Worksheet: Structural induction – *Some answers*

(1) (a)

$$\begin{aligned} \mathsf{nodes}(\mathbf{leaf}) &= 1 \\ \mathsf{nodes}(\mathbf{Branch}(T_1, T_2)) &= \mathsf{nodes}(T_1) + \mathsf{nodes}(T_2) + 1 \end{aligned}$$

(b)

$$\begin{aligned} \mathsf{height}(\mathbf{leaf}) &= 0 \\ \mathsf{height}(\mathbf{Branch}(T_1, T_2)) &= \max(\mathsf{height}(T_1), \mathsf{height}(T_2)) + 1 \end{aligned}$$

Here max is a function which returns the maximum of two natural numbers.

(c) Let $P(T)$ be the property: $\mathsf{nodes}(T) \leq 2^{\mathsf{height}(T)+1} - 1$. We prove $P(T)$ is true of every binary tree T, using structural induction on T.

There are two cases:

- **Base case:** Here we have to show $P(\mathbf{leaf})$ is true; that is $\mathsf{nodes}(\mathbf{leaf}) \leq 2^{\mathsf{height}(\mathbf{leaf})+1} - 1$.
  This follows by calculation since by definition $\mathsf{nodes}(\mathbf{leaf}) = 1$ and $\mathsf{height}(\mathbf{leaf}) = 0$, and $1 \leq 2^{0+1} - 1$.
- **Inductive case:** Here we assume $P(T_1)$ and $P(T_2)$ are true for some arbitrary trees $T_1,\ T_2$. This we call the *inductive hypothesis* IH, which means we are assuming

$$\begin{aligned} \mathsf{nodes}(T_1) &\leq 2^{\mathsf{height}(T_1)+1} - 1 \\ \mathsf{nodes}(T_2) &\leq 2^{\mathsf{height}(T_2)+1} - 1 \end{aligned}$$

  Under this assumption we have to show that $P(\mathbf{Branch}(T_1, T_2))$ follows. For clarity let us denote $\mathsf{nodes}(\mathbf{Branch}(T_1, T_2))$ by N and $\mathsf{height}(\mathbf{Branch}(T_1, T_2))$ by H. This means we have to deduce $N \leq 2^{H+1} - 1$. See the sequence of deductions below:

$$\begin{aligned} N &= \mathsf{nodes}(T_1) + \mathsf{nodes}(T_2) + 1 && \text{by definition} \\ &\leq 2^{\mathsf{height}(T_1)+1} - 1 + 2^{\mathsf{height}(T_1)+1} - 1 + 1 && \text{by IH} \\ &\leq 2^H - 1 + 2^H - 1 + 1 && \text{since } \mathsf{height}(T_i) + 1 \leq H \\ &= 2^H + 2^H - 1 \\ &= 2^{H+1} - 1 \end{aligned}$$

(2) The function plusses is easy to define:

$$\begin{aligned} \mathsf{plusses}(\mathtt{n}) &= 0 \\ \mathsf{plusses}((E_1 + E_2)) &= \mathsf{plusses}(E_1) + \mathsf{plusses}(E_2) + 1. \end{aligned}$$

As usual, in the case for the compound expression $(E_1 + E_2)$, we are allowed to make use of $\mathsf{plusses}(E_1)$ and $\mathsf{plusses}(E_2)$, since E_1 and E_2 are sub-expressions of the expression we're interested in.

(3) The function nums is only a minor variation:

$$\begin{aligned} \mathsf{nums}(\mathtt{n}) &= 1 \\ \mathsf{nums}((E_1 + E_2)) &= \mathsf{nums}(E_1) + \mathsf{nums}(E_2). \end{aligned}$$

Let $P(E)$ be the property $\mathsf{plusses}(E) < \mathsf{nums}(E)$. We will show by structural induction on E that $P(E)$ is true for every expression E.

Base case: Here E is a numeral, say $\mathtt{n}$, and we need only look up the definitions of the two functions:

$$\mathsf{plusses}(\mathtt{n}) = 0 \quad < \quad 1 = \mathsf{nums}(\mathtt{n})$$

Inductive Step: Here E has the form $E_1 + E_2$ and we may assume that the statement P is true of E_1 and E_2. So we may assume

$$\begin{aligned} \mathsf{plusses}(E_1) &< \mathsf{nums}(E_1) \\ \mathsf{plusses}(E_2) &< \mathsf{nums}(E_2) \end{aligned}$$

We refer to these assumptions as IH.

