

- (1) Use mathematical induction to prove that $2^n > n$, for every natural number n .

Let $P(n)$ denote the statement: $2^n > n$. We show $P(n)$ is true for every natural number n .

Base case: We need to show that $P(0)$ is true. But this is obvious, since by definition 2^0 is 1, and therefore

$$2^0 = 1 > 0$$

Inductive case: Here we assume $P(k)$ to be true:

$$2^k > k \quad (\text{IH})$$

From this assumption we have to derive that $P(k+1)$ is true, namely

$$2^{(k+1)} > k+1$$

This requires a tiny bit of algebraic manipulation:

$$\begin{aligned} 2^{(k+1)} &= 2^k + 2^k && \text{(well-known fact)} \\ &> k + 2^k && \text{(applying IH)} \\ &\geq k + 1 && \text{(well-known fact)} \end{aligned}$$

The last step relies on the fact that 2^k is always greater than or equal to 1.

Since we have proved the **base case** and the **inductive case**, it follows that $P(n)$ is true for every natural number n .

- (2) Prove that $n^3 - n$ is divisible by 6, for every natural number n .

Let $P(n)$ be the statement: $n^3 - n$ is divisible by 6. We use mathematical induction to show $P(n)$ is true for every natural number n .

Base case: We need to show $P(0)$ is true. This is simple:

$$\begin{aligned} 0^3 - 0 &= 0 - 0 \\ &= 0 \end{aligned}$$

and we all know 0 is divisible by 6. (Because 0 is 6 times 0.)

Inductive case: Here we assume $P(k)$ to be true, that is $k^3 - k$ is divisible by 6; this we refer to as IH. We need to show $P(k+1)$ to be true, based on that assumption. That is we need to show $(k+1)^3 - (k+1)$ is divisible by 6. As usual we massage the requirement, until we can get to a term in which the inductive hypothesis can be usefully applied.

$$\begin{aligned} (k+1)^3 - (k+1) &= k^3 + 3k^2 + 3k + 1 - k - 1 && \text{(well-known fact)} \\ &= k^3 + 3k^2 + 2k && \text{(simplifying)} \\ &= (k^3 - k) + 3k^2 + 2k + k \\ &= (k^3 - k) + 3k(k+1) && \text{(re-arrange)} \end{aligned}$$

We have now rearranged the term so that we can use IH. We know $(k^3 - k)$ is divisible by 6. So we need only show that the other term, $3k(k + 1)$ is also divisible by 6. It is certainly divisible by 3. But it is also divisible by 2 since $k(k + 1)$ is always even. (This is because either k or $k + 1$ is always even, and any multiple of an even number is even.)

By mathematical induction we can now conclude that the statement $P(n)$ is true for every natural number n , as required.

- (3) Show that for every natural number greater than 0, the n^{th} odd natural number is $2n - 1$.

Let $P(n)$ be the statement:

$$n = 0 \text{ or } \text{Odd}(n) = 2n - 1$$

where, for $n \geq 1$, $\text{Odd}(n)$ denotes the n^{th} odd natural number. We prove, using mathematical induction, that $P(n)$ is true for every natural number n .

Base case: We have to show $P(0)$ is true: this is trivial.

Inductive case: Assume $P(k)$ is true, namely

$$k = 0 \text{ or } \text{Odd}(k) = 2k - 1 \quad (\text{IH})$$

From this assumption we have to show how to deduce $P(k + 1)$, namely

$$k + 1 = 0 \text{ or } \text{Odd}(k + 1) = 2(k + 1) - 1$$

Obviously $k + 1 = 0$ is never going to be true, so we end up deriving $\text{Odd}(k + 1) = 2(k + 1) - 1$.

What do we know about $\text{Odd}(k + 1)$? Well it is two greater than $\text{Odd}(k)$. So we can reason as follows:

$$\begin{aligned} \text{Odd}(k + 1) &= \text{Odd}(k) + 2 && \text{(definition of Odd)} \\ &= 2k - 1 + 2 && \text{(applying IH)} \\ &= 2(k + 1) - 1 && \text{(rearrangement)} \end{aligned}$$

This last line was what we are required to establish.

