Maps between Posets

(Homo|iso|epi|mono|endo|auto)-morphisms

- A morphism (or homomorphism) is an application $f \in S_1 \rightarrow S_2$ between two sets $S_1$ and $S_2$ equipped with operations

$$g \in S_1^n \rightarrow S_1$$
$$g' \in S_2^n \rightarrow S_2$$

such that $\forall x_1, \ldots, x_n \in S_1$:

$$f(g(x_1, \ldots, x_n)) = g'(f(x_1), \ldots, f(x_n))$$

- If $n = 1$ then $f \circ g = g' \circ f$, diagramatically:

- an isomorphism is a bijective morphism
- an epimorphism is an onto/surjective morphism
- an monomorphism is a one-to-one/injective morphism
- an endomorphism has $S_1 = S_2$
- an automorphism is a bijective endomorphism
– The morphism may be relative to relations \( r \subseteq S_1^n \) and \( r' \subseteq S_2^n \) such that for all \( (x_1, \ldots, x_n) \in S_1^n \):

\[
\langle x_1, \ldots, x_n \rangle \in r \implies \langle f(x_1), \ldots, f(x_n) \rangle \in r'
\]

– For binary relations:

\[
x_1 r x_2 \implies f(x_1) r' f(x_2)
\]

Complete (homo|iso|epi|mono|endo|auto)-morphisms

– A **complete morphism** (or **homomorphism**) is an application \( f : S_1 \rightarrow S_2 \) between two sets \( S_1 \) and \( S_2 \) equipped with operations

\[
G \in \wp(S_1) \rightarrow S_1
\]

\[
G' \in \wp(S_2) \rightarrow S_2
\]

such that \( \forall X \subseteq S_1 \):

\[
f(G(X)) = G'(f(X)) \text{ where } f(X) \overset{\text{def}}{=} \{ f(x) \mid x \in X \}
\]

Monotone maps

– Let \( \langle P, \leq \rangle \) and \( \langle Q, \sqsubseteq \rangle \) be two posets. A map \( f \in P \rightarrow Q \) is **monotone** iff

\[
\forall x, y \in P : (x \leq y) \implies (f(x) \sqsubseteq f(y))
\]

– Alternatives

  - order-preserving
  - isotone
  - increasing
- order morphism
- ...
- Example:

Monotony \(^1\) is self-dual (the dual of “monotone” is “monotone”)

\(^1\) Also “Monotonicity”.

Characterization of monotone maps using lubs

**Theorem.** Let \(\langle P, \leq \rangle\) and \(\langle Q, \sqsubseteq \rangle\) be two posets and \(f \in P \mapsto Q\). If \(f\) is monotone then whenever \(S \subseteq P\) and both lubs \(\bigvee S\) exists in \(P\) and \(\bigcup f(S)\) exists in \(Q\) then:

\[
\bigcup f(S) \sqsubseteq f(\bigvee S)
\]

The reciprocal is false but holds for join-semi-lattices.

**Proof.** Assume \(f\) is monotone, \(\bigvee S\) and \(\bigcup f(S)\) exist. Then \(\forall s \in S : s \leq \bigvee S\) so by monototny \(f(s) \leq f(\bigvee S)\) whence \(\bigcup f(S) \sqsubseteq f(\bigvee S)\) by def. lub.

Antitone (decreasing) maps

- Let \(\langle P, \leq \rangle\) and \(\langle Q, \sqsubseteq \rangle\) be two posets. A map \(f \in P \mapsto Q\) is **antitone** iff

\[
\forall x, y \in P : (x \leq y) \implies (f(x) \sqsubseteq f(y))
\]

- Alternatives
  - order-inversing
  - decreasing
  - ...
- Self-dual notion

- A counter-example to the reciprocal is

Conversely, for a join-semi-lattice, if \(\bigcup f(S) \sqsubseteq f(\bigvee S)\) whenever \(\bigvee S\) and \(\bigcup f(S)\) exist then when \(x \leq y\) and \(S = \{x, y\}\) we have \(\bigvee S = x \lor y = y\) so \(f(x) \sqcup f(y)\) exists in the join-semi-lattice and \(f(x) \sqcup f(y) = \bigcup f(S) \sqsubseteq f(\bigvee S) = f(y)\) whence \(f(x) \sqcup f(y) = f(y)\) which implies \(f(x) \sqsubseteq f(y)\). 

\[\square\]
The inclusion can be strict, as shown by the following example

- $f$ is monotone
- $\biggoplus f\{a, b\} = f(a) \biggoplus f(b) = x \biggoplus x = x$
- $\bigwedge z = f(c) = f(a \lor b)$

Characterization of monotone maps using glbs

**Theorem.** Let $\langle P, \leq \rangle$ and $\langle Q, \sqsubseteq \rangle$ be two posets and $f \in P \mapsto Q$. If $f$ is monotone then whenever $S \subseteq P$, the glbs $\bigwedge S$ exists in $P$ and $\bigcap f(S)$ exists in $Q$, we have:

$\bigcap f(S) \sqsubseteq f(\bigwedge S)$.

