Algorithmic Graph Theory

Part III

Perfect Graphs and Their Subclasses

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What we'll do

1. THE BASICS.
2. PERFECT GRAPHS.
3. COGRAPHS.
4. CHORDAL GRAPHS.
5. SPLIT GRAPHS.
6. THRESHOLD GRAPHS.
7. INTERVAL GRAPHS.

THE BASICS.

Induced Subgraphs

Recall:

Definition

Given two graphs $G = (V, E)$ and $G' = (V', E')$,
we say that G is an induced subgraph of G' if
$V \subseteq V'$
and $E = \{uv \in E' \ : \ u, v \in V\}$.

Equivalently: G can be obtained from G' by deleting vertices.

Notation: $G < G'$

Hereditary Graph Properties

Hereditary graph property (hereditary graph class)
= a class of graphs closed under deletion of vertices
= a class of graphs closed under taking induced subgraphs

Formally:
a set of graphs X such that

$$G \in X \text{ and } H < G \quad \Rightarrow \quad H \in X$$

.

Hereditary Graph Properties

Hereditary graph property (Hereditary graph class)
= a class of graphs closed under deletion of vertices
= a class of graphs closed under taking induced subgraphs

Examples:

- forests
- complete graphs
- line graphs
- bipartite graphs
- planar graphs
- graphs of degree at most Δ
- triangle-free graphs
- perfect graphs

Hereditary Graph Properties

Why hereditary graph classes?

- Vertex deletions are very useful for developing algorithms for various graph optimization problems.
- Every hereditary graph property can be described in terms of *forbidden induced subgraphs*.

Hereditary Graph Properties

H-free graph = a graph that does not contain *H* as an induced subgraph
Free(*H*) = the class of *H*-free graphs

$Free(\mathcal{M}) := \bigcap_{H \in \mathcal{M}} Free(H)$
$\mathcal{M}$-free graph = a graph in $Free(\mathcal{M})$

Proposition

X hereditary $\iff X = Free(\mathcal{M})$ *for some* $\mathcal{M}$

- $\mathcal{M} = \{$all (minimal) graphs not in $X\}$

The set $\mathcal{M}$ is the set of **forbidden induced subgraphs** for X.

Examples

$\mathcal{M}$ can be **finite**:

- cographs
  $= P_4$-free graphs
- line graphs
- claw-free graphs $= K_{1,3}$-free graphs
- triangle-free graphs $= K_3$-free graphs
- graphs of degree at most Δ

…or **infinite**:

- forests $= \{\text{cycles}\}$-free graphs
- bipartite graphs = $\{\text{odd cycles}\}$-free graphs
- chordal graphs = $\{\text{cycles of order} \geq 4\}$-free graphs
- perfect graphs
- planar graphs

Comparing Hereditary Graph Classes

Proposition

For every two sets $\mathcal{M}_1$ and $\mathcal{M}_2$ of graphs, it holds that:

$$Free(\mathcal{M}_1) \subseteq Free(\mathcal{M}_2)$$

if and only if

$$(\forall G_2 \in \mathcal{M}_2)(\exists G_1 \in \mathcal{M}_1)(G_1 < G_2)\,.$$

Exercise

Prove the above equivalence.

Example:
$\mathcal{M}_1 = \{P_4, C_4\}$,
$\mathcal{M}_2 = \{C_4, C_5, C_6, \ldots\}$.

Recognition Problems

For a given graph class X we can define the following problem:

RECOGNITION OF GRAPHS IN X
Input: A graph G.
Question: Is $G \in X$?

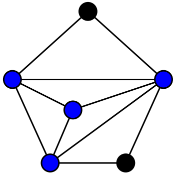
Examples:

- If $X =$ the class of all 3-colorable graphs, the recognition problem is NP-complete.
- If $X =$ the class of graphs G such that $\chi(G) = \max_{H \subseteq G} \delta(H) + 1$, the recognition problem is NP-complete.
- If $X = Free(\mathcal{M})$ where $\mathcal{M}$ is finite then the recognition problem is in P. (Why?)

