Algorithmic Graph Theory
Part II - Graph Colorings

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What we'll do

1. THE CHROMATIC NUMBER OF A GRAPH.
2. HADWIGER'S CONJECTURE.
3. BOUNDS ON χ.
4. EDGE COLORINGS.
5. LIST COLORINGS.
6. ALGORITHMIC ASPECTS OF GRAPH COLORING.
7. APPLICATIONS OF GRAPH COLORING.

THE CHROMATIC NUMBER OF A GRAPH.

Chromatic Number

Definition

A k-coloring of a graph $G = (V, E)$ is a mapping

$$c : V \to \{1, \ldots, k\}$$

such that

$$uv \in E \ \Rightarrow\ c(u) \neq c(v)\,.$$

G is k-colorable if there exists a k-coloring of it.

$\chi(G)$ = chromatic number of G = the smallest number k such that G is k-colorable.

Graph Coloring

Example:

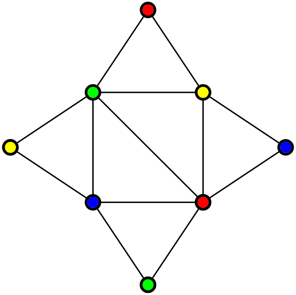


Figure: A 4-coloring of a graph

k-coloring $\equiv$ partition $V = I_1 \cup \ldots \cup I_k$,
where I_j is a (possibly empty) independent set

independent set = a set of pairwise non-adjacent vertices

Graph Coloring

Example:

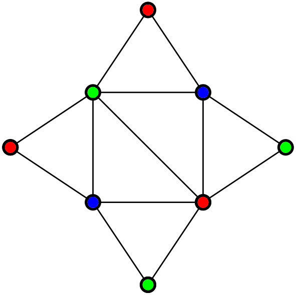


Figure: A 3-coloring of the same graph

k-coloring $\equiv$ partition $V = I_1 \cup \ldots \cup I_k$,
where I_j is a (possibly empty) independent set

independent set = a set of pairwise non-adjacent vertices

Graph Coloring

Examples:

complete graphs: $\chi(K_n) = n$.

cycles: $\chi(C_n) = \begin{cases} 2, & \text{if } n \text{ is even;} \\ 3, & \text{if } n \text{ is odd.} \end{cases}$

$\chi(G) \leq 2 \;\Leftrightarrow\; G$ is bipartite.

Greedy Coloring

How could we color, in a simple way, the vertices of a graph?

Order the vertices of G linearly, say $(v_1, \ldots, v_n)$.
for $i = 1, \ldots, n$ **do**
    $c(v_i) :=$ smallest available color
end for

Greedy Coloring

How many colors are used?

For every vertex v_i, at most $d(v_i)$ different colors are used on its neighbors.

The algorithm produces a coloring with at most $\Delta(G)+1$ colors.

$\Delta(G) = \max_{v \in V(G)} d(v)$ is the maximum vertex degree

Time complexity: $O(|V|+|E|)$

Proposition

For every graph G,

$$\chi(G) \leq \Delta(G)+1\,.$$

Greedy Coloring

Remarks:

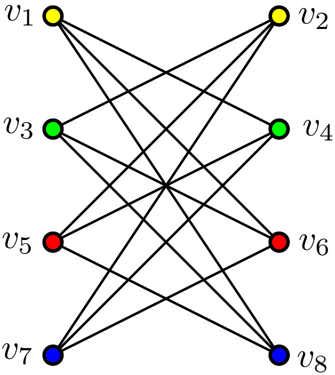
1. The gap between $\chi(G)$ and $\Delta(G)$ can be arbitrarily large.

   Example: Take $G = K_{1,n}$, a star.

   $\chi(K_{1,n}) = 2$,
   while
   $\Delta(K_{1,n}) = n$.
2. The greedy method can perform arbitrarily badly.

   Example:
   G = complete bipartite graph $K_{n,n}$ minus a perfect matching.