The reciprocal is false but holds for meet-semi-lattices.

**Proof.** By duality.

Order embedding

- Let $\langle P, \leq \rangle$ and $\langle Q, \sqsubseteq \rangle$ be two posets. A map $f \in P \mapsto Q$ is an order embedding (written $f \in P \mapsto Q$ or $f \in P \mapsto Q$) iff

$$\forall x, y \in P : x \leq y \iff f(x) \sqsubseteq f(y)$$

- Example:

An order embedding is injective

**Theorem.** Let $\langle P, \leq \rangle$ and $\langle Q, \sqsubseteq \rangle$ be two posets and $f \in P \mapsto Q$ be an order-embedding. $f$ is injective.

**Proof.**

$$f(x) = f(y) \implies f(x) \sqsubseteq f(y) \land f(y) \sqsubseteq f(x)$$

$$x \leq y \land y \leq x$$

$$x = y \quad \text{and so}$$

$$x \neq y \implies f(x) \neq f(y)$$

$\square$
**Order isomorphism**

- Let \( \langle P, \leq \rangle \) and \( \langle Q, \sqsubseteq \rangle \) be posets. An **order-isomorphism** is an order-embedding which is onto (whence bijective).

- Example:
  
  ![Diagram](image.png)

Example of order isomorphism: boolean encoding of finite sets

**Theorem.** Let \( X = \{x_1, x_2, \ldots, x_n\} \) be a finite set. Define

\[
\varphi : \varphi(X) \mapsto 2^n \\
\varphi(S) \overset{\text{def}}{=} \lambda i. (x_i \in S \iff \text{tt})
\]

The \( \varphi \) is an order-isomorphism between \( \langle \varphi(X), \subseteq \rangle \) and \( \langle 2^n, \preceq \rangle \) where \( \preceq \) is the componentwise ordering based on \( \text{ff} \leq \text{ff} < \text{tt} \leq \text{tt} \).

**Proof.**

- \( x \subseteq Y \)

\[
\iff \forall i \in [1, n] : x_i \in X \implies x_i \in Y \\
\iff \forall i \in [1, n] : \varphi(X)_i \leq \varphi(Y)_i \\
\iff \varphi(X) \preceq \varphi(Y) \text{ on } 2^n
\]

- If \( X \neq Y \) then there is a \( x_i \in X \) not in \( Y \) (or inversely) so \( \varphi(x_i) = \text{tt} \) and \( \varphi(Y)_i = \text{ff} \) (or inversely), proving that \( \varphi(X) \neq \varphi(Y) \) hence \( \varphi \) is injective.

- Given \( \langle b_1, \ldots, b_n \rangle \in 2^n \), we take \( S = \{x_i \in S \mid b_i = \text{tt}\} \) so that \( \varphi(S) = \langle b_1, \ldots, b_n \rangle \) proving that \( \varphi \) is onto. \( \square \)

Used to encode finite sets as bit vectors.
Embedding of a poset in its powerset

**Theorem.** Let \( \langle P, \leq \rangle \) be a poset. Then there is a set \( Q \subseteq \wp(P) \) of subsets of \( P \) such that \( \langle P, \leq \rangle \) is order-isomorphic to \( \langle Q, \subseteq \rangle \).

**Proof.**
- Define \( Q = \{ \downarrow x \mid x \in P \} \)
- Define \( \varphi \in P \mapsto Q \) by \( \varphi(x) = \downarrow x \)
- \( \varphi \) is a bijection
- \( (x \leq y) \iff (\downarrow x \subseteq \downarrow y) \)

Example:
- It follows that for a join preserving map and a finite subset \( X \subseteq P \) for which \( \lor X \) does exist:
  \[
  f(\lor X) = \bigsqcup f(X)
  \]

- The dual notion is that of meet preserving map:
  \[
  f(\land X) = \bigsqcap f(X)
  \]
  for all finite subsets \( X \subseteq P \) such that \( \land X \) exists.

Join/meet preserving maps

- let \( \langle p, \leq \rangle \) and \( \langle Q, \subseteq \rangle \) be two posets. The map \( f \in P \mapsto Q \) is called join preserving whenever if \( x, y \in P \) and the lub \( x \lor y \) exists in \( P \) then the lub \( f(x) \sqcup f(y) \) does exist in \( Q \) and is such that:
  \[
  f(x \lor y) = f(x) \sqcup f(y)
  \]

- Example:
  - \( f(c \lor d) = f(e) = z = y \lor z = f(c) \sqcup f(d) \)
  - \( b \lor c \) does not exists so there is no requirement on \( f(b) \sqcup f(c) \)

Join/meet preserving maps are monotone

**Theorem.** A join or meet preserving map is monotone

**Proof.**
- if \( x \sqsubseteq y \) then \( x \sqcup y = y \) does exists. So \( f(s \sqcup y) = f(x) \) hence \( f(x) \sqcup f(y) = f(y) \) since \( f \) preserves existing, proving that \( f(x) \sqsubseteq f(y) \) by def. of lubs.
- By duality a meet-preserving maps is monotone (since the dual of monotone is monotone)
Not all monotone maps preserve lubs/glbs