PERFECT GRAPHS.

Recall:
$\omega(G)$: clique number of G = the maximum size of a clique in G.

clique = a subset of pairwise adjacent vertices



$\alpha(G)$: max size of an independent set in G

- C is a clique in $G \Leftrightarrow C$ is independent in $\overline{G}$:
- $\omega(G) = \alpha(\overline{G})$

Perfect Graphs

Example:

$\omega(K_n) = n$,

$$\omega(C_n) = \begin{cases} 3, & \text{if } n = 3; \\ 2, & \text{otherwise.} \end{cases}$$

Recall the inequality:

$$\chi(G) \geq \omega(G)\,.$$

Definition

A graph G is perfect, if

$$\chi(H) = \omega(H)$$

holds for every induced subgraph H of G.

Clearly, the class of perfect graphs is hereditary.

Perfect Graphs

Theorem (Lovász 1972, Perfect Graph Theorem)

A graph G is perfect if and only if its complement $\overline{G}$ is perfect.

Examples of non-perfect graphs:

- odd cycles of order at least 5: $C_5, C_7, C_9, \ldots$

  $\chi(C_{2k+1}) = 3$

  $\omega(C_{2k+1}) = 2.$

- their complements: $\overline{C_5}, \overline{C_7}, \overline{C_9}, \ldots$

  $\chi(\overline{C_{2k+1}}) =$ smallest number of pairwise disjoint cliques covering all vertices of $C_{2k+1} = k + 1$

  $\omega(\overline{C_{2k+1}}) = \alpha(C_{2k+1}) = k$

Berge Graphs

Berge graph: a $\{C_5, C_7, \overline{C_7}, C_9, \overline{C_9}, \ldots\}$-free graph.

Claude Berge, 1926–2002, a French mathematician

He was also a sculptor,
collector and expert on primitive art,
founding member of the literary group **Oulipo**,
a Hex and chess player.

The Strong Perfect Graph Theorem

Berge graph: a $\{C_5, C_7, \overline{C_7}, C_9, \overline{C_9}, \dots\}$-free graph.

Clearly, every perfect graph is Berge.

Conjecture (Berge 1963)

A graph G is perfect if and only if it is Berge.

Strong Perfect Graph Theorem (Chudnovsky, Robertson, Seymour, Thomas 2002)

A graph G is perfect if and only if it is Berge.

Total length of the proof $\approx$ 150 pages.

The Strong Perfect Graph Theorem

Theorem

Let G be a Berge graph. Then either:

- *G belongs to a basic class; that is, either:*
  - *G or $\overline{G}$ is bipartite, or*
  - *G or $\overline{G}$ is the line graph of a bipartite graph, or*
  - *G is a double split graph,*

  or G admits one of the following:
  - *a 2-join,*
  - *a complement of 2-join,*
  - *a balanced skew partition.*

The Strong Perfect Graph Theorem

Why does SPGT follow from the decomposition theorem?

Suppose the SPGC is false.
There is a smallest counterexample G.
G is not in any of the basic classes, since those graphs are perfect.
G does not admit any of the four types of decomposition since each of these decompositions preserves perfectness.
Contradiction. □

Algorthmic Aspects of Perfect Graphs

Some important NP-complete graph algorithmic problems are solvable in polynomial time for perfect graphs:

- COLORABILITY,
- INDEPENDENT SET,
- CLIQUE.

These results are due to Grötschel-Lovász-Schrijver (1984) and are not combinatorial.

- They are based on semidefinite programming and the ellipsoid method.

Existence of combinatorial algorithms is an open problem.

Recognizing Perfect Graphs

Theorem (Chudnovsky, Cornuéjols, Liu, Seymour, Vuković 2005)

There is a polynomial-time algorithm for recognizing Berge graphs.

- $O(|V|^9)$
- 36 pages
- independent of the proof of SPGT

Graphs Without Odd Holes

Does the input graph contain an odd cycle?

