $E \to^(k+1)E'$ implies $E + F \to^{(k+1)} E' + F$.
So suppose $E \to^{(k+1)} E'$. Looking up the definition of $\to^{(k+1)}$ in Slide 34, this means that there is some G such that

(a) $E \to^k G$

(b) $G \to E'$

But now we can apply IH to (a) to get $E + F \to^k G + F$. We can also apply the rule (S-LEFT) on Slide 16, to (b), to obtain $G + F \to E' + F$.
Now we can once more apply the definition on Slide 34 to obtain $G + F \to^{(k+1)} E' + F$.

Because we have proved both the base case and the inductiove step, we can now conclude that $P(n)$ is true for every natural number n. So $E \to^n E'$ implies $E + F \to^n E' + F$.

5. The proof is very similar in structure to that of the last question. But we use $P(n)$ defined by

$$E \to^n E' \textit{ implies } \mathtt{m} + E \to^n \mathtt{m} + E'$$

Also in the inductive case an application of the rule (S-LEFT) is used, instead of (S-RIGHT).

6. Suppose

(a) $E_1 \to^* \mathtt{n_1}$

(b) $E_2 \to^* \mathtt{n_1}$

So from (a) we know that there is some k_1 such that $E_1 \to^{k_1} \mathtt{n_1}$. Applying the last result but one to this, we have that

$$E_1 + E_2 \to^{k_1} \mathtt{n_1} + E_2$$

Also (b) means that for some k_2, $E_2 \to^{k_2} \mathtt{n_2}$. Applying the last result to this we get we get

$$\mathtt{n_1} + E_2 \to^{k_2} \mathtt{n_1} + \mathtt{n_2}$$

Putting these two evaluations together we obtain

$$E_1 + E_2 \to^{(k_1+k_2)} \mathtt{n_1} + \mathtt{n_2}$$

A final application of the rule (S-ADD) gives

$$E_1 + E_2 \rightarrow^{(k_1+k_2)+1} \mathtt{n}$$

because $n_1 + n_2 = n$.

7. Omitted

8. We are asked to prove that whenever a reduction $E \rightarrow E'$ is derivable, $\mathsf{plusses}(E) = \mathsf{plusses}(E') + 1$. Let us call this statement $P(E)$; We shall prove $P(E)$ to be true by structural induction induction on E.

Base Case: The base case is that E is a numeral, say $\mathtt{n}$. Here $P(E)$ is vacuously true since $\mathtt{n} \rightarrow E'$ for no E'; in other words $\mathsf{plusses}(\mathtt{n}) = \mathsf{plusses}(E') + 1$ for every E' such that $\mathtt{n} \rightarrow E'$.

Inductive Step: Here we can assume that E is $E_1 + E_2$ and that $P(E_1)$ and $P(E_2)$ are both true; and from this we have to prove $P(E_1 + E_2)$ to be true.
So let

$$E_1 + E_2 \rightarrow E' \tag{1}$$

We must show, using $P(E_1)$ and $P(E_2)$, that $\mathsf{plusses}(E_1 + E_2) = \mathsf{plusses}(E') + 1$. From the definition of this function we know

$$\mathsf{plusses}(E_1 + E_2) = 1 + \mathsf{plusses}(E_1) + \mathsf{plusses}(E_2)$$

and so we have to prove

$$\mathsf{plusses}(E_1) + \mathsf{plusses}(E_2) = \mathsf{plusses}(E') \tag{2}$$

Let us look at how E' is generated from (1) above. There are three possible ways in which this move could have been generated.

(a) E' actually is $E_1' + E_2$ and $E_1 \rightarrow E_1'$.
Here

$$\begin{aligned}
\mathsf{plusses}(E_1) + \mathsf{plusses}(E_2) &= \mathsf{plusses}(E_1') + 1 + \mathsf{plusses}(E_2) \qquad \text{using } P(E_1) \\
&= \mathsf{plusses}(E_1' + E_2) \qquad \text{by the definition of } \mathsf{plusses} \\
&= \mathsf{plusses}(E')
\end{aligned}$$

This is what we are required to prove in (2) above.