Counter-example:

- \( f \) is monotone
- \( f(x \lor y) = f(z) = b \)
- \( f(x) \sqcup f(y) = a \sqcup a = a \neq b \)

Complete join preserving maps

- Let \( \langle P, \leq \rangle \) and \( \langle Q, \subseteq \rangle \) be two posets. The map \( f \in P \Rightarrow Q \) is a **complete join preserving** whenever it preserves existing lubs:
  \[ \forall X \subseteq P : \exists \bigvee X \in P \implies f(\bigvee X) = \bigvee f(X) \]

- The dual notion is that of **complete meet preserving** map:
  \[ \forall X \subseteq P : \exists \bigwedge X \in P \implies f(\bigwedge X) = \bigwedge f(X) \]

Not all finite join/meet preserving maps are complete

- Example of finite join preserving map which is not a complete join preserving map:
Continuous and co-continuous maps

- A map \( f \in P \mapsto Q \) from a poset \( \langle P, \leq \rangle \) into a poset \( \langle Q, \sqsubseteq \rangle \) is continuous (or upper-continuous) if an only if for all chains \( C \) of \( P \) such that \( \bigvee C \) exists then \( \bigcup f(C) \) exists and we have
  \[
  f(\bigvee C) = \bigcup f(C)
  \]
- Often this hypothesis is needed only for denumerable chains. \( f \) is \( \omega \)-continuous iff for all increasing chains \( x_0 \leq x_1 \leq \ldots \leq x_n \leq \ldots \) of \( P \) such that \( \bigvee_{i \in \mathbb{N}} x_i \) exists then \( \bigcup_{i \geq 0} f(x_i) \) exists and
  \[
  f(\bigvee_{i \in \mathbb{N}} x_i) = \bigcup_{i \in \mathbb{N}} f(x_i)
  \]

Example (\( \varphi \)) and counter-example (\( \psi \):

- The reciprocal is not true. A monotone map may not be \( \omega \)-continuous, as shown by the following counter-example:
  - \( f(x) = x + 1, \ x \leq \omega \)
  - \( f(\omega + 1) = \omega + 1 \)
  - \( f \) is monotone
  - \( f \) is not continuous since
    \[
    f(\bigcup_{n<\omega} n) = f(\omega) = \omega + 1
    \]
    \[
    \bigcup_{n<\omega} f(n) = \bigcup_{n<\omega} (n + 1) = \bigcup \omega = \omega
    \]
Chain conditions and continuity

**Theorem.** Let \( \langle P, \leq \rangle \) be a poset statisfying the ascending chain condition (ACC) and \( \langle Q, \sqsubseteq \rangle \) be a poset. Then any monotone map \( f \in P \mapsto Q \) is continuous. \( \blacksquare \)

**Proof.** Let \( \langle x_\delta, \delta \in \Omega \rangle \) be an increasing chain of elements of \( P \). By the ACC, \( \exists k < \omega : \forall \delta > k : x_\delta = x_k \) so that \( \bigvee_{\delta \in \Omega} x_\delta = x_k \). It follows that \( f(\bigvee_{\delta \in \Omega} x_\delta) = f(x_k) \). Since \( \forall \delta \in \Omega : x_\delta \leq x_k \) and \( f \) is monotone, we have \( f(x_\delta) \subseteq f(x_k) \) whence \( \bigcup_{\delta \in \Omega} f(x_\delta) \subseteq f(x_k) \). But \( f(x_k) \in \{ f(x_\delta) \mid \delta \in \Omega \} \) so \( f(x_\delta) \subseteq \bigcup_{\delta \in \Omega} f(x_\delta) \) and by antisymmetry \( \bigcup_{\delta \in \Omega} f(x_\delta) = f(x_k) \). It follows that \( \bigcup_{\delta \in \Omega} f(x_\delta) = f(x_k) = f(\bigvee_{\delta \in \Omega} x_\delta) \), proving continuity. \( \Box \)

By duality, if \( \langle P, \leq \rangle \) is a poset satisfying the descending chain condition (DCC) and \( \langle Q, \sqsupseteq \rangle \) is a poset then any monotone map \( f \in P \mapsto Q \) is co-continuous.