Example:



Improved Greedy Coloring

Order the vertices of G linearly as $(v_1, \ldots, v_n)$ so that each v_i is a vertex of minimum degree in the graph induced by first i vertices

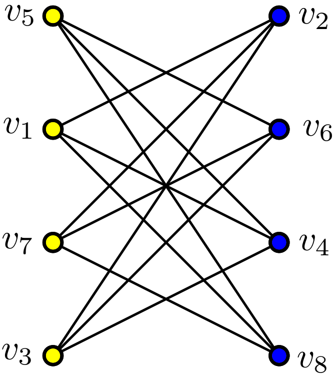
$$H_i = G[\{v_1, \ldots, v_i\}] .$$

(First determine v_n, then v_{n-1}, etc.)

for $i = 1, \ldots, n$ **do**
    $c(v_i) :=$ smallest available color
end for

Improved Greedy Coloring

Example:



Analysis of the Improved Greedy Coloring

At the time of coloring v_i, at most $d_{H_i}(v_i)$ different colors are used among its neighbors, where $H_i = G[\{v_1, \ldots, v_i\}]$.

By the choice of v_i, we have

$$d_{H_i}(v_i) = \delta(H_i)\,,$$

minimum vertex degree in H_i.

The algorithm uses at most $k + 1$ colors, where

$$k = \max_{H \subseteq G} \delta(H)$$

is the **degeneracy** of G.

Improved Greedy Coloring

Proposition

For every graph G,

$$\chi(G) \leq \max_{H \subseteq G} \delta(H) + 1\,.$$

Exercise

Construct a family of bipartite graphs showing that the improved greedy coloring can perform arbitrarily badly.

Coloring Planar Graphs

Example:

Every planar graph is 5-degenerate and hence 6-colorable.

- It follows from Euler's formula that every planar graph with $n \geq 3$ vertices has at most $3n - 6$ edges.
- In particular, every planar graph has a vertex of degree at most 5.

With not much extra work, it is possible to show that every planar graph is 5-colorable.

Every Planar Graph is 5-colorable

Proof by contradiction:
Take a smallest counterexample G.
Pick a vertex v of minimum degree in G.
We know that $d_G(v) \leq 5$. From minimality, $d_G(v) = 5$.
Consider a planar embedding of G and let $v_1, \ldots, v_5$ be the clockwise ordering of the neighbors of v.
Let c be a 5-coloring of $G - v$. We may assume that $c(v_i) = i$ for $i = 1, \ldots, 5$, otherwise we can 5-color G.
If v_1 and v_3 are in different component of the subgraph induced by vertices colored 1 and 3, we can 5-color G (by switching the two colors in the component of v_1, and setting $c(v) = 1$.).
Similarly for v_2 and v_4.
So, there is a v_1-v_3 path colored 1 and 3, and there is a v_2-v_4 path colored 2 and 4.
Contradiction to planarity. $\square$

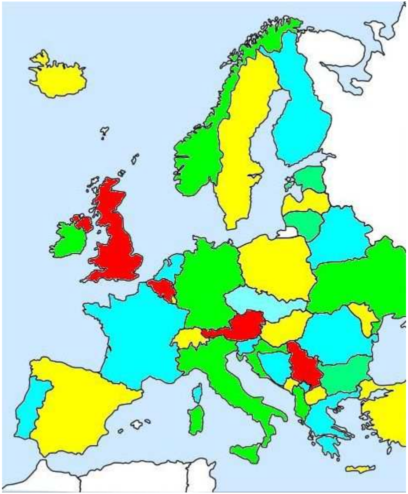
Every Planar Graph is 4-colorable

This is the famous **Four color theorem**
(Appel-Haken 1976, Robertson-Sanders-Seymour-Thomas 1997).

Every Planar Map is Four Colorable.

- First stated (as a question) in 1852.
- Proved only 124 years later (after several false proofs).
- Computer-assisted proof.

A 4-colored Map of Europe



A 4-colored Map of Central Europe

Source: `http://www.mathsisfun.com/activity/coloring.html`

Hadwiger's Conjecture.

Graph Minors

We say that a graph H is a **minor** of a graph G if H can be obtained from G by a sequence of

- vertex deletions,
- edge deletions,
- edge contractions.

Planar graphs are closed under minors.

Theorem (Wagner, 1937)

A graph is planar if and only if it has no K_5 or $K_{3,3}$ minor.

Hadwiger's Conjecture

Conjecture (Hadwiger, 1943)

For every $k \geq 2$, every graph with no K_k minor is $(k-1)$-colorable.

- A far-reaching generalization of the four color theorem.
- Still open.
- Bollobás, Catlin and Erdős call it **one of the deepest unsolved problems in graph theory**.
- A "list coloring" generalization of it was disproved recently.