Now we look up the definition of the functions applied to E and $P(E)$ follows by simple calculation:

$$\begin{aligned} \mathsf{plusses}(E) &= \mathsf{plusses}(E_1) + \mathsf{plusses}(E_2) + 1 && \text{(definition)} \\ &< \mathsf{nums}(E_1) + \mathsf{nums}(E_2) && \text{(IH)} \end{aligned}$$

Question: Can you justify the last step ?

By structural induction we may now conclude the $P(E)$ is true for every expression E.

(4) (a) To define a function $f : \mathit{BinNum} \to \mathbb{N}$ is it sufficient to

- **Base case:** explain what it means to apply f to the binary numeral $\mathtt{0}$ and what it means to apply it to $\mathtt{1}$
- **Inductive case:** Assuming we know what $f(b)$ is, describe what it means to
  - apply f to $b\mathtt{0}$
  - apply f to $b\mathtt{1}$

So there are two base cases and two inductive cases.

The function number : $\mathit{BinNum} \to \mathbb{N}$ is defined by

$$\begin{aligned} \mathsf{number}(\mathtt{0}) &= 0 && \text{a base case} \\ \mathsf{number}(\mathtt{1}) &= 1 && \text{the second base case} \\ \mathsf{number}(b\mathtt{0}) &= 2 \times \mathsf{number}(b) && \text{an inductive case} \\ \mathsf{number}(b\mathtt{1}) &= 2 \times \mathsf{number}(b) + 1 && \text{the second inductive case} \end{aligned}$$

Similarly the function $\mathsf{sum} : \mathit{BinNum} \to \mathbb{N}$ is defined by:

$$
\begin{array}{rcll}
\mathsf{sum}(\mathtt{0}) & = & 0 & \text{a base case} \\
\mathsf{sum}(\mathtt{1}) & = & 1 & \text{the second base case} \\
\mathsf{sum}(b\mathtt{0}) & = & \mathsf{number}(b) & \text{an inductive case} \\
\mathsf{sum}(b\mathtt{1}) & = & \mathsf{number}(b) + 1 & \text{the second inductive case}
\end{array}
$$

(b) The structural induction principle for *BinNum* is as follows:
Let $P(b)$ be a property of binary numerals. To show that $P(b)$ holds for all binary numerals b it is sufficient to:

(i) **A base case:** prove $P(\mathtt{0})$ is true

(ii) **A base case:** prove $P(\mathtt{1})$ is true

(iii) **Inductive cases:** assuming the *inductive hypothesis* $P(b)$ prove
- $P(b\mathtt{0})$ follows
- $P(b\mathtt{1})$ follows.

As an example of this principle let us show that the property

$$P(b) \qquad : \mathsf{sum}(b) \leq \mathsf{number}(b)$$

is true for every binary numeral b. To do so we have to establish four facts:

(i) **A base case:** We have to show $P(\mathtt{0})$ is true, that is $\mathsf{sum}(\mathtt{0}) \leq \mathsf{number}(\mathtt{0})$. This follows by definition of the two functions; $\mathsf{sum}(\mathtt{0}) = 0 = \mathsf{number}(\mathtt{0})$.

(ii) **Another base case:** We have to show $P(\mathtt{1})$ is true; this is similar to the first case.

(iii) **The inductive cases:** here we assume $P(b)$ holds, that is $\mathsf{sum}(b) \leq \mathsf{number}(b)$; this is the inductive hypothesis, which we call (IH). From this we have to deduce two consequences:

- $P(b\mathtt{0})$ follows. Some simple calculations suffice:

$$
\begin{array}{rcll}
\mathsf{sum}(b\mathtt{0}) & = & \mathsf{sum}(b) & \text{by definition} \\
 & \leq & \mathsf{number}(b) & \text{(IH)} \\
 & \leq & 2 \times \mathsf{number}(b) & \text{maths} \\
 & & \mathsf{number}(b\mathtt{0}) & \text{by definition}
\end{array}
$$

- $P(b\mathtt{1})$ follows. Similar to the previous inductive case. Make sure you can write it out correctly.