Boolean lattice morphism

- Let \( \langle P, \lor, \land \rangle \) and \( \langle Q, \sqcup, \sqcap \rangle \) be lattices. A **lattice morphism** \( f \in P \mapsto Q \) satisfies:

\[
\begin{align*}
    f(x \lor y) &= f(x) \sqcup f(y) \\
    f(x \land y) &= f(x) \sqcap f(y)
\end{align*}
\]

- Let \( \langle P, 0, 1, \lor, \land, \neg \rangle \) and \( \langle Q, \bot, \top, \sqcup, \sqcap, \neg \rangle \) be boolean algebras. Assume \( f \) is a lattice morphism. \( f \in P \mapsto Q \) if and only if:

- \( f \) is a lattice morphism
- \( f(0) = \bot \)
- \( f(1) = \top \)
- \( f(\neg x) = f(x)' \)

On the conditions defining the Boolean lattice morphisms

**Theorem.** Let \( \langle P, 0, 1, \lor, \land, \neg \rangle \) and \( \langle Q, \bot, \top, \sqcup, \sqcap, \neg \rangle \) be boolean algebras. Assume \( f \) is a lattice morphism.

\[
\begin{align*}
    (i) & \quad (a) \ f(0) = \bot \text{ and } f(1) = \top \\
    & \quad \iff (b) \ f(\neg a) = (f(a))', \forall a \in P \\
\end{align*}
\]

(ii) If \( f(\neg a) = (f(a))' \), then

\[
\begin{align*}
    (c) & \quad f(a \lor b) = f(a) \sqcup f(b) \\
    & \quad \iff (d) \ f(a \land b) = f(a) \sqcap f(b)
\end{align*}
\]

\( \blacksquare \)
Proof. (i) Assume (a), then:
\[ \bot = f(0) = f(a \land \neg a) = f(a) \cap f(\neg a) \]
\[ \top = f(1) = f(a \lor \neg a) = f(a) \cup f(\neg a) \]
proving that \( f(\neg a) = (f(a))' \) whence (b)
Assume (b), then
\[ f(0) = f(a \land \neg a) = f(a) \land f(\neg a) \]
\[ f(1) = f(a \lor \neg a) = f(a) \lor f(\neg a) \]
proving (a)
(ii) Assume \( f \) preserves complement and join.
\[ f(a \land b) = f(\neg (\neg a \lor \neg b)) \]
\[ = (f(\neg a) \lor f(\neg b))' \]
\[ = ((f(a))' \lor (f(b))')' \]
\[ = f(a) \cap f(b) \]
\[ \square \]

Notations for monotone, lub/glb preserving and (co-)continuous maps

Let \( \langle P, \leq \rangle \) and \( \langle Q, \sqsubseteq \rangle \) be posets. We define:
\( \langle P, \leq \rangle \xrightarrow{m} \langle Q, \sqsubseteq \rangle \) (or \( P \xrightarrow{m} Q \)) if \( \leq \) and \( \sqsubseteq \) are understood to be the set of \textit{monotone} maps of \( P \) into \( Q \)
\( \langle P, \leq \rangle \xrightarrow{u} \langle Q, \sqsubseteq \rangle \) (or \( P \xrightarrow{u} Q \)) if \( \leq \) and \( \sqsubseteq \) are understood to be the set of \textit{complete lub-preserving} maps of \( P \) into \( Q \)
\( \langle P, \leq \rangle \xrightarrow{m} \langle Q, \sqsubseteq \rangle \) (or \( P \xrightarrow{m} Q \)) if \( \leq \) and \( \sqsubseteq \) are understood to be the set of \textit{complete glb-preserving} maps of \( P \) into \( Q \)

\( \langle P, \leq \rangle \xrightarrow{u} \langle Q, \sqsubseteq \rangle \) (or \( P \xrightarrow{u} Q \)) if \( \leq \) and \( \sqsubseteq \) are understood to be the set of \textit{\( \omega \)-upper-continuous} maps of \( P \) into \( Q \)
\( \langle P, \leq \rangle \xrightarrow{l} \langle Q, \sqsubseteq \rangle \) (or \( P \xrightarrow{l} Q \)) if \( \leq \) and \( \sqsubseteq \) are understood to be the set of \textit{\( \omega \)-lower-continuous} maps of \( P \) into \( Q \)
We use \( \rightarrow \) for \textit{injective} maps
\( \mapsto \) for \textit{surjective} maps
\( \leftrightarrow \) for \textit{bijective} maps

The complete lattice of pointwise ordered maps on a complete lattice

Theorem. Let \( P \) be a set and \( \langle Q, \sqsubseteq, \bot, \top, \sqcap, \sqcup \rangle \) be a complete lattice. Let \( \sqsubseteq \) be the \textit{pointwise ordering} of maps \( f \in P \mapsto L: f \sqsubseteq g \iff \forall x \in P : f(x) \sqsubseteq g(x) \). Then \( \langle P \mapsto Q, \sqsubseteq, \bot, \top, \sqcap, \sqcup \rangle \) (where \( \bot \) \textit{def} \( \lambda x. \bot \), \( \top \) \textit{def} \( \lambda x. \top \), \( \sqcup F \) \textit{def} \( \lambda x. \sqcup_{f \in F} f(x) \) and \( \sqcap F \) \textit{def} \( \lambda x. \sqcap_{f \in F} f(x) \)) is a complete lattice. \( \blacksquare \)
The complete lattice of pointwise ordered monotone maps on a complete lattice