Solvable in P.

hole: a cycle of order at least 4

Does the input graph contain an odd hole?

Open!

Theorem (Bienstock 1991)

Testing whether a graph contains an odd hole through a given vertex is NP*-complete.*

Classes of Perfect Graphs

Some classes of perfect graphs:

- bipartite graphs and their complements
- line graphs of bipartite graphs (and their complements)
- cographs
- chordal graphs
- split graphs
- threshold graphs
- interval graphs

Cographs.

Cographs

Definition

Cographs:

- K_1 is a cograph
- If G and H are cographs, then so is their disjoint union.
- If G and H are cographs, then so is their join.
- There are no further cographs.

Exercise

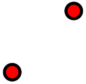
Prove that the class of cographs is hereditary.

Cographs

Example:

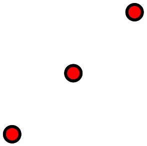
Cographs

Example:



Cographs

Example:



Cographs

Example:

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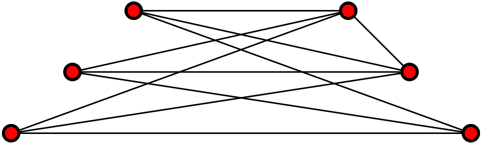

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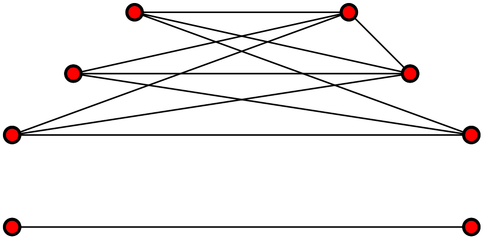

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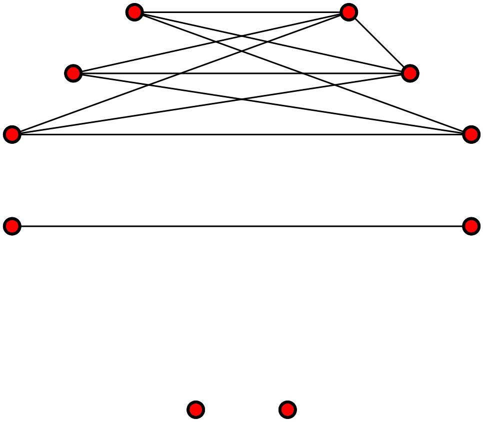

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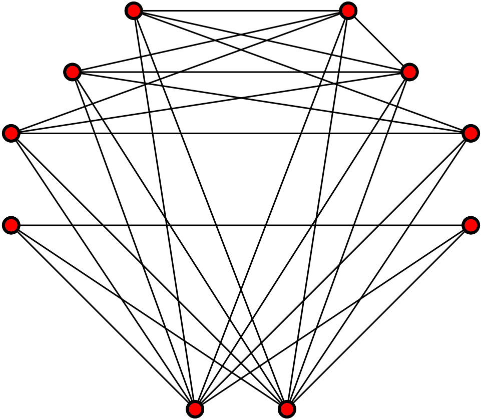


For every cograph $G \neq K_1$, either G or $\overline{G}$ is disconnected.

Exercise

Show that every cograph is perfect, using only the definitions of the two classes.

Properties of Cographs

The following graph is not a cograph:

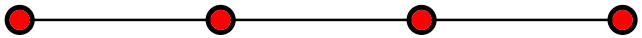


Figure: P_4: a self-complementary connected graph

Theorem

G is a cograph if and only if G is P_4-free.

Corollary

Recognition of cographs is in P.

Properties of Cographs

Theorem (Corneil, Perl, and Stewart 1985)

Cographs can be recognized in linear time.

The recognition algorithm uses modular decomposition.

Theorem

G is a cograph if and only if G is P_4-free.

Proof

It can be proved by induction on the number of vertices that *every cograph is P_4-free*.

We prove that every P_4-free graph is a cograph.

