Dollar or no dollar, that is the question

New combinatorial results on the Burrows-Wheeler-Transform

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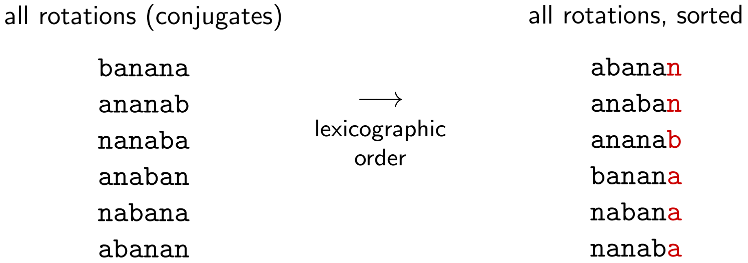
Part I:

Introduction

The Burrows-Wheeler-Transform

Ex.: T = `banana`. The BWT is a permutation of T: `nnbaaa`



all rotations (conjugates)

```
banana
ananab
nanaba
anaban
nabana
abanan
```

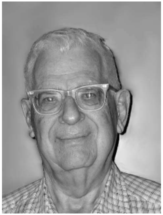
$\longrightarrow$ lexicographic order

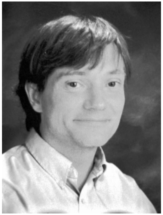
all rotations, sorted

```
abanan
anaban
ananab
banana
nabana
nanaba
```

Take a string (word) T, list all of its rotations, sort them lexicographically, concatenate last characters: bwt(T).

BWT history





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  as a lossless text compression algorithm
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source: Adjeroh, Bell, Mukerjee: The Burrows-Wheeler-Transform, Springer, 2008

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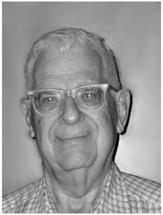

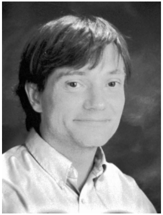

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Reversing the BWT

input: nnbaaa, 3 bwt(T), i: where $0 \leq i < n$

output: (wanted) banana. T: i'th rotation lex.ly

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Recall: BWT-matrix (F: first column, L: last column)

$$\begin{array}{cccccccc}
 & \text{F} & & & & & \text{L} \\
0 & \texttt{a} & \texttt{b} & \texttt{a} & \texttt{n} & \texttt{a} & \texttt{n} \\
1 & \texttt{a} & \texttt{n} & \texttt{a} & \texttt{b} & \texttt{a} & \texttt{n} \\
2 & \texttt{a} & \texttt{n} & \texttt{a} & \texttt{n} & \texttt{a} & \texttt{b} \\
3 & \texttt{b} & \texttt{a} & \texttt{n} & \texttt{a} & \texttt{n} & \texttt{a} \\
4 & \texttt{n} & \texttt{a} & \texttt{b} & \texttt{a} & \texttt{n} & \texttt{a} \\
5 & \texttt{n} & \texttt{a} & \texttt{n} & \texttt{a} & \texttt{b} & \texttt{a}
\end{array}$$

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	F		L
0	a	bana	n
1	a	naba	n
2	a	nana	b
3	b	anan	a
4	n	aban	a
5	n	anab	a