**Theorem.** Let \( \langle P, \leq \rangle \) be a poset and \( \langle Q, \subseteq, \bot, \top, \sqcap, \sqcup \rangle \) be a complete lattice. The set of monotonic maps of \( P \) into \( Q \) is a complete lattice \( \langle P \rightarrow_{m} Q, \subseteq, \bot, \top, \sqcap, \sqcup \rangle \).

\[\text{Proof.} \quad \text{The ordering} \quad f \sqsubseteq g \iff \forall x \in P : f(x) \subseteq g(x) \text{ makes} \quad (P \rightarrow_{m} Q, \subseteq) \quad \text{a complete lattice} \]

- Since \( (P, \rightarrow_{m}) \subseteq (P, \rightarrow_{0}) \), is follows that \( (P, \rightarrow_{m}, \subseteq) \) is a poset
- The lub in \( (P \rightarrow_{0}, \subseteq) \) is \( \bigsqcup_{i \in \Delta} \) such that \( \bigcup_{i \in \Delta} f_{i}(x) = \bigcup_{i \in \Delta} (f_{i}(x)) \)
- Observe that \( \bigcup_{i \in \Delta} f_{i} \) is monotone since \( x \leq y \implies \forall i \in \Delta : f_{i}(x) \subseteq f_{i}(y) \)
- Since \( f_{i} \in P \rightarrow_{m} Q \) so \( \forall i \in \Delta : f_{i}(x) \subseteq \bigcup_{i \in \Delta} f_{i}(y) \) proving \( \bigcup_{i \in \Delta} f_{i}(x) = \bigcup_{i \in \Delta} f_{i}(y) \) that is \( \bigcup_{i \in \Delta} f_{i} \in P \rightarrow_{m} Q \)
- It follows that \( \bigcup_{i \in \Delta} f_{i} \) is also the lub in \( P \rightarrow_{m} Q \)

\( \square \)
Claude Elwood Shannon

Randal E. Bryant

Reference

Encoding Maps between Posets

Claude Elwood Shannon

Randal E. Bryant

Reference

Encoding of Boolean functions by Boolean terms
Boolean terms

- Let $\langle B, 0, 1, \lor, \land, \neg \rangle$ be a boolean algebra.
- Let $\mathcal{V}$ be a set of variables and $\langle x_1, \ldots, x_n \rangle \in \mathcal{V}^n$.
- The boolean terms $\text{Bt}(B, \langle x_1, \ldots, x_n \rangle)$ are defined by the following grammar:

\[
T ::= x_i \mid 0 \mid 1 \mid T_1 \lor T_2 \mid T_1 \land T_2 \mid \neg T_1 \mid (T_1)
\]

Encoding of Boolean functions by Boolean terms

- The encoding of $v = \langle v_1, \ldots, v_n \rangle \in 2^n$ over variables $\langle x_1, \ldots, x_n \rangle$ is:

\[
\text{Te}(v)(x_1, \ldots, x_n) = \langle v_1 = 1 \iff x_1 \land \ldots \land (v_n = 1 \iff x_n = \neg x_n)\rangle
\]

- The encoding of $f \in 2^n \mapsto 2$ over variables $\langle x_1, \ldots, x_n \rangle$ is:

\[
\text{Te}(f)(x_1, \ldots, x_n) = \bigvee \{\text{Te}(v)(x_1, \ldots, x_n) \mid v \in 2^n \land f(v) = 1\}
\]

The interpretation of Boolean terms

- The semantics or interpretation $S[T] \in 2^n \mapsto 2$ of $T \in \text{Bt}(B, \langle x_1, \ldots, x_n \rangle)$ is defined by:

\[
S[x_i](v_1, \ldots, v_n) \overset{\text{def}}{=} v_i,
S[0](v_1, \ldots, v_n) \overset{\text{def}}{=} 0,
S[1](v_1, \ldots, v_n) \overset{\text{def}}{=} 1,
S[T_1 \lor T_2](v_1, \ldots, v_n) \overset{\text{def}}{=} S[T_1](v_1, \ldots, v_n) \lor S[T_2](v_1, \ldots, v_n),
S[T_1 \land T_2](v_1, \ldots, v_n) \overset{\text{def}}{=} S[T_1](v_1, \ldots, v_n) \land S[T_2](v_1, \ldots, v_n),
S[\neg T_1](v_1, \ldots, v_n) \overset{\text{def}}{=} \neg S[T_1](v_1, \ldots, v_n),
S[(T_1)](v_1, \ldots, v_n) \overset{\text{def}}{=} S[T_1](v_1, \ldots, v_n).
\]

Theorem.