For a contradiction, let $G = (V, E)$ be a minimal counterexample.
(G is a P_4-free graph on n vertices that is not a cograph, while every P_4-free graph on less than n vertices is a cograph.)

Both G and $\overline{G}$ are connected.

Let $x \in V(G)$. Then $G - x$ is a cograph.

Since $n > 2$, we may assume that $G - x$ is disconnected (else replace G with its complement).

Proof

Since $\overline{G}$ is connected, there exists a vertex y not adjacent to x.

Let C be the component of $G - x$ containing y.

Since G is connected, x has a neighbor z in C.

We can then find two adjacent vertices u and v in C such that $ux \in E$ and $vx \notin E$.

Let D be a component of $G - x$ different from C.

Let w be a neighbor of x in D.

G contains an induced P_4 on the vertices (v, u, x, w).

Contradiction.

Two Exercises

Exercise 1:

What are the P_3-free graphs?

Exercise 2:

What are the bipartite P_4-free graphs?

Cographs: Algorithmic Aspects

Proposition

The following problems are polynomially solvable for cographs:

(a) INDEPENDENT SET,
(b) CLIQUE,
(c) DOMINATING SET.
(d) COLORABILITY.

For example, $\alpha(G)$ can be computed recursively as follows:

- $\alpha(K_1) = 1$
- If K is the disjoint union of G and H then

$$\alpha(K) = \alpha(G) + \alpha(H)\,.$$

- If K is the join of G and H then

$$\alpha(K) = \max\{\alpha(G), \alpha(H)\}\,.$$

CHORDAL GRAPHS.

Chordal Graphs

Definition

A graph is chordal if every cycle on at least 4 vertices contains a chord.

chord: an edge connecting two non-consecutive vertices of the cycle.

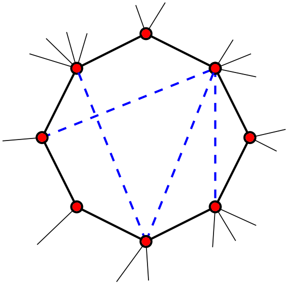
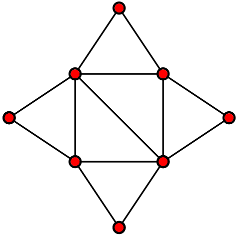


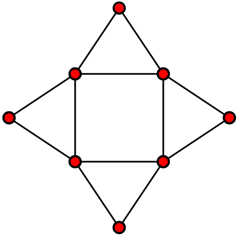
Figure: A cycle with four chords.

Chordal Graphs

Example:



chordal



not chordal

Perfectness of Chordal Graphs

A graph is chordal if and only if it is $\{C_4, C_5, \ldots\}$-free.

Proposition

Every chordal graph is perfect.

Proof: We apply the SPGT.
If a chordal graph G is not perfect then
$G \notin \overline{Free}(\{C_5, C_7, \overline{C_7}, C_9, \overline{C_9}, \ldots\})$.
$\Rightarrow \overline{C_{2k+1}} < G$ for some $k \geq 3$.
Since $C_4 < \overline{C_{2k+1}}$, it follows that $C_4 < G$. Contradiction. $\square$

Chordal Graphs: the Intersection Model

Theorem (Gavril, 1974)

Chordal graphs are precisely the vertex-intersection graphs of subtrees in a tree.

Chordal Graphs: the Intersection Model

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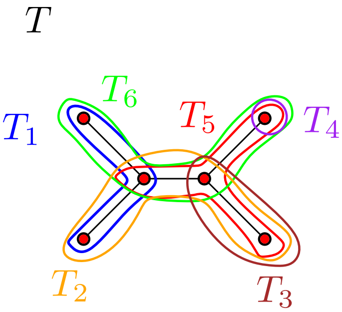
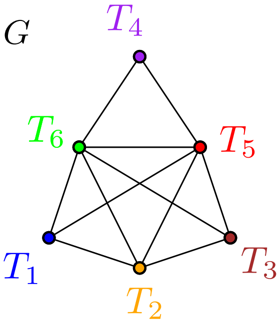
Example:





Chordal Graphs: Structural Properties

A cutset: a set of vertices $X \subseteq V$ such that the graph $G - X$ is disconnected.