  J. Barát, G. Joret, D. Wood, Disproof of the list Hadwiger conjecture, Electronic Journal of Combinatorics 18 (2011) P232.

Hadwiger's Conjecture

Conjecture (Hadwiger, 1943)

For every $k \geq 2$, every graph with no K_k minor is $(k-1)$-colorable.

What is known?

- $k = 2$: trivial
- $k = 3$: **exercise**
- $k = 4$: known to be true
- $k = 5$: equivalent to the four color theorem
- $k = 6$: proved in 1993 by Robertson, Seymour and Thomas
- $k \geq 7$: open

BOUNDS ON χ.

Upper Bounds

Greedy coloring shows:

$$\chi(G) \leq \Delta(G) + 1\,.$$

Improved greedy coloring shows:

$$\chi(G) \leq \max_{H \subseteq G} \delta(H) + 1\,.$$

Brooks' Theorem

Is the bound $\chi(G) \leq \Delta(G) + 1$ tight?

- If $G = K_n$ is a complete graph, then $\chi(G) = n = \Delta(G) + 1$.
- If $G = C_{2k+1}$ is an odd cycle, then $\chi(G) = 3 = \Delta(G) + 1$.

Theorem (Brooks, 1941)

For every connected graph G other than a complete graph or an odd cycle, we have

$$\chi(G) \leq \Delta(G)\,.$$

- Proof is a bit trickier than the proof of $\chi(G) \leq \Delta(G) + 1$, but still based on greedy coloring.
- Brooks' Theorem characterizes graphs for which equality holds in the upper bound achieved by the greedy algorithm.

Analogue for Improved Greedy Coloring?

In contrast with this, graphs for which equality holds in the upper bound achieved by the improved greedy algorithm are not easy to recognize.

Theorem (Zhu, 2011)

*It is co-*NP*-complete to determine whether a given graph G satisfies*

$$\chi(G) = \max_{H \subseteq G} \delta(H) + 1 \,.$$

X. Zhu, Graphs with chromatic numbers strictly less than their colouring numbers, Ars Mathematica Contemporanea 4 (2011) 25–27.

A Lower Bound

$$\chi(G) \geq \frac{n}{\alpha(G)}.$$

$\alpha(G)$ = the **independence number** of G = maximum size of an independent set in G.

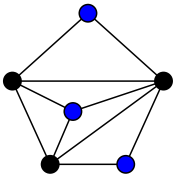


Figure: A graph with $\alpha(G) = 3$.

Exercise

Show that the bound $\chi(G) \geq \frac{n}{\alpha(G)}$ can be arbitrarily bad.

Another Lower Bound

$$\chi(G) \geq \omega(G)\,.$$

$\omega(G)$: **clique number** of G = the maximum size of a clique in G.

clique = a subset of pairwise adjacent vertices

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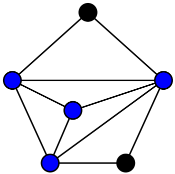


Figure: A graph with clique number 4

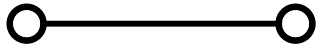
Another Lower Bound

The bound $\chi(G) \geq \omega(G)$ can be arbitrarily bad.

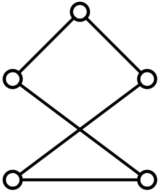
Mycielski construction.

Given a graph G with vertices $v_1, \ldots, v_n$, we define a supergraph $M(G)$.
$V(M(G)) = V(G) \cup \{u_1, \ldots, u_n; w\}$,
$E(M(G)) = E(G) \cup \{u_i v : 1 \leq i \leq n, v \in N_G(v_i) \cup \{w\}\}$.

Example:



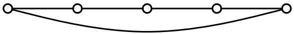
G



$M(G)$

Mycielski Construction

Example:



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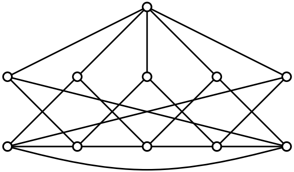
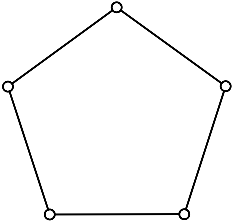


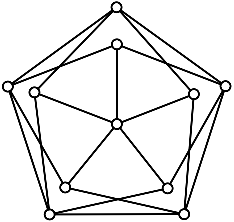
$M(G)$

Mycielski Construction

Example:



G



$M(G)$

Mycielski Construction

Proposition

(i) If G is triangle-free, then so is $M(G)$.
(ii) If G has chromatic number k then $M(G)$ has chromatic number $k+1$.