- **Obs. 1:** F = all characters of T in lex. order: aaabnn

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    F    L
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1   anaban
2   ananab
3   banana
4   nabana
5   nanaba
```

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- **Obs. 2:** for all i: L_i precedes F_i in T:

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1	a	n	a	b	a	n
2	a	n	a	n	a	b
3	b	a	n	a	n	a
4	n	a	b	a	n	a
5	n	a	n	a	b	a

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Thm. (LF-property): The j'th occurrence of character x in L is the j'th occurrence of character x in F.

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2	a	b
3	b	a
4	n	a
5	n	a

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5	n	a

a

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5	n	a

n a

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3	b	a
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5	n	a

a n a

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banana

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2 occ's of `ana`

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`abanan`	`abanan`
`anaban`	`anaban`
`ananab`	`ananab`
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`nabana`	`nabana`
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2 occ's of ana	2 occ's of na preceded by a	2 occ's of a preceded by n
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anaban	anaban	anaban
ananab	ananab	ananab
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So: we get a run of a's of length 2, and a run of n's of length 2

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Ex.: $T =$ banana has 2 occurrences of the pattern ana

2 occ's of ana	2 occ's of na preceded by a	2 occ's of a preceded by n
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So: we get a run of a's of length 2, and a run of n's of length 2 ($2 =$ no. occ's).

Of course, things are a bit more complicated:

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rotation	BWT
`he caverns measureless to man, And sank in tumult to a ...`	t
`he caves. It was a miracle of rare device, A sunny pleasure-...`	t
`he dome of pleasure Floated midway on the waves; Where was ...`	t
`he fountain and the caves. It was a miracle of rare device,...`	t
`he green hill athwart a cedarn cover! A savage place! as ...`	t
`he hills, Enfolding sunny spots of greenery. But oh! that ...`	t
`he milk of Paradise.`	t
`he mingled measure From the fountain and the caves. It was a ...`	t
`he on honey-dew hath fed, And drunk the milk of Paradise.  ...`	␣
`he played, Singing of Mount Abora. Could I revive within me ...`	s
`he sacred river ran, Then reached the caverns measureless ...`	t
`he sacred river, ran Through caverns measureless to man ...`	t
`he sacred river. Five miles meandering with a mazy motion ...`	t
`he shadow of the dome of pleasure Floated midway on the waves ...`	T
`he thresher’s flail: And mid these dancing rocks at once and ...`	t
`he waves; Where was heard the mingled measure From the ...`	t

Kubla Kahn by Samuel Coleridge

- many `the`’s, some `he, she, The`

Compression with the BWT

- in original paper: using Move-to-front and Huffman/arithmetic coding

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  Ex.: $r(\texttt{banana}) = 3$ recall: $\mathrm{bwt}(\texttt{banana}) = \texttt{nnbaaa}$

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- for repetitive strings, r is small (more on this later)

Pattern matching with the BWT

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Given a string T of length n (the text) and a string P of length m (the pattern), find all occurrences of P in T as a substring.
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- without additional data structures, time $\Omega(n + m)$ (read the input)
- exist algorithms achieving $\Theta(n + m)$ worst-case (Knuth-Morris-Pratt)

Pattern matching with the BWT