For all $a = \langle a_1, \ldots, a_n \rangle \in 2^n$ and $b = \langle b_1, \ldots, b_n \rangle \in 2^n$:

\[
S[\text{Te}(a)(x_1, \ldots, x_n)]b = 1 \quad \text{iff} \quad b = a,
S[\text{Te}(a)(x_1, \ldots, x_n)]b = 0 \quad \text{iff} \quad b \neq a
\]

Proof.

\[
S[\text{Te}(a)(x_1, \ldots, x_n)]b = \langle a_1 = 1 \land S[x_1]b : \neg S[x_1]b \land \ldots \land (a_n = 1 \land S[x_n]b : \neg S[x_n]b) \rangle
= \langle a_1 = 1 \land b_1 = \neg b_1 \land \ldots \land (a_n = 1 \land b_n = \neg b_n) \rangle
= \langle a_1 = b_1 \land \ldots \land a_n = b_n \rangle
= a = b
= \begin{cases} 1 & \text{iff } a = b \\ 0 & \text{iff } a \neq b \end{cases}
\]

\[\square\]
Bijection between Boolean functions and their encodings by Boolean terms

**Theorem.** $2^n \leftrightarrow 2^2$ and $\{T_e(f)(x_1, \ldots, x_n) | f \in 2^n \leftrightarrow 2\}$ are isomorphic by $(S, T_e)$.

**Proof.**
- Let $T \in \{T_e(f)(x_1, \ldots, x_n) | f \in 2^n \leftrightarrow 2\}$. We must show that $T_e(S[T]) = T$. Given $f \in 2^n \leftrightarrow 2$, we have $T_e(S[T_e(f)(x_1, \ldots, x_n)]) = T_e(f)$, Q.E.D. □

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**Boolean terms in disjunctive normal forms**

- A Boolean term over $\{x_1, \ldots, x_n\}$ is in **disjunctive normal form (DNF)** iff it is in the form

\[
\bigvee_{i=1}^{k} \bigwedge_{j=1}^{n} \ell_{ij} \quad \text{where } \ell_{ij} \text{ is } x_j \text{ or } \neg x_j
\]

- Any boolean term $T$ can be put in equivalent DNF.⁴

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**Example (conditional)**

\[
f(x, y, z) = (x \lor y \land z) = (x \land y) \lor (\neg x \land z)
\]

\[
= (\neg x \land z) \lor (y \lor \neg y) \lor ((x \land y) \land (z \lor \neg z))
\]

\[
= (\neg x \land y \land z) \lor (\neg x \land y \land z) \lor (x \land y \land \neg z) \lor (x \land y \land z)
\]

in so called “disjunctive normal form”.

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**Algorithm:**
- Use De Morgan’s laws to reduce the term to meets and joins of literals $x_j$ or $\neg x_j$
- Use the distributive laws, with the lattice identities to obtain a join of meets of literals
- Finally, each $x_j$ (or $\neg x_j$) should appear once and only once in each meet term:
  1. Drop any meet term containing $x_i$ and $\neg x_i$ for some $i = 1, \ldots, n$
  2. If neither $x_j$ nor $\neg x_j$ occurs in $\bigwedge_{k \in K} x^k_j$ (where $e_k \in \{0, 1\}$, $x^1 = x$, $x^0 = \neg x$) then:

\[
\bigwedge_{k \in K} x^k_j = \left( \bigwedge_{k \in K} x^k_j \right) \lor (x_j \lor \neg x_j)
\]

Repeating this process for each missing variable will lead to a term in DNF.
Encoding of Boolean functions by BDDs


Example of Shannon trees

A BDD (Binary Decision Diagram) discovered by Randal Bryant in 1986 is a compact representation of a Shannon tree of a boolean expression.
Example:

\[- f(x, y, z) = (x \land y) \land (y \land \neg z) \lor (z \lor \neg y) \]

- Table representation:

| x | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 |
| y | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 |
| z | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 |
| f | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 1 |


- Shannon tree representation (with \( x < y < z \))

(1) Sharing: merge redundant subtrees (to get a Directed Acyclic Graph — DAG)
– Let \( B = \bigcup_{n \in \mathbb{N}} B_n \)

Shannon expansion theorem:

**Theorem.** Let \( f(x_1, \ldots, x_n) \in B_n. \forall i \in [1, n] : \exists! (f_{\bar{x}_i}, f_i) \) ^\text{5} \in \mathbb{B}_{n-1} \times \mathbb{B}_{n-1} \) such that

\[
f(x_1, \ldots, x_n) = (\neg x_i \land f_{\bar{x}_i}) \lor (x_i \land f_i)
\]

**Proof.** Choose:

\[
f_{\bar{x}_i}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) = f(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n)
\]

\[
f_{\bar{x}_i}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) = (x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n)
\]

\( \square \)

---

**Shannon decomposition of Boolean functions**

- Let \( \langle \text{Var}, <^v \rangle \) be a totally strictly ordered set of variables
- Let \( \text{Var}_n = \{ V \subseteq \text{Var} \mid |V| = n \} \) be the set of \( n \) variables \( \{x_1, \ldots, x_n\} \) where, by convention, \( x_1 <^v \ldots <^v x_n \)
- Let \( B_n = \text{Var}_n \times \{\{0, 1\}^n \mapsto \{0, 1\}\} \) be the set of pairs \( \langle \{x_1, \ldots, x_n\}, f \rangle \) denoted \( f(x_1, \ldots, x_n) \) which value at point \( x_1 = b_1, \ldots, x_n = b_n \) is \( f(b_1, \ldots, b_n) \)
- Let \( V(f(x_1, \ldots, x_n)) = \{x_1, \ldots, x_n\} \) where \( x_1 <^v \ldots <^v x_n \)