Exercise

Prove the above proposition.

Graphs of Large Girth and Large Chromatic Number

Theorem (Erdős, 1959)

For every two positive integers g and k, there exists a graph with girth more than g and chromatic number more than k.

girth = the shortest length of a cycle

Paul Erdős was the first to prove this result, using the probabilistic method:

- A random graph on n vertices, formed by choosing independently whether to include each edge with probability $p < n^{(1-g)/g}$, has (almost surely) at most $n/2$ cycles of length g or less, but no independent set of size $n/2k$.
- Removing one vertex from each short cycle leaves a smaller graph with girth greater than g, in which each color class of any coloring must be small.
- The resulting graph requires more than k colors in any coloring.

EDGE COLORINGS.

Edge Colorings

Definition

A k-edge-coloring of a graph $G = (V, E)$ is a mapping

$$c : E \to \{1, \ldots, k\}$$

such that

$$e \cap f \neq \emptyset,\ e \neq f \ \Rightarrow\ c(e) \neq c(f)\,.$$

G is k-edge-colorable if there exists a k-edge coloring of it.

$\chi'(G)$ = chromatic index of G = the smallest number k such that G is k-edge-colorable.

Edge Colorings

k-edge-coloring $\equiv$ partition $E = M_1 \cup \ldots \cup M_k$,
where M_j is a (possibly empty) matching

matching = a set of pairwise disjoint edges

Example:

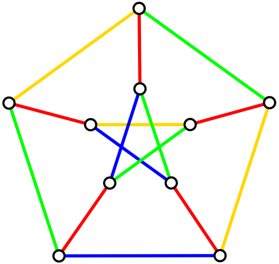


Figure: A 4-edge-coloring of the Petersen graph

Exercise: Prove that $\chi'(\mathrm{Pet}) = 4$.

Edge Colorings: Bounds

For every graph G,

$$\Delta(G) \leq \chi'(G) \leq 2\Delta(G) - 1\,.$$

- Lower bound:
  Consider the edges meeting at a point of max degree.
- Upper bound:
  Every edge is incident with at most $2\Delta(G) - 2$ other edges. Use greedy coloring.

Example:

cycles: $\chi'(C_n) = \begin{cases} 2, & \text{if } n \text{ is even;} \\ 3, & \text{if } n \text{ is odd.} \end{cases}$

Exercise

Determine the chromatic index of complete graphs.

Edge Colorings as Vertex Colorings

Edge colorings of $G \equiv$ vertex colorings of $L(G)$, the **line graph** of G:

- $V(L(G)) = E(G)$,
- two vertices $e, f \in V(L(G))$ are adjacent in $L(G)$ if and only if e and f share an endpoint.

Example:

- $L(C_n) = C_n$.

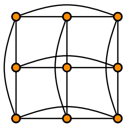
Edge Colorings as Vertex Colorings

Edge colorings of G $\equiv$ vertex colorings of $L(G)$, the **line graph** of G:

- $V(L(G)) = E(G)$,
- two vertices $e, f \in V(L(G))$ are adjacent in $L(G)$ if and only if e and f share an endpoint.

Example:

- $L(C_n) = C_n$.
- $L(K_{3,3})$:



Vizing and Kőnig's Theorems

The upper bound of $\chi'(G) \leq 2\Delta(G) - 1$ can be improved:

Theorem (Vizing, 1964)

For every (simple) graph G,

$$\chi'(G) \leq \Delta(G) + 1\,.$$

Theorem (Kőnig, 1916)

For every **bipartite** *multigraph G,*

$$\chi'(G) = \Delta(G)\,.$$

LIST COLORINGS.

List Coloring

Definition

Given sets S_v for all $v \in V$, a coloring

$$c : V \to \cup_{v \in V} S_v$$

is an S-coloring if $c(v) \in S_v$ (for all v).