Backward search [Ferragina and Manzini, 2000]

1. process pattern back-to-front
2. $Occ(xU) \subseteq Occ(U) - 1$

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$(Occ(\texttt{a}) = \{1, 3, 5\}, Occ(\texttt{na}) = \{2, 4\}, Occ(\texttt{ana}) = \{1, 3\})$.

all occ's of a	all occ's of na	all occ's of ana
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Pattern matching with the BWT

Magic! Backward search can be done on the BWT directly (with some additional magic. . .):

Ex.: ~~T = banana~~ and P = ana.

bwt(T) = nnbaaa.

all occ's of a	all occ's of na	all occ's of ana
n	n	n
n	n	n
b	b	b
a	a	a
a	a	a
a	a	a

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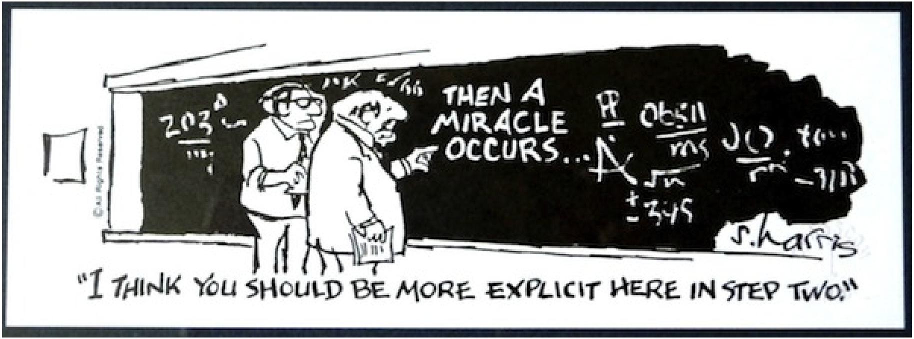
a	a	a
a	a	a

Thm. Pattern matching on bwt(T) (decision and counting) can be implemented in $O(m \log \sigma)$ time, using only $o(n)$ additional bits.

σ = alphabetsize

BWT magic



copyright: Sydney Harris

BWT magic

The BWT ...

- requires same space as T in bits: $n \log \sigma$ bits $\sigma =$ alphabetsize
  (suffix array: $n \log n$ bits, suffix tree: much more — still $\mathcal{O}(n)$)

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We have seen:

- lossless: BWT is reversible: `nnbaaa`,3 ↦ `banana`
- easier to compress than T, if T repetitive
- pattern matching in $\mathcal{O}(m \log \sigma)$ time $m = |P|$
  (on T : $\mathcal{O}(n + m)$ time) $n = |T|$

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We have not seen:

- reversible in linear time $\mathcal{O}(n)$ $n = |T|$
- computable in linear time $\mathcal{O}(n)$
- can replace text (suffix array, suffix tree: no)

Compressed data structures for strings

The amount of (just HTML) online text material in the Web was estimated, in 2002, to exceed by 30-40 times what had been printed during the whole history of mankind.

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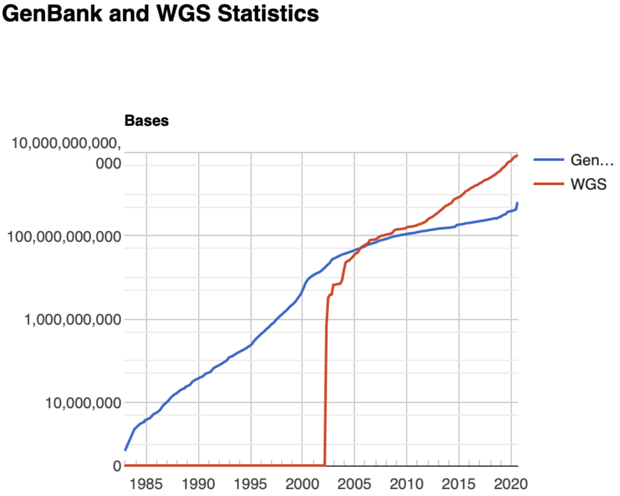
N.B. And this was in 2002!

Let's look at biological sequences . . .



GenBank and WGS Statistics

source: NCBI website

Compressed data structures for strings

So we need efficient ways of . . .

- storing,
- querying,
- mining,
- searching,
- . . .

. . . very large amounts of textual data.

Compressed data structures for strings

Some data structures based on the BWT:

- FM-index [Ferragina and Manzini, FOCS 2000]
- RLFM-index [Mäkinen and Navarro, CPM 2005]
- *r*-index [Gagie et al, JACM 2020; Bannai et al. TCS 2020]
- some recent developments on *r*-index [Rossi et al. JCB 2022; Giuliani et al. SEA 2022; Cobas et al. CPM 2021; Boucher et al. SPIRE 2021]

Some tools in bioinformatics (aligners):

bwa [Durbin and Li, 2009]	ca. 41,000 cit.
bowtie [Langmead and Salzberg, 2010]	ca. 36,000 cit.
soap2 [Li et al., 2009]	
...	

The parameter r

Def. String T, $r =$ number of runs of bwt(T).

- size of data structures $\mathcal{O}(r)$
- algorithms' running time ideally a function of r (not of $n = |T|$)
- increasingly used as a repetitiveness measure of T
- some papers on r:
    - Manzini: "An analysis of the Burrows-Wheeler-Transform" [JACM 2001]
    - Kempa and Kociumaka: "Resolution of the Burrows-Wheeler Transform Conjecture" [FOCS 2020]
    - Navarro: "Indexing Highly Repetitive String Collections, Part I: Repetitiveness Measures" [ACM Comp. Surv., 2021]
    - Mantaci et al.: "Measuring the clustering effect of BWT via RLE" [TCS 2017]

BWT from a combinatorial perspective

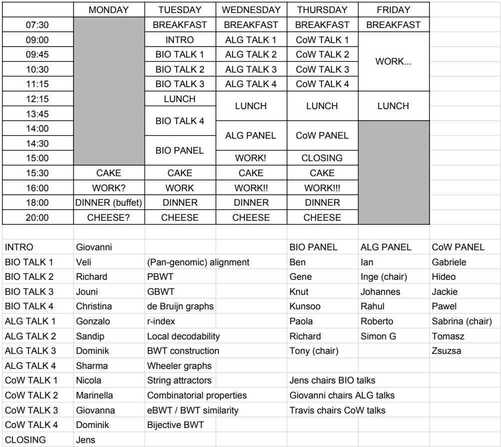
- special case of the Gessel-Reutenauer-bijection [Crochemore, Désarménien, Perrin, 2004]
- introduction of the extended BWT (eBWT), a generalization of the BWT to multisets of strings [Mantaci et al. 2007]
- strings T with fully clustering BWTs (e.g. $\text{bwt}(T) = \texttt{bbbbaaccc}$)
    - full characterization for $\sigma = 2$ [Mantaci et al., 2003]
    - partial characterization for $\sigma > 2$ [Puglisi et al., 2008]
    - characterization via interval exchanges [Ferenczi et al., 2013]
