THE REDUCED ROW ECHELON FORM OF A MATRIX IS UNIQUE

EG

In what follows, when A is a matrix, we denote its columns by A_i. All matrices we shall consider will be $m \times n$; if a matrix is assumed to be invertible, then it will be $m \times m$.

Proposition 1. *If A and B are matrices with $B = FA$, where F is invertible, then the following holds:*

(1) *if a set of columns of B is linearly independent, then so is the set of the corresponding columns of A;*
(2) *if the column B_i of B is a linear combination of other columns of B, then the column B_i of A is a linear combination of the corresponding columns of A, with the same coefficients.*

This fact is very easily proved by noting that $B_i = FA_i$. Of course the same is true with A and B interchanged, since F is invertible.

Definition 2. A column A_k in the matrix A is *dominant* if it is not a linear combination of the columns A_i for $1 \leq i < k$.

The following proposition has a very straightforward proof.

Proposition 3. (1) *The elementary row operation of multiplying the i-th row of a matrix A by c ($c \neq 0$), then the result is the same as doing $E_i(c)A$, where $E_i(c)$ is the matrix obtained from the $m \times m$ identity matrix by performing the same operation.*

(2) *The elementary row operation of adding to the i-th row of a matrix A the j-th row multiplied by d ($i \neq j$), then the result is the same as doing $E_{ij}(d)A$, where $E_{ij}(d)$ is the matrix obtained from the $m \times m$ identity matrix by performing the same operation.*

(3) *The elementary row operation of switching the i-th and j-th rows of a matrix A ($i \neq j$), then the result is the same as doing $E_{ij}A$, where E_{ij} is the matrix obtained from the $m \times m$ identity matrix by performing the same operation.*

It is clear that the matrices $E_i(c)$, $E_{ij}(d)$ and E_{ij} are all invertible. Thus the matrix B resulting from A by performing any sequence of elementary row operations can be written as $B = FA$ for a suitable invertible matrix F. Two matrices are *row-equivalent* if one can be obtained from the other by doing a sequence of elementary row operations.

Proposition 4. *If A and B are row-equivalent, then their sets of dominant columns are the same.*

To avoid misunderstandings, what we mean by the above statement is that if, for instance, the dominant columns in A are the first, second and fourth, then the dominant columns in B are the first, second and fourth. Of course these columns may be different, but the index sets are the same.

Definition 5. An $m \times n$ matrix $U = [u_{ij}]$ is in *row echelon form* if the following holds for all $i = 1, 2, \dots, m$:

> *if the leftmost nonzero coefficient in row i is u_{ij}, then $u_{kl} = 0$ for all $l < j$ and all $k \geq i$.*

The leftmost coefficient in a row of a matrix in row echelon form is called a *pivot* and we call the column in which the pivot is a *pivot column*.

A matrix is in *reduced row echelon form* if, in addition to being in row echelon form, all its pivots are 1 and, if a pivot is in position (i, j), then all coefficients u_{kj} are zero, for $k < i$.

It is quite evident that any pivot column in a matrix in row echelon form is dominant and every nonpivot column is nondominant: the linear system for writing a column as a linear combination of the preceding columns cannot have solution in the case of a pivot column and surely has one otherwise.

In the case of a matrix in reduced row echelon form this is even easier: the pivot columns are exactly $e_1, e_2, \dots, e_r$, the first r columns of the $m \times m$ identity matrix, where r is the rank of the matrix.

It is well know that any matrix is row-equivalent to a matrix in reduced row echelon form: the Gauss-Jordan elimination does precisely this.

Theorem 6. *A matrix is row-equivalent to a unique matrix in reduced row echelon form.*

Proof. Elementary row operation cannot change the set of dominant columns, which are so predetermined by the matrix A we start with. So it is sufficient to examine a nondominant column of a matrix U in reduced row echelon form which is row-equivalent to A; assume it is the j-th column. Then it will be

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_k \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

where k is the number of pivot columns that precede it. If the pivot columns are indexed by $i_1, i_2, \dots, i_k$, this amounts to saying that

$$U_j = \alpha_1 U_{i_1} + \alpha_2 U_{i_2} + \cdots + \alpha_k U_{i_k}$$

and, by proposition 1, this means that A_j can be written *uniquely* as

$$A_j = \alpha_1 A_{i_1} + \alpha_2 A_{i_2} + \cdots + \alpha_k A_{i_k}.$$

Thus the coefficients $\alpha_1, \alpha_2, \dots, \alpha_k$ are univocally determined by A. □