List Coloring

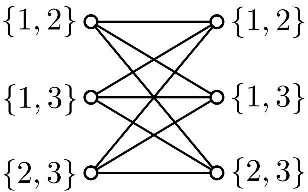
Definition

Given sets S_v for all $v \in V$, a coloring

$$c : V \to \cup_{v \in V} S_v$$

is an *S*-coloring if $c(v) \in S_v$ (for all v).

Example:



There is no S-coloring.

List Coloring

Definition

list-chromatic number (choice number / choosability) of a graph $G = (V, E)$:

$$\chi_\ell(G) = \min\{t \ : \ \text{if } |S_v| \geq t \ \forall v \text{ then there is an } S\text{-coloring}\}\,.$$

For every graph G,

$$\chi_\ell(G) \geq \chi(G)\,.$$

Example:

$\chi_\ell(K_{3,3}) = 3$.

- We showed $\chi_\ell(K_{3,3}) \geq 3$.
- $\chi_\ell(K_{3,3}) \leq 3$: Brooks' theorem also works for list colorings.

List Edge-Colorings

Analogous definitions can be made for edge colorings.

$\chi'_\ell(G)$, list-chromatic index of a graph G:

$$\chi'_\ell(G) = \min\{t \ : \text{ if } |S_e| \geq t \ \forall e \in E \text{ then there is an } S\text{-edge-coloring}\}\,.$$

List coloring conjecture (Vizing, 1976)

For every multigraph G,

$$\chi'_\ell(G) = \chi'(G)\,.$$

Dinitz' Conjecture

Special case: Dinitz' conjecture (1979)

Dinitz' conjecture

Given n^2 lists of size n each,

$$|S_{ij}| = n, \quad i,j = 1,\ldots,n\,,$$

it is always possible to fill the entries s_{ij} of an $n \times n$ array so that for all $i \neq i', j \neq j'$, we have $s_{ij} \neq s_{i'j}$ and $s_{ij} \neq s_{ij'}$.

- Such arrays are called partial Latin squares.
- Very hard to construct already for $n = 3$.

Dinitz' Conjecture

Example:

Given is the following 3×3 lists of 3 elements from the set $\{1, 2, 3, 4, 5, 6\}$:

$$\begin{matrix} \{1,2,3\} & \{1,3,4\} & \{2,5,6\} \\ \{2,3,5\} & \{1,2,3\} & \{4,5,6\} \\ \{4,3,6\} & \{3,5,6\} & \{2,3,5\} \end{matrix}$$

The following partial Latin square can be constructed:

$$\begin{matrix} 1 & 3 & 2 \\ 2 & 1 & 5 \\ 6 & 5 & 3 \end{matrix}$$

Special case: Dinitz' conjecture (1979)

Dinitz' conjecture

Given n^2 lists of size n each

$$|S_{ij}| = n, \quad i,j = 1,\ldots,n\,,$$

it is always possible to fill the entries s_{ij} of an $n \times n$ array so that for all $i \neq i'$, $j \neq j'$, we have $s_{ij} \neq s_{i'j}$ and $s_{ij} \neq s_{ij'}$.

Dinitz' conjecture is equivalent to: $\chi'_\ell(K_{n,n}) = n$.

Galvin's Theorem

Theorem (Galvin, 1995)

List coloring conjecture is true for bipartite multigraphs.

- Galvin gave an elegant proof based on the notion of kernels in digraphs.

Corollary

Dinitz' conjecture is true.

ALGORITHMIC ASPECTS OF GRAPH COLORING.

The Colorability Problem

COLORABILITY
Input: Graph G, integer k.
Question: Is $\chi(G) \leq k$?

Theorem

The COLORABILITY *problem is* NP*-complete.*

NP-completeness of Colorability

We will show a polynomial time reduction (due to Schrijver) from the following NP-complete problem:

INDEPENDENT SET
Input: Graph G, integer k.
Question: Is $\alpha(G) \geq k$?

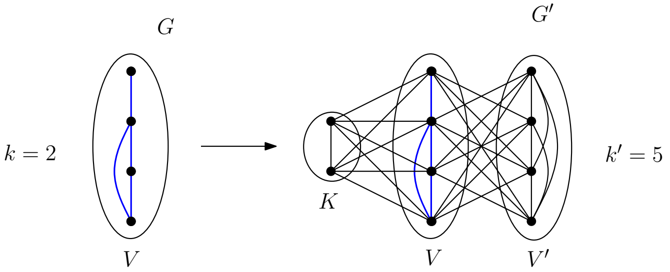
NP-completeness of Colorability

INDEPENDENT SET $\propto$ COLORABILITY:

Let $(G = (V, E), k)$ be an input for INDEPENDENT SET. We need to decide whether $\alpha(G) \geq k$.