- fixpoints of the BWT [Mantaci et al., 2017]
- characterization of BWT images [Likhomanov and Shur, 2011]

Good overview: Rosone and Sciortino: "The Burrows-Wheeler Transform between Data Compression and Combinatorics on Words." [CiE 2013]

- two research communities working on the BWT
- (1) data structures and algorithms on strings and
  (2) combinatorics on words
- little interaction until . . .

Dagstuhl workshop "25 years of the Burrows-Wheeler-Transform" (2019) organized by T. Gagie, G. Manzini, G. Navarro, J. Stoye

The schedule:

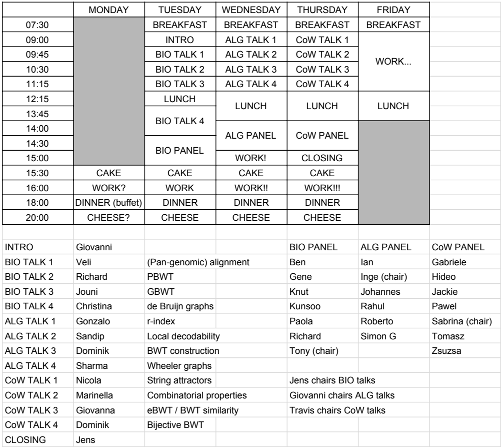


	MONDAY	TUESDAY	WEDNESDAY	THURSDAY	FRIDAY
07:30		BREAKFAST	BREAKFAST	BREAKFAST	BREAKFAST
09:00		INTRO	ALG TALK 1	CoW TALK 1	WORK...
09:45		BIO TALK 1	ALG TALK 2	CoW TALK 2	
10:30		BIO TALK 2	ALG TALK 3	CoW TALK 3	
11:15		BIO TALK 3	ALG TALK 4	CoW TALK 4	
12:15		LUNCH	LUNCH	LUNCH	LUNCH
13:45		BIO TALK 4			
14:00			ALG PANEL	CoW PANEL	
14:30		BIO PANEL			
15:00			WORK!	CLOSING	
15:30	CAKE	CAKE	CAKE	CAKE	
16:00	WORK?	WORK	WORK!!	WORK!!!	
18:00	DINNER (buffet)	DINNER	DINNER	DINNER	
20:00	CHEESE?	CHEESE	CHEESE	CHEESE	

INTRO	Giovanni			BIO PANEL	ALG PANEL	CoW PANEL
BIO TALK 1	Veli	(Pan-genomic) alignment		Ben	Ian	Gabriele
BIO TALK 2	Richard	PBWT		Gene	Inge (chair)	Hideo
BIO TALK 3	Jouni	GBWT		Knut	Johannes	Jackie
BIO TALK 4	Christina	de Bruijn graphs		Kunsoo	Rahul	Pawel
ALG TALK 1	Gonzalo	r-index		Paola	Roberto	Sabrina (chair)
ALG TALK 2	Sandip	Local decodability		Richard	Simon G	Tomasz
ALG TALK 3	Dominik	BWT construction		Tony (chair)		Zsuzsa
ALG TALK 4	Sharma	Wheeler graphs				
CoW TALK 1	Nicola	String attractors		Jens chairs BIO talks		
CoW TALK 2	Marinella	Combinatorial properties		Giovanni chairs ALG talks		
CoW TALK 3	Giovanna	eBWT / BWT similarity		Travis chairs CoW talks		
CoW TALK 4	Dominik	Bijective BWT				
CLOSING	Jens					

At the workshop, the communities were called

The schedule:



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CLOSING	Jens				

At the workshop, the communities were called ALG, BIO, and CoW (sic!)

But: The two communities use slightly different definitions of the BWT:

- ALG (incl. BIO): It is assumed that each string terminates with an end-of-string character (denoted \$, smaller than all others) $T = \texttt{banana\$}$
- CoW: no such assumption $T = \texttt{banana}$

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This talk is about the implications of this difference.

Part II:

Dollar or no dollar, that is the question

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In particular:

1. the transform itself

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1. the transform itself
2. BWT construction

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1. the transform itself
2. BWT construction
3. BWT images

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This talk is about the implications of this difference.

In particular:

1. the transform itself
2. BWT construction
3. BWT images
4. BWT of string collections

1. The transform itself

Different transforms

banana

abanan
anaban
ananab
banana
nabana
nanaba

nnbaaa

banana$

$banana
a$banan
ana$ban
anana$b
banana$
na$bana
nana$ba

annb$aa

Different transforms

	without dollar (banana)	with dollar (banana$)
the transform	nnbaaa	annb$aa

Different transforms

	without dollar (banana)	with dollar (banana$)
the transform	nnbaaa	annb$aa
remove $	nnbaaa	annbaa

Different transforms

	without dollar (banana)	with dollar (banana$)
the transform	nnbaaa	annb$aa
remove $	nnbaaa	annbaa
# runs r	3	4

Different transforms

	without dollar (banana)	with dollar (banana$)
the transform	nnbaaa	annb$aa
remove $	nnbaaa	annbaa
$\#$ runs r	3	4

- **Thm.** There exist strings for which the difference in r is $\Theta(\log n)$.

[Giuliani, Inenaga, L., Sciortino, 2022, forthcoming]

Different transforms

	without dollar (banana)	with dollar (banana$)