Theorem (Dirac, 1961)

Every minimal cutset in a chordal graph is a clique.

Chordal Graphs: Structural Properties

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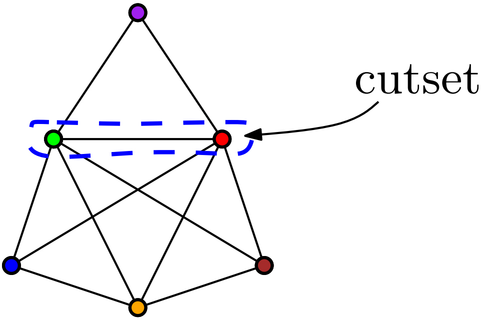


Proof

By contradiction. Suppose X is a minimal cutset in G containing two non-adjacent vertices x and y.

Choose two components C and D of the (disconnected) graph $G - X$.

By the minimality of X, every vertex of X has a neighbor in every component of $G - X$.

Let P be a shortest x-y path all of whose internal vertices belong to C.
Let Q be a shortest x-y path all of whose internal vertices belong to D.

Then $P \cup Q$ is a chordless cycle on at least 4 vertices.

Contradiction. $\square$

Chordal Graphs: Structural Properties

A vertex is simplicial if its neighborhood is a clique.

Corollary

Let G be a chordal graph. Then,

(i) *G is either complete or it contains a pair of non-adjacent simplicial vertices.*

(ii) *G contains a simplicial vertex.*

Theorem (Fulkerson and Gross, 1965)

A graph is chordal if and only if it has a perfect elimination ordering.

A permutation $(v_1, \ldots, v_n)$ of the vertices of a graph G is a perfect elimination ordering if each v_i is a simplicial vertex of $G[v_i, \ldots, v_n]$.

Chordal Graphs: Algorithmic Aspects

Theorem

Every chordal graph contains a simplicial vertex.

If G is chordal and $v \in V(G)$ then $G - v$ is chordal.

With iterative deleting of simplicial vertices, it is easy to develop polynomial time algorihtms for the following problems on chordal graphs:

- CLIQUE,
- COLORABILITY,
- INDEPENDENT SET.

Chordal Graphs: Algorithmic Aspects

Suppose v is a simplicial vertex in a chordal graph G.

- CLIQUE:

$$\omega(G) = \max\{d(v) + 1, \omega(G - v)\}\,.$$

- COLORABILITY:

$$\chi(G) = \max\{d(v) + 1, \chi(G - v)\}\,.$$

  Apply the greedy coloring algorithm to the vertices in the reverse of a perfect elimination ordering.

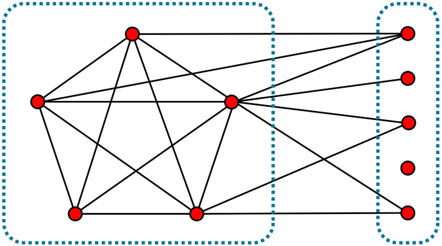
- INDEPENDENT SET:

$$\alpha(G) = 1 + \alpha(G - N[v])\,.$$

Split Graphs.

Definition

A graph is split if there exists a partition of its vertex set into a clique and an independent set.

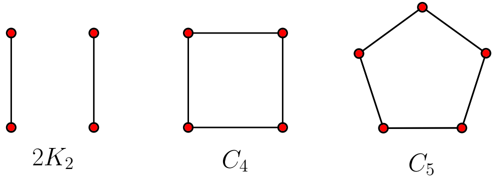


Source: http://en.wikipedia.org/wiki/Split_graph

Forbidden Induced Subgraphs

Theorem (Földes and Hammer, 1977)

A graph is split if and only if it is $\{2K_2, C_4, C_5\}$*-free.*



Exercise

Prove the *if* part of the theorem.