---

**Shannon tree**

- A Shannon tree over variables \( x_1 <^v \ldots <^v x_n \) is
  - if \( n = 0 \) then 1 or 0
  - if \( n > 0 \) then \( \langle x_1, t_1, t_2 \rangle \) where \( t_1, t_2 \) are Shannon trees over \( x_2 <^v \ldots <^v x_n \)
- Example \( x_1 = x <^v x_2 = y \)

\[
\langle x, \langle y, 1, 0 \rangle, \langle y, 1, 1 \rangle \rangle
\]
Isomorphism between Shannon trees and Boolean functions

- A Shannon tree $t$ over variables $x_1 ^< v \ldots ^< v x_n$ represents a Boolean function
  
  $f(t)(x_1, \ldots, x_n) = \text{match } t \text{ with}$
  
  \[
  \begin{cases}
  0|1 \to t \quad \text{case } n = 0 \\
  \langle x_1, t_1, t_2 \rangle \to (x_1 \land f(t_1)(x_2, \ldots, x_n)) \\
  \lor (\neg x_1 \land f(t_2)(x_2, \ldots, x_n))
  \end{cases}
  \]

- The Shannon tree representing a Boolean function $f(x_1, \ldots, x_n)$ with $x_1 ^< v \ldots ^< v x_n$ is:

  $\text{Sh}(f(x_1, \ldots, x_n)) = \langle n = 0 \ ? \ f() :$
  
  $\langle x_1, \text{Sh}(\lambda x_2, \ldots, x_n \cdot f(0, x_2, \ldots, x_n)),$
  
  $\text{Sh}(\lambda x_2, \ldots, x_n \cdot f(1, x_2, \ldots, x_n))\rangle$

Definition of Boolean Decision Diagrams (BDD)

The BDDs are recursively defined as follows:

- 0 is a BDD
- 1 is a BDD
- if $b_1, b_2$ are BDDs, $x \in \text{Var}$ is a variable then $b = \langle x, b_1, b_2 \rangle$ is a BDD (with $\text{var}(b) = x$, $\text{left}(b) = b_1$, $\text{right}(b) = b_2$)

Example:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>0</th>
<th>1</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>x</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

$\langle x, \langle y, 1, 0 \rangle, \langle y, 1, 1 \rangle \rangle$

Example:

$\begin{align*}
  b_0 &= 0 \\
  b_1 &= 1 \\
  b_2 &= \langle z, b_1, b_0 \rangle \\
  b_3 &= \langle z, b_0, b_1 \rangle \\
  b_4 &= \langle y, b_3, b_2 \rangle \\
  &= \langle y, \langle z, 0, 1 \rangle, \langle z, 1, 0 \rangle \rangle
\end{align*}$
Ordered Boolean Decision Diagram (OBDD)

- Let \( \langle \text{Var}, <^v \rangle \) be a totally strictly ordered set of variables
- A BDD \( t \) is ordered (ordered \( b = \text{tt} \)) if and only if either \( b \in \{0, 1\} \) or
- If \( \text{left}(b) \not\in \{0, 1\} \) then \( \text{var}(b) <^v \text{var}(\text{left}(b)) \)
- If \( \text{right}(b) \not\in \{0, 1\} \) then \( \text{var}(b) <^v \text{var}(\text{right}(b)) \)
- \( \text{left}(b) \neq \text{right}(b) \)

Counter-examples:

---

Representation of a Shannon tree by an Ordered Boolean Decision Diagram (OBDD)

- The OBDD \( \text{obdd}(t) \) representing a Shannon tree \( t \) is defined as follows
  \[
  \text{obdd}(t) = \text{match } t \text{ with }\\
  \begin{array}{l}
  | 0|1 \rightarrow t \\
  | (x, t_1, t_2) \rightarrow (t_1 = t_2 \land \text{obdd}(t_1) \lor (x, \text{obdd}(t_1), \text{obdd}(t_2)))
  \end{array}
  \]

---

Boolean functions represented by an Ordered Boolean Decision Diagram (OBDD)

- An OBDD no longer represents one function of \( B \) but rather all functions whose results are the same regardless of the assignment of additional variables absent in the BDD
- **Example:** If \( \forall x, y, z : f(x, y, z) = g(y) \) then
  \[
  \text{obdd}(\text{sh}(f(x, y, z))) = \text{obdd}(\text{sh}(g(y)))
  \]
  For example if \( g(y) = \neg y \) then this OBDD is
– If this does not matter, then it is sufficient to memorize the OBDD as well as the corresponding set of variables \( \{x, y, z\} \) or \( \{y\} \) in the above example.

