Algorithmic Graph Theory

A very brief and informal overview of some basic notations and definitions

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I. Graphs and graph parameters

- **graph**: $G = (V, E)$ where V is a finite set of **vertices** and E is a subset of pairs of vertices (elements of E are referred to as **edges**)
  $N(v) = \{u \in V \ : \ uv \in E\}$ is the set of **neighbors** of a vertex v, $d(v) = d_G(v) = |N(v)|$ is its **degree**
  $\Delta(G)$: maximum degree of a vertex in G
  $\delta(G)$: minimum degree of a vertex in G
- K_n: a complete graph on n vertices; C_n: a cycle on n vertices; P_n: a path on n vertices; $K_{m,n}$: a complete bipartite graph with $m + n$ vertices.
- **independent set** (**stable set**): a subset of pairwise non-adjacent vertices in a graph
  $\alpha(G)$ = **independence number** of G: maximum size of an independent set in G
- **clique**: a subset of pairwise adjacent vertices in a graph
  $\omega(G)$ = **clique number** of G: maximum size of a clique in G
- **dominating set**: a subset of vertices such that every vertex not in the set has a neighbor in the set
  $\gamma(G)$ = **domination number** of G: minimum size of a dominating set in G
- **vertex cover**: a subset of vertices such that every edge of the graph has at least one of its end-points in the set
  $\tau(G)$ = **vertex cover number** of G: minimum size of a vertex cover in G
- **matching**: a subset of pairwise disjoint edges
  perfect matching: a matching covering all vertices of the graph
  $\nu(G)$ = **matching number** of G: maximum size of a matching in G
- k**-(vertex) coloring**: a mapping $c : V \to \{1, \ldots, k\}$ such that for all $u, v \in V$ such that $\{u, v\} \in E$, it holds $c(u) \neq c(v)$
  k**-colorable graph**: a graph that admits a k-coloring
  $\chi(G)$ = **chromatic number** of G: minimum k such that G is k-colorable
- k**-edge coloring**: a mapping $c : E \to \{1, \ldots, k\}$ such that for all $e, f \in E$ such that $e \neq f$ and $e \cap f \neq \emptyset$, it holds that $c(e) \neq c(f)$
  k**-edge colorable graph**: a graph that admits a k-edge coloring
  $\chi'(G)$ = **chromatic index** of G: minimum k such that G is k-edge colorable
- **list-chromatic number** of G: minimum k such that for all choices of sets S_v, $v \in V$, with $|S_v| \geq k$, there exists an S-coloring (a mapping $c : V \to \cup_{v \in V} S_v$ such that $c(v) \in S_v$ for all $v \in V$ and $c(u) \neq c(v)$ for all $uv \in E$)
- **simplicial vertex**: a vertex whose neighborhood is a clique

II. Graph operations

- $H = (V', E')$ is a **subgraph** of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$
- $H = (V', E')$ is an **induced subgraph** of $G = (V, E)$ if $V' \subseteq V$ and $E' = \{e \in E \mid e \subseteq V'\}$; notation: $H < G$
- **disjoint union** of two graphs G and H: the graph obtained by adding to G a disjoint copy of H and no additional edges
- **join** of two graphs G and H: the graph obtained by adding to G a disjoint copy of H and all possible edges between G and H

- **complement** of a graph $G = (V, E)$: a graph $\overline{G}$ with vertex set V in which two vertices are adjacent if and only if they are non-adjacent in G
- the **line graph** of a graph $G = (V, E)$ is the graph with vertex set E in which two distinct edges e and f are adjacent if and only if they have a common endpoint in G

III. Graph classes

- **hereditary graph class**: a set of graphs closed under vertex deletions (equivalently, closed under induced subgraphs)
- **forest**: a graph without cycles; **tree**: a connected forest
- **bipartite graph**: a graph such that there exists two disjoint sets A and B with $V = A \cup B$ such that $|e \cap A| = |e \cap B| = 1$ for every edge $e \in E$; equivalently, a 2-colorable graph
- **perfect graph**: a graph such that $\chi(H) = \omega(H)$ for all its induced subgraphs H; equivalently: a $\{C_5, C_7, \overline{C_7}, C_9, \overline{C_9}, \ldots\}$-free graph
- **threshold graph**: a graph such that there exists non-negative weights $w : V \to \mathbb{R}_+$ and a threshold t such that for every $I \subseteq V$, $\sum_{v \in I} w(v) \leq t$ if and only if I is an independent set; equivalently: a $\{2K_2, C_4, C_5\}$-free graph
- **split graph**: a graph that admits a partition of its vertex set into a clique and an independent set; equivalently: a $\{2K_2, C_4, C_5\}$-free graph
- **cograph**: a graph that can be recursively built from copies of the one-vertex graph by iteratively applying the operations of disjoint union and join; equivalently: a P_4-free graph
- **chordal graph**: a graph in which every cycle of lenth at least 4 has a **chord** (an edge connecting two non-consecutive vertices of the cycle); equivalently: a $\{C_4, C_5, \ldots\}$-free graph
- **interval graph**: intersection graph of closed intervals on the real line
- **planar graph**: a graph that can be drawn in the plane without edge crossings

IV. Graph problems

- Independent Set: Given a graph G and an integer k, is $\alpha(G) \geq k$?
- Clique: Given a graph G and an integer k, is $\omega(G) \geq k$?
- Vertex Cover: Given a graph G and an integer k, is $\tau(G) \leq k$?
- Matching: Given a graph G and an integer k, is $\nu(G) \geq k$?
- Dominating Set: Given a graph G and an integer k, is $\gamma(G) \leq k$?
- Independent Dominating Set: Given a graph G and an integer k, is there an independent dominating set of size at most k?
- Colorability: Given a graph G and an integer k, is $\chi(G) \leq k$?
- k-Colorability: Given a graph G, is $\chi(G) \leq k$?
- Chromatic Index: Given a graph G and an integer k, is $\chi'(G) \leq k$?
- Recognition of Graphs in X: Given a graph G, is $G \in X$?