Since we have proved the **base case** and the **inductive case** we can now conclude that $P(n)$ is true for every natural number n . In particular, for every $n \geq 1$, the n^{th} odd number is $2n - 1$.

- (4) Prove that $n! > 2^n$ whenever n is greater than or equal to 4. Proof omitted.
- (5) Prove that every natural number greater than or equal to 4 can be written as a sum of 2s and/or 5s.

Here we let $P(n)$ be the statement: n can be written as a sum of 2s and/or 5s. We wish to prove that $P(n)$ is true for every natural number greater than or equal to 4.

Base case: Here the base case is when $n = 4$. $P(4)$ is true because $4 = 2 + 2$.

Inductive step: Here we assume $P(k)$ to be true. So we know that k can be written as a sum consisting of only 2s and 5s. On the basis of this assumption we need to show that $P(k + 1)$ is true, that is $k + 1$ can be written as a sum of 2s and 5s. How can we use the *inductive assumption*? Well there are two cases. First suppose that the rendition of k uses at least one 5. We have

$$k = 5 + \dots$$

But then

$$\begin{aligned} k + 1 &= 5 + 1 + \dots \\ &= 2 + 2 + 2 + \dots \end{aligned}$$

In other words $k + 1$ can also be written in the required form.

The other possibility is that no 5 is used. But since k is greater than or equal to 4 we have

$$k = 2 + 2 + \dots$$

So

$$\begin{aligned} k + 1 &= 2 + 2 + 1 + \dots \\ &= 5 + \dots \end{aligned}$$

again in the required form.

(6) Show that

$$\sum_{i=0}^n i^2 = \frac{n(n+1)(2n+1)}{6}.$$

Again, we go by mathematical induction on n .

Base case: The case $n = 0$ is dealt with immediately:

$$\sum_{i=0}^0 i^2 = 0 = \frac{0 \times 1 \times 1}{6} = \frac{n(n+1)(2n+1)}{6}.$$

Inductive step: The case $n = k + 1$ is handled by making use of the inductive hypothesis, which in this case is

$$\sum_{i=0}^k i^2 = \frac{k(k+1)(2k+1)}{6}.$$

So, to handle $k + 1$, we calculate

$$\begin{aligned} \sum_{i=0}^{k+1} i^2 &= (k+1)^2 + \sum_{i=0}^k i^2 \\ &= (k+1)^2 + \frac{k(k+1)(2k+1)}{6} \quad \text{using IH.} \end{aligned}$$

We now just need to show that

$$(k + 1)^2 + \frac{k(k + 1)(2k + 1)}{6} = \frac{(k + 1)(k + 2)(2k + 3)}{6}$$

which is indeed true. This is just an algebraic calculation, which you should not worry about if you don't know how to do it. The important point to understand is where and how the inductive hypothesis is used.

(7) Prove that for all natural numbers $n \geq 1$,

$$\sum_{i=1}^n \frac{1}{i(i + 1)} = \frac{n}{n + 1}.$$

Again, we use induction. In this case, the smallest number we're interested in is 1, so that forms the base case.

Base case: The case $n = 1$. We have

$$\sum_{i=1}^1 \frac{1}{i(i + 1)} = \frac{1}{2} = \frac{n}{n + 1}$$

which is what we need.

Inductive step: We now deal with the case $n = k + 1$ using the inductive hypothesis, which is the same statement in the case $n = k$:

$$\sum_{i=1}^k \frac{1}{i(i + 1)} = \frac{k}{k + 1}.$$

We can now calculate as follows.

$$\begin{aligned} \sum_{i=1}^{k+1} \frac{1}{i(i + 1)} &= \frac{1}{(k + 1)(k + 2)} + \sum_{i=1}^k \frac{1}{i(i + 1)} \\ &= \frac{1}{(k + 1)(k + 2)} + \frac{k}{k + 1} \\ &= \frac{1 + (k + 2)k}{(k + 1)(k + 2)} \\ &= \frac{k^2 + 2k + 1}{(k + 1)(k + 2)} \\ &= \frac{(k + 1)(k + 1)}{(k + 1)(k + 2)} \\ &= \frac{k + 1}{k + 2} \end{aligned}$$

as required. Again, this is mostly algebraic calculation. Try to understand where, how and why the inductive hypothesis is used.

(8) Omitted.

(9) Omitted.