the transform	nnbaaa	annb$aa
remove $	nnbaaa	annbaa
# runs r	3	4

- **Thm.** There exist strings for which the difference in r is $\Theta(\log n)$. [Giuliani, Inenaga, L., Sciortino, 2022, forthcoming]
- This is asymptotically tight: here $r = O(1)$, and upper bound is $\mathcal{O}(\log r \log n)$. [Akagi, Funakoshi, Inenaga, 2021]

Different transforms

Thm. There exist strings for which the difference in r is $\Theta(\log n)$.

- $r(T\$)$ increases by $\log n$: Fibonacci words of even order $T = \mathit{Fib}(2k), r(T) = 2, r(T\$) = 2k - 1$

  ex.:
  $r(\mathit{Fib}(8)) = 2, r(\mathit{Fib}(8)\$) = 7$
  $r(\mathit{Fib}(12)) = 2, r(\mathit{Fib}(12)\$) = 11$

- $r(T\$)$ decreases by $\log n$: Fibonacci words of odd order without the first character $T = \mathit{Fib}(2k+1)[1:], r(T) = 2k, r(T\$) = 5$

  ex:
  $r(\mathit{Fib}(13)[1:]) = 12, r(\mathit{Fib}(13)[1:]\$) = 5$
  $r(\mathit{Fib}(15)[1:]) = 14, r(\mathit{Fib}(15)[1:]\$) = 5$

Different transforms

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  $r(Fib(13)[1 :]) = 12, r(Fib(13)[1 :]\$) = 5$
  $r(Fib(15)[1 :]) = 14, r(Fib(15)[1 :]\$) = 5$

- both additive and multiplicative difference

2. BWT construction

BWT construction

Most BWT construction algorithms first construct the Suffix Array (SA), then construct the BWT from the SA, using: $L_i = T_{SA[i]-1}$ (recall Obs. 2).

ex. $T = \underset{0}{\texttt{b}}\underset{1}{\texttt{a}}\underset{2}{\texttt{n}}\underset{3}{\texttt{a}}\underset{4}{\texttt{n}}\underset{5}{\texttt{a}}\underset{6}{\texttt{\$}}$.

SA	
6	$
5	a$
3	ana$
1	anana$
0	banana$
4	na$
2	nana$

BWT construction

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SA	
6	$
5	a$
3	ana$
1	anana$
0	banana$
4	na$
2	nana$

SA		L
6	$banan	a
5	a$bana	n
3	ana$ba	n
1	anana$	b
0	banana	$
4	na$ban	a
2	nana$b	a

BWT construction

Most BWT construction algorithms first construct the Suffix Array (SA), then construct the BWT from the SA, using: $L_i = T_{SA[i]-1}$ (recall Obs. 2).

ex. $T = \underset{0\,1\,2\,3\,4\,5\,6}{\texttt{banana\$}}$.

SA	
6	$
5	a$
3	ana$
1	anana$
0	banana$
4	na$
2	nana$

SA	L
6	$banana
5	a$banan
3	ana$ban
1	anana$b
0	banana$
4	na$bana
2	nana$ba

Thus: SA-construction in $\mathcal{O}(n)$ time ⇒ BWT-construction in $\mathcal{O}(n)$ time.

BWT construction without dollar

- This works well if there is a \$.
- What if there is no dollar?

BWT construction without dollar

- This works well if there is a $.
- What if there is no dollar?

banana
0 1 2 3 4 5

SA	
5	a
3	ana
1	anana
0	banana
4	na
2	nana

BWT construction without dollar

- This works well if there is a $.
- What if there is no dollar?

banana
0 1 2 3 4 5

SA		SA	L
5	a	5	abanan
3	ana	3	anaban
1	anana	1	ananab
0	banana	0	banana
4	na	4	nabana
2	nana	2	nanaba

nnbaaa ✓

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0 1 2 3 4 5

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banana
0 1 2 3 4 5

SA	
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SA		L
5	abana	n
3	anaba	n
1	anana	b
0	banan	a
4	naban	a
2	nanab	a

nnbaaa ✓

anaban
0 1 2 3 4 5

SA	
2	aban
4	an
0	anaban
3	ban
5	n
1	naban

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banana
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3	ban	3	banana
5	n	5	nabana
1	naban	1	nabana

nbnaaa ✗

BWT construction without dollar

- This works well if there is a $.
- What if there is no dollar?

banana
012345

SA		SA	L
5	a	5	abanan
3	ana	3	anaban
1	anana	1	ananab
0	banana	0	banana
4	na	4	nabana
2	nana	2	nanaba

nnbaaa ✓

anaban
012345

SA		SA	L
2	aban	2	abanan
4	an	4	ananab
0	anaban	0	anaban
3	ban	3	banana
5	n	5	nabana
1	naban	1	nabana

nbnaaa ✗

Problem 1: $suf_i < suf_j \Leftrightarrow conj_i < conj_j$ does not hold in general!

Thus: We need the CA (conjugate array), not the SA!

BWT construction without dollar

Problem 2: If T not primitive, then CA not defined (several identical rotations):

$\underset{0\,1\,2\,3\,4\,5}{\texttt{nanana}} = (\texttt{na})^3$

CA