We construct the following input (G', k') for COLORABILITY:

NP-completeness of Colorability



- $V(G') = V \cup V' \cup K$ (disjoint union), where $V' = \{v' \;:\; v \in V\}$ and $|K| = k$,
- two vertices in V are adjacent in G' $\Leftrightarrow$ they are adjacent in G
- V' and K are cliques in G',
- every vertex in V is adjacent with every vertex in $V' \cup K$, except with its copy in V',
- there are no edges between V' and K,
- $k' = n + 1$, where $n = |V|$.

We will show: $\alpha(G) \geq k \;\Leftrightarrow\; \chi(G') \leq k'$.

$(\Rightarrow)$:

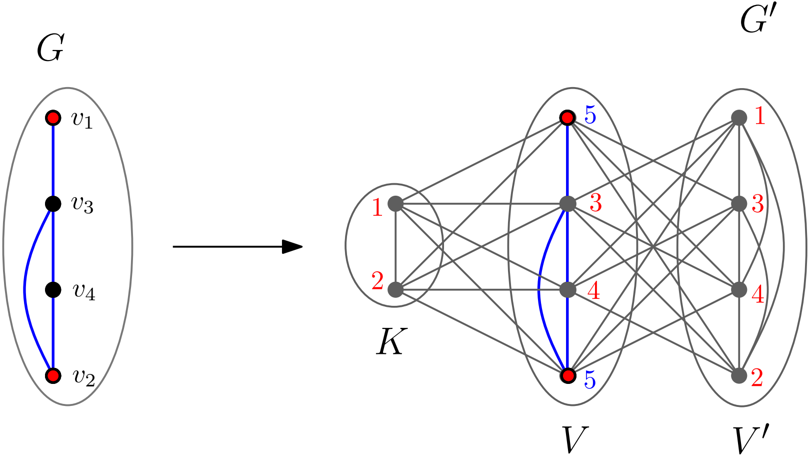
Let $\alpha(G) \geq k$ and let I be an independent set of size k in G.

Write
$I = \{v_1, \ldots, v_k\}$,
$V \setminus I = \{v_{k+1}, \ldots, v_n\}$, and
$K = \{w_1, \ldots, w_k\}$.

k'-coloring $c : V(G') \to \{1, \ldots, n+1\}$ of G' is given by:

$$c(v) = \begin{cases} i, & \text{if } v \in \{w_i, v_i'\} \text{ for some } 1 \leq i \leq k; \\ i, & \text{if } v \in \{v_i, v_i'\} \text{ for some } k+1 \leq i \leq n-k; \\ n+1, & \text{for } v \in I. \end{cases}$$

Example:



NP-completeness of Colorability

$(\Leftarrow)$:
Let $c : V(G') \to \{1, \ldots, n+1\}$ be a $(n+1)$-coloring of G'.
W.l.o.g. we may assume $c(V') = \{1, \ldots, n\}$.
Consider the set

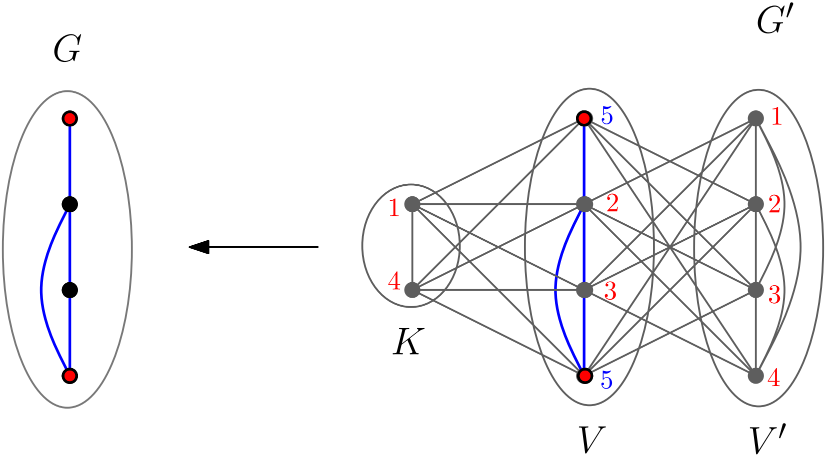
$$I = \{v \in V \ : \ c(v) \neq c(v')\}\,.$$