(b) Here E_1 is a numeral, say $\mathtt{n_1}$, E' is $\mathtt{n_1} + E_2'$, where $E_2 \rightarrow E_2'$.
Now we do some calculations, using $P(E_2)$:

$$\begin{aligned}
\mathsf{plusses}(E_1) + \mathsf{plusses}(E_2) &= 0 + \mathsf{plusses}(E_2) \\
&= 0 + \mathsf{plusses}(E_2') \qquad \text{using } P(E_2) \\
&= \mathsf{plusses}(E') \qquad \text{by the definition of } \mathsf{plusses}
\end{aligned}$$

(c) The third possibility is that both E_1 and E_2 are numerals, say $\mathtt{n_1}$ and $\mathtt{n_2}$ respectively. Here E' must also be a numeral, say $\mathtt{n_3}$, where $n_3 = n_1 + n_2$.
Here the calculations are straightforward: $\mathsf{plusses}(E_1) + \mathsf{plusses}(E_2)$ is the same as $\mathsf{plusses}(E')$ since the number of plusses in all of E_1, E_2, E_3 is zero.

9. We must prove, by induction on the structure of expressions, that for any expression E,

$$\text{if } E \text{ is not a numeral, then } E \rightarrow E' \text{ for some } E'.$$

Let us denote this property by $P(E)$.

Base Case: This is the case of a numeral. There is nothing to prove, since the property only talks about expressions which are not numerals.

Inductive Step: The case of an expression $(E_1 + E_2)$. Here we may assume that both $P(E_1)$ and $P(E_2)$ are true. These we will refer to as the *inductive hypotheses*, $IH(E_1)$ and $IH(E_2)$.

Let us first apply the inductive hypothesis to $IH(E_1)$. This means that, if E_1 is not a numeral, then $E_1 \rightarrow E_1'$ for some E_1'. But then by applying a rule, we can deduce that $(E_1 + E_2) \rightarrow (E_1' + E_2)$. So, if E_1 is not a numeral, we have done what was needed.

However E_1 may be a numeral, say $\mathtt{n_1}$. In this case we cannot use this argument since the inductive hypothesis $IH(E_1)$ does not tell us anything. But we still have $IH(E_2)$ to apply. This means that if E_2 is not a numeral, then $E_2 \rightarrow E_2'$. Since E_1 is the numeral $\mathtt{n_1}$, we can apply the other rule to deduce that $\mathtt{n_1} + E_2 \rightarrow \mathtt{n_1} + E_2'$, which is what we need.

We still have to consider the case when E_2 is also a numeral, call it $\mathtt{n_2}$. Again here $IH(E_2)$ tells us nothing. But in this case, the axiom tells us that $\mathtt{n_1} + \mathtt{n_2} \rightarrow \mathtt{n_3}$ where $n_3 = n_1 + n_2$, which is what we needed.

So, in all cases, the expression $(E_1 + E_2) \rightarrow$ something, as required. That is we have proved $P(E_1 + E_2)$, under the assumptions $P(E_1)$ and $P(E_2)$.

By structural induction we may now conclude the $P(E)$ is true for every expression E, that for every E, $E \rightarrow$ something.

10. We are asked to combine the above two observations to argue that for any expression E, there is a numeral $\mathtt{n}$ such that $E \rightarrow^* \mathtt{n}$.

By question (9), every expression which is not a numeral can be reduced. So, the only way it is possible for an expression E to fail to reach a numeral after many steps of reduction is if it can be reduced forever, that is, if there is an infinite sequence

$$E \rightarrow E_1 \rightarrow E_2 \rightarrow E_3 \rightarrow \cdots$$

By question (8), this is impossible: since $\mathsf{plusses}(E)$ is a finite number and it reduces by one every time we perform a step of evaluation using $\rightarrow$, every evaluation sequence starting at E must be finite. It follows that every such sequence eventually reaches a numeral, as required.

This argument is quite interesting: we have used the function $\mathsf{plusses}$ to measure how far an expression is from reaching a final answer. In real programming languages, it is not possible to do this, especially when infinite loops are possible!

11. To give a similar argument for the larger language incorporating $\times$ as well as $+$, the best plan is

- Define a function $\mathsf{operations}$ which counts the number of operations in an expression. That is, it counts both the $+$ symbols and the $\times$ symbols together.
- Show that every reduction $E \rightarrow E'$ deals with exactly one operation.
- Show that every non-numeral can be reduced.
- Use the same argument to show that there are no infinite reduction sequences and hence that every expression eventually reaches a numeral.