```
1  ananan
3  ananan
5  ananan
0  nanana
2  nanana
4  nanana
```

Linear-time BWT construction without dollar

- For \$-terminated strings, neither problem exists.
- Same for Lyndon words (primitive and $<$ all their rotations).
- All previous BWT-construction algorithms either use \$ or Lyndon rotations.

Our algorithm [Boucher, Cenzato, L., Rossi, Sciortino, SPIRE, 2021]:

- first linear-time BWT-construction algorithm which uses neither \$ nor Lyndon rotations
- adaptation of the SAIS-algorithm for SA-construction [Nong et al., 2011]
- previously, SAIS had been adapted for $T\$$ [Okanohara and Sadakane 2009], and to the bijective BWT [Bannai et al., 2021]

Our algorithm for BWT construction

1. assign circular types to positions
2. sort LMS-substrings
3. assign new names to LMS-substrings
4. construct new string, solve recursively
5. induce CA from relative order of LMS-positions

Step 1

0	1	2	3	4	5
b	a	n	a	n	a
L	S	L	S	L	S
	*		*		*

Step 2

	a	*b*	*n*
S^*	1 3 5		
L		0	2 4
S	5 1 3		
	5 1 3	0	2 4

Step 3

5	a b a	A
1	a n a	B
3	a n a	B

Step 4

0	1	2
A	B	B
S	L	L
*		

A	*B*
0	
	2 1
0	2 1

Step 5

	a	*b*	*n*
	5 3 1		
		0	4 2
CA	5 3 1	0	4 2
BWT	n n b	a	a a

3. BWT images

BWT images

The BWT-mapping $\mathrm{bwt} : \Sigma^n \to \Sigma^n$, $T \mapsto \mathrm{bwt}(T)$ is not bijective:

- $\mathrm{bwt}(T) = \mathrm{bwt}(T') \iff T$ and T' are conjugates.
- Thus, not every word W is a BWT-image.
- Characterization of BWT-images exists (next)

BWT images

Idea: If a word *W* is a BWT-image, then it can be reversed:

	F	*L*
0	a	b
1	a	a
2	a	n
3	b	a
4	n	n
5	n	a

[1] a.k.a. standard permutation

BWT images

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4	n	n
5	n	a

	0	1	2	3	4	5
L	b	a	n	a	n	a
F	a	a	a	b	n	n

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F	a	a	a	b	n	n

We get: aab, of length $< n = 6$. ✗

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	F	L
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1	a	a
2	a	n
3	b	a
4	n	n
5	n	a

	0	1	2	3	4	5
L	b	a	n	a	n	a
F	a	a	a	b	n	n

We get: aab, of length $< n = 6$. ✗

In other words, the permutation defined by the LF-mapping[1] has more than one cycle: $\left(\begin{smallmatrix}0&1&2&3&4&5\\3&0&4&1&5&2\end{smallmatrix}\right) = (0,3,1)(2,4,5)$.

[1] a.k.a. standard permutation

BWT images

Def. Given a word W, its standard permutation π is defined by: $\pi(i) < \pi(j)$ iff (a) $W[i] < W[j]$ or (b) $W[i] = W[j]$ and $i < j$.

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Ex. `banana`, runlengths: 1,1,1,1,1,1, gcd $= 1$, π has 2 cycles: ✗

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Ex. `nnbaaa`, runlengths: 2,1,3, gcd $= 1$,
$\pi = \left(\begin{smallmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 3 & 0 & 1 & 2 \end{smallmatrix}\right) = (0, 4, 1, 5, 2, 3)$ has 1 cycle: ✓ bwt(`banana`)

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Ex. `nnnaaa`, runlengths: 3,3, gcd $= 3$,
$\pi = \left(\begin{smallmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 0 & 1 & 2 \end{smallmatrix}\right) = (0,3)(1,4)(3,5)$ 3 cycles: ✓ bwt(`ananan`)

BWT images with dollar

And with dollar?

- W has exactly one occurrence of \$ $\Longrightarrow$ $\text{gcd} = 1$.
- Thm. of Likhomanov and Shur: W is a BWT-image iff π is cyclic.
- Note that W has at most one pre-image (\$ is at the end).

When a dollar makes a BWT

[Giuliani, L., Masillo, Rizzi, TCS, 2021]