Other Properties

Corollary

A graph is split if and only if its complement is a split graph.
A graph G is a split graph if and only if both G and $\overline{G}$ are chordal.

Theorem

Split graphs are precisely the vertex-intersection graphs of subtrees of a star.

Theorem

Let $d_1 \geq d_2 \geq \ldots \geq d_n$ be the degree sequence of a graph G. Also, let $m = \max\{i : d_i \geq i - 1\}$. Then, G is a split graph if and only if $\sum_{i=1}^{m} d_i = m(m-1) + \sum_{i=m+1}^{n} d_i$.

Algorithmic Aspects

Split graphs can be recognized in linear time.

Other algorithmic problems on split graphs:

- COLORABILITY? In P.
- CLIQUE? In P.
- INDEPENDENT SET? In P.
- DOMINATING SET? NP-complete.

Open Problem. Give the forbidden induced subgraph characterization of graphs that can be partitioned into a clique and a graph of maximum degree at most 1.

THRESHOLD GRAPHS.

Definition

Definition

A graph $G = (V, E)$ is threshold if there exist positive real vertex weights $w(v)$ for all $v \in V$ and a threshold $t \in \mathbb{R}$ such that for every vertex set $X \subseteq V$,

$$X \text{ is independent} \quad \text{if and only if} \quad \sum_{v \in X} w(v) \leq t \,.$$

Definition

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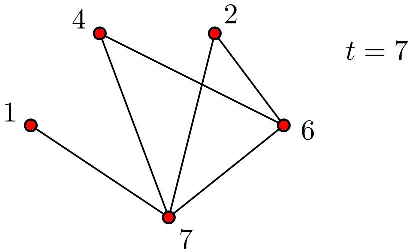
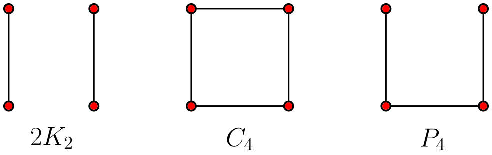


Forbidden Induced Subgraphs

Theorem (Chvátal, Hammer 1977)

A graph is threshold if and only if it is $\{2K_2, C_4, P_4\}$-free.



Further Characterizations

Theorem

The following properties are equivalent for a graph G:

1. *G is threshold.*
2. *G is a split cograph.*
3. *There exist positive real vertex weights $w(v)$ for all $v \in V$ and a threshold $t \in \mathbb{R}$ such that for every two distinct vertices $u, v \in V$,*

$$uv \in E \quad \textit{if and only if} \quad w(u) + w(v) \geq t\,.$$

4. *G can be constructed from the one-vertex graph by repeated applications of the following two operations:*
   - *Addition of a single isolated vertex to the graph.*
   - *Addition of a single dominating vertex to the graph.*

Algorithmic Aspects

Threshold graphs can be recognized in linear time.

Other algorithmic problems on threshold graphs:

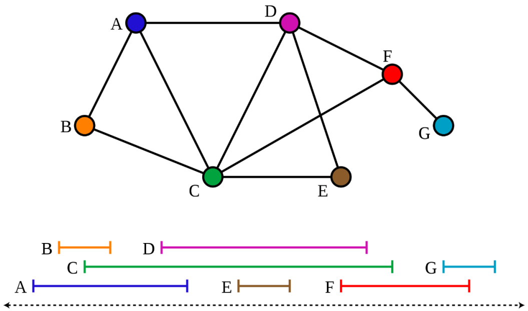
- COLORABILITY? In P.
- CLIQUE? In P.
- INDEPENDENT SET? In P.
- DOMINATING SET? In P.

INTERVAL GRAPHS.

Definition

Definition

A graph is an interval graph if its vertices can be put into one-to-one correspondence with a set of intervals on the real line such that two vertices are connected by an edge if and only if their corresponding intervals have nonempty intersection.