By construction of G', we have $c(v) = n+1$ for all $v \in I$.
$\Rightarrow I$ is an independent set in G.
For all $v \in V \setminus I$ we have $c(v) = c(v')$.
Since V' is a clique, all the $n - |I| + 1$ colors used on V are distinct.
Hence, c can use at most

$$n + 1 - (n - |I|) - 1 = |I|$$

colors on K.
Since $\chi(K) = k$, we have $|I| \geq k \Rightarrow \alpha(G) \geq k$.

Example:



k-COLORABILITY
Input: Graph G.
Question: Is $\chi(G) \leq k$?

Theorem

The k-COLORABILITY problem is NP-complete for every $k \geq 3$.

The Edge Colorability Problem

The EDGE-COLORABILITY problem is defined analogously:

EDGE-COLORABILITY
Input: Graph G, integer k.
Question: Is $\chi'(G) \leq k$?

- It is NP-complete.
- $\chi'(G) \in \{\Delta(G), \Delta(G) + 1\}$.
- Determining whether $\chi'(G) = \Delta(G)$ or $\chi'(G) = \Delta(G) + 1$ is NP-complete.

Approximability

The chromatic number is also hard to approximate.

Theorem (Zuckerman, 2007)

For all $\epsilon > 0$, approximating the chromatic number within $n^{1-\epsilon}$ is NP*-hard.*

APPLICATIONS OF GRAPH COLORING.

Storing Chemicals

- A set of chemicals needs to be stored in a warehouse.
- Certain pairs of chemicals react with each other and should not be stored in the same box.
- How many boxes do we need in order to store the chemicals safely?

Scheduling

Example 1:

- A tourist agency is organizing excursions for n groups.
- For each group we know the starting and finishing time of their excursion.
- At least how many tourist guides does the company need?

Example 2:

- An airline company needs to schedule n flights from the central hub and back with known departure and arrival times.
- What is the smallest number of aircrafts the company needs?

The resulting conflict graphs are interval graphs, and the coloring problem can be solved efficiently.

Scheduling

Example 3:

- Assume that we have a set of processors (machines) and a set of tasks.
- Each task has to be executed on two preassigned processors simultaneously.
- A processor cannot work on two jobs at the same time.
- In how many rounds can we perform all the jobs?

This is an example of an **edge coloring problem** (of multigraphs).

Register Allocation in Compiler Optimization

- A compiler is a computer program that translates one computer language into another.
- Variables in registers can be accessed much quicker than those not in registers.
- However, two variables in use at the same time cannot be assigned to the same register without corrupting its value.
- How many registers are needed to store the variables?

Assignment of Radio Frequencies

- Assume that we have a number of radio stations, identified by x and y coordinates in the plane.
- We have to assign a frequency to each station, but due to interferences, stations that are close to each other have to receive different frequencies.
  - Such problems arise in frequency assignment of base stations in cellular phone networks.

The resulting conflict graph is a unit disk graph.

- The colorability problem is 3-approximable for unit disk graphs.

Other Variants of Colorings

- Total coloring
- Harmonious coloring
- Complete coloring
- Exact coloring
- Acyclic coloring
- Star coloring
- Strong coloring
- Strong edge coloring
- Equitable coloring
- Interval edge-coloring
- T-coloring
- Rank coloring
- Circular coloring
- Path coloring
- Fractional coloring
- Oriented coloring
- Cocoloring
- Subcoloring
- Defective coloring
- Weak coloring
- Sum-coloring

What we'll do – Week 1

1. Tue March 5: Review of basic notions in graph theory, algorithms and complexity ✓
2. Wed March 6: Graph colorings ✓
3. Thu March 7: Perfect graphs and their subclasses, part 1
4. Fri March 8: Perfect graphs and their subclasses, part 2

What we'll do – Week 2

1. Tue March 19: Further examples of tractable problems, part 1
2. Wed March 20:
   Further examples of tractable problems, part 2
   Approximation algorithms for graph problems
3. Thu March 21: Lectio Magistralis lecture, *“Graph classes: interrelations, structure, and algorithmic issues”*