But we can ask a more complex question now:

Let $\mathrm{bwt}_{\$} : \Sigma^n \to \Sigma^n, T \mapsto \mathrm{bwt}(T\$)$ without the dollar.

ex. `banana` ↦ `annbaa`, since bwt(`banana$`) = `annb$aa`.

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Questions:

- Is $\mathrm{bwt}_{\$}$ bijective? (no)

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- If W is a $\text{bwt}_\$$-image, how many distinct T's map to it?

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Questions:

- Is $\mathrm{bwt}_{\$}$ bijective? (no)
- Can we characterize $\mathrm{bwt}_{\$}$-images?
- If W is a $\mathrm{bwt}_{\$}$-image, how many distinct T's map to it?
- How can we find these T's?

When a dollar makes a BWT

Question: Is W a $\text{bwt}_{\$}$-image? In other words, can we insert \$ somewhere to make it a BWT?

When a dollar makes a BWT

Question: Is W a bwt$_\$$-image? In other words, can we insert \$ somewhere to make it a BWT?

Ex.: $W =$ annbaa.

0	\$annbaa	-
1	a\$nnbaa	-
2	an\$nbaa	-
3	ann\$baa	-
4	annb\$aa	bwt(banana\$)
5	annba\$a	-
6	annbaa\$	bwt(nabana\$)

We call 4 and 6 nice positions.

annbaa is a bwt$_\$$-image ✓
with 2 nice positions.

When a dollar makes a BWT

Question: Is W a bwt$_\$$-image? In other words, can we insert \$ somewhere to make it a BWT?

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5	annba\$a	-
6	annbaa\$	bwt(nabana\$)

We call 4 and 6 nice positions.

annbaa is a bwt$_\$$-image ✓
with 2 nice positions.

Ex.: $W =$ banana.

0	\$banana	-
1	b\$anana	-
2	ba\$nana	-
3	ban\$ana	-
4	bana\$na	-
5	banan\$a	-
6	banana\$	-

banana is no bwt$_\$$-image. ✗

Computing nice positions

- Simple algorithm: for every i, $0 \leq i < n$, try reversing: $\mathcal{O}(n^2)$ time
- Our algorithm: $\mathcal{O}(n \log n)$ time
- def.: π_i standard permutation of W with \$ in position i
- idea: compute π_{i+1} directly from π_i in $\mathcal{O}(\log n)$ time
- smart data structure for maintaining permutations

Our algorithm

Lemma: We can get π_{i+1} from π_i with one transposition:

$$\pi_{i+1} \quad = \quad (\pi_i(i), \pi_i(i+1)) \circ \pi_i \underset{\text{\$ is in position } i}{=} (0, \pi_i(i+1)) \circ \pi_i .$$

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Algorithm **findNicePositions(W)**:

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$\pi_0 \leftarrow$ standard permutation of $\$W$
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$\mathcal{I} \leftarrow \emptyset$
For each position $i$, $0 \leq i < n$:
    if $i+1$ and $i$ in the same cycle then
        $c \leftarrow c+1$                                // split
    otherwise
        $c \leftarrow c-1$                                // merge
    update $\pi_i$ to $\pi_{i+1}$
    if $c = 1$: add $i+1$ to $\mathcal{I}$
return $\mathcal{I}$
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Analysis

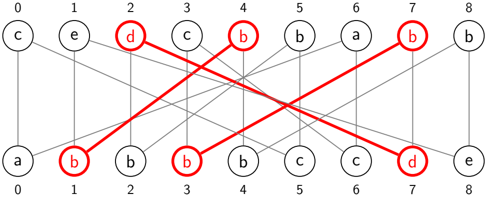
- Using **splay trees** [Sleator and Tarjan, 1985]:
    - decide whether i and $i+1$ in the same cycle in amortized $\mathcal{O}(\log n)$ time
    - update π_i in amortized $\mathcal{O}(\log n)$ time
- Altogether $\mathcal{O}(n \log n)$ time

Characterizing nice positions

Def.
$P = P_{left} \dot{\cup} P_{right}$ is called **pseudo-cycle** if $P_{left} < P_{right}$ and $\pi(P) = (P_{left} - 1) \cup P_{right}$.