Source: http://en.wikipedia.org/wiki/Interval_graph

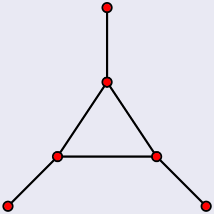
Two Exercises

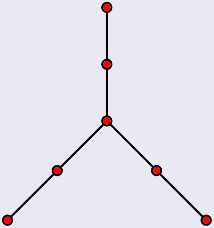
Exercise 1

Prove that interval graphs are chordal.

Exercise 2

Prove that the following two graphs are not interval:





Algorithmic Aspects

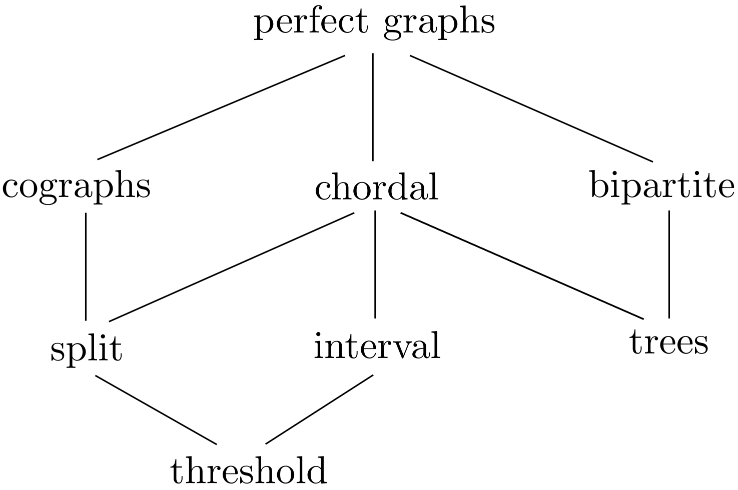
Theorem (Booth and Lueker 1976)

Interval graphs can be recognized in linear time.

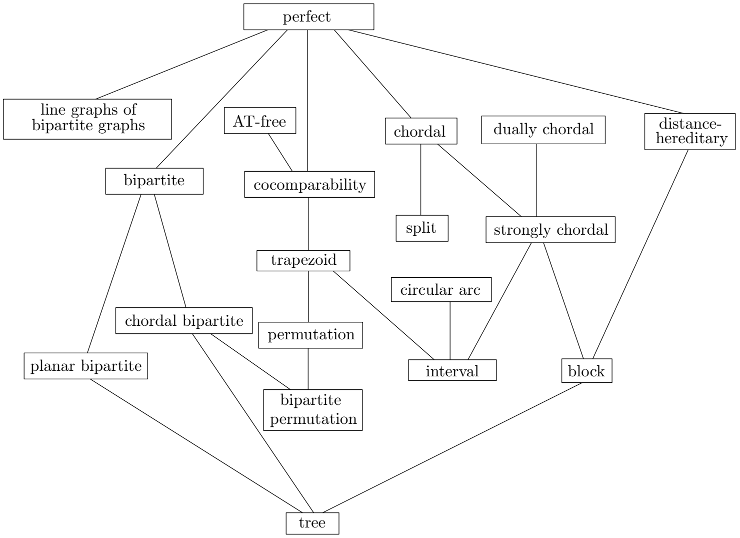
Other algorithmic problems on interval graphs:

- COLORABILITY? In P.
- CLIQUE? In P.
- INDEPENDENT SET? In P.
- DOMINATING SET? In P.

Hasse Diagram of Some Classes of Perfect Graphs



Hasse Diagram of Some Classes of Perfect Graphs



What we'll do – Week 1

1. Tue March 5: Review of basic notions in graph theory, algorithms and complexity ✓
2. Wed March 6: Graph colorings ✓
3. Thu March 7–8: Perfect graphs and their subclasses✓

What we'll do – Week 2

1. Tue March 19: Further examples of tractable problems, part 1
2. Wed March 20:
   Further examples of tractable problems, part 2
   Approximation algorithms for graph problems
3. Thu March 21: Lectio Magistralis lecture, *“Graph classes: interrelations, structure, and algorithmic issues”*