ex.: $W =$ `cedcbbabb`, then $\pi = \left(\begin{smallmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 8 & 7 & 6 & 1 & 2 & 0 & 3 & 4 \end{smallmatrix}\right)$.

$P = \{2,4,7\}$, $\pi(P) = \{1,3,7\}$, $P_{left} = \{2,4\}$, $P_{right} = \{7\}$



Characterizing nice positions

Why are pseudo-cycles bad?

`cedcbbabb`

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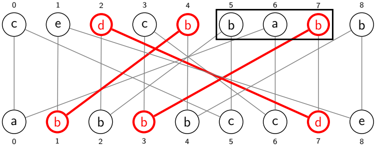

critical interval $= \{5, 6, 7\}$.

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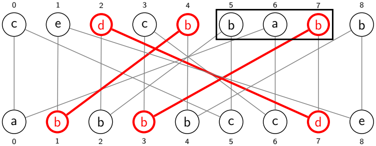

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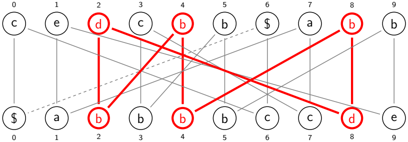

Red edges become cyles in π_6

Characterizing nice positions

Thm. Position i is nice iff there is no pseudo-cycle in π whose critical interval contains i.

4. BWT of string collections

How to compute the BWT of a set of strings?

[Cenzato and L., CPM 2022]

ex. $\mathcal{M} = \{\texttt{ATATG}, \texttt{TGA}, \texttt{ACG}, \texttt{ATCA}, \texttt{GGA}\}$

It turns out that there are many non-equivalent methods in use:

variant (our terminology)	result on example	tools
eBWT	CGGGATGTACGTTAAAAA	pfpebwt
dollarEBWT	GGAAACGG$$$TTACTGT$AAA$	G2BWT, pfpebwt, msbwt
multidolBWT	GAGAAGCG$$$TTATCTG$AAA$	BCR, ropebwt2, nvSetBWT, Merge-BWT, eGSA, eGAP, bwt-lcp-parallel, gsufsort
concatBWT	$AAGAGGGC$#$TTACTGT$AAA$	BigBWT, tools for single strings
colexBWT	AAAGGCGG$$$TTACTGT$AAA$	ropebwt2

The different BWT variants

1. eBWT($\mathcal{M}$): the extended BWT of $\mathcal{M}$ of Mantaci et al. (2007)
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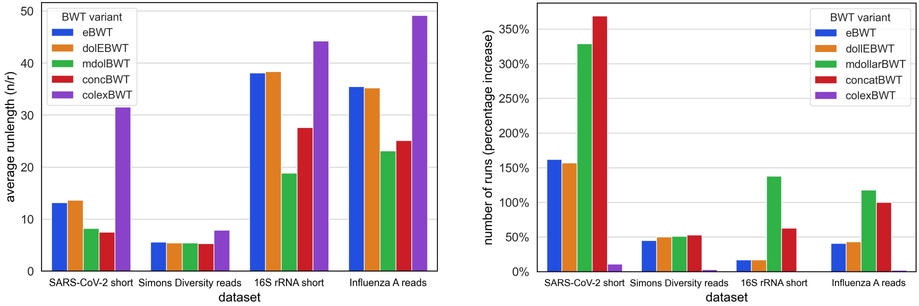
5. colexBWT($\mathcal{M}$) $= \text{multidol}(\mathcal{M}, \gamma)$, where γ is the permutation corresponding to the colexicographic ('reverse lexicographic').

The different BWT variants

BWT variant	example	order of shared suffixes
non-sep.based		
eBWT($\mathcal{M}$)	CGGGATGTACGTTAAAAA	omega-order of strings
separator-based		
dollarEBWT($\mathcal{M}$)	GGAAACGG$$$TTACTGT$AAA$	lexicographic order of strings
multidolBWT($\mathcal{M}$)	GAGAAGCG$$$TTATCTG$AAA$	input order of strings
concatBWT($\mathcal{M}$)	AAGAGGGC$$$TTACTGT$AAA$	lexicographic order of subsequent strings in input
colexBWT($\mathcal{M}$)	AAAGGCGG$$$TTACTGT$AAA$	colexicographic order

The different BWT variants

Results regarding r on short sequence datasets, of all BWT variants.



Left: average runlength (n/r). Right: number of runs r (percentage increase with respect to the optimal BWT of [Bentley et al., ESA 2020]).

The different BWT variants

- BWT variants differ significantly among each other ($>$ 11% Hamming distance on some data sets)
- we theoretically explained these differences ("interesting intervals")
- differences especially high on large sets of short sequences
- multidolBWT and concatBWT depend on the input order
- differences extend to parameter r (number of runs of the BWT) (up to a factor of 4.2 in our experiments)

Part III:

Conclusion

Dollar or no dollar, that is the question.

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4. BWT of string collections: several non-equivalent methods in use

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- Use pseudo-cycles for computing nice positions (first steps in [Giuliani, L., Masillo, ICTCS 2022])

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Thank you for your attention!

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