Typed Shannon tree

– The idea of typed Shannon tree [2] came from the remark that

\[
\neg f = (\neg x \land \neg f_x) \lor (\neg x \land \neg f_x)
\]

so that the Shannon trees \( Sh(f) \) and \( Sh(\neg f) \) of \( f \) and \( \neg f \) are identical except at the leaves where 0 and 1 are exchanged.

– So one can use \( +Sh(f) \) for \( Sh(f) \) and \( -Sh(f) \) for \( Sh(\neg f) \) with \(+1 = 1 \) and \(-1 = 0 \).

Reference


Boolean functions represented by a Typed Shannon tree

– The Boolean function \( bf(t) \) represented by a typed Shannon tree \( t \) over \( x_1 <^v \ldots <^v x_n \) is

\[
bf(t) = \text{match } t \text{ with } \\
0|1 \rightarrow \lambda t \rightarrow \text{ case } n = 0 \\
\langle x, \langle s_1, t_1 \rangle, \langle s_2, t_2 \rangle \rangle \rightarrow \\
\text{let } f_1(x_2, \ldots, x_n) = bf(t_1) \\
\text{and } f_2(x_2, \ldots, x_n) = bf(t_2) \text{ in } \\
\lambda x_1, \ldots, x_n. (x_1 \land \text{bo}(s_1)(f_1(x_2, \ldots, x_n))) \\
\quad \lor (\neg x_1 \land \text{bo}(s_2)(f_2(x_2, \ldots, x_n)))
\]

where \( \text{bo}(+) (b) = b \) while \( \text{bo}(−)(b) = −b \)
Typed Shannon trees representing a Boolean function

Let \( f(x_1, \ldots, x_n) \in B_n \) be a Boolean function over the variables \( x_1 <^v \cdots <^v x_n \). The typed Shannon tree encoding \( f \) is:

\[
\text{tsh}(f(x_1, \ldots, x_n)) = \begin{cases} 
(n = 1 ? \langle x, f(0) ? (+, 1) ; (-, 1) \rangle, \\
(f(1) ? (+, 1) ; (-, 1)) \end{cases}
\]

\(
\text{let } \langle s_1, t_1 \rangle = \langle f(0,1,\ldots,1) = 1 \rangle ? \\
\langle +, \text{tsh}(\lambda x_2, \ldots, x_n : f(0,x_2,\ldots,x_n)) \rangle \\
\langle -, \text{tsh}(\lambda x_2, \ldots, x_n : f(0,x_2,\ldots,x_n)) \rangle \end{cases}
\)

and \( \langle s_2, t_2 \rangle = \langle f(1,1,\ldots,1) = 1 \rangle ? \\
\langle +, \text{tsh}(\lambda x_2, \ldots, x_n : f(1,x_2,\ldots,x_n)) \rangle \\
\langle -, \text{tsh}(\lambda x_2, \ldots, x_n : f(1,x_2,\ldots,x_n)) \rangle \)

in \( \langle x_1, \langle s_1, t_1 \rangle, \langle s_2, t_2 \rangle \rangle \)

Encoding of a Typed Shannon tree by a Typed Decision Graph (TDG)

If \( t \) is a typed Shannon tree, the the corresponding TDG is obtained by applying the previous sharing and elimination rules:

\[
\text{tdg}(t) = \begin{cases} 
(t = \langle s, 1 \rangle ? \langle s, 1 \rangle, \\
\text{if } t = \langle x, \langle s_1, t_1 \rangle, \langle s_2, t_2 \rangle \rangle ? \\
\langle (s_1 = s_2 \land t_1 = t_2) ? (s_1 = + ? t_1 ? -t_1) \\
\langle x, \langle s_1, \text{tdg}(t_1) \rangle, \langle s_2, \text{tdg}(t_2) \rangle \rangle 
\end{cases}
\]
Example 1: \( f(x, y, z) = (x \land y) \lor (y \land \neg z) \lor (z \land \neg y) \)

Example 2: \( f(x, y, z) = (y \land x) \lor (x \land \neg z) \lor (z \land \neg x) \)

Boolean functions represented by a Typed Decision Graph (TDG)

The Boolean function \( bf(t) \) represented by a TDG \( t \) over variables \( x_1, \ldots, x_n \) is

\[
bf(t)(x_1, \ldots, x_n) = \text{match } t \text{ with }
\]
\[
\begin{align*}
1 & \rightarrow 1 \\
\langle x, \langle s_1, t_1 \rangle, \langle s_2, t_2 \rangle \rangle & \rightarrow \\
(x = x_1 \ ? \text{ let } f_1(x_2, \ldots, x_n) = bf(t_1)(x_2, \ldots, x_n) \\
\text{ and } f_2(x_2, \ldots, x_n) = bf(t_2)(x_2, \ldots, x_n) \\
\text{ in } (x_1 \land bo(s_1)(f_1(x_2, \ldots, x_n))) \\
\lor (\neg x_1 \land bo(s_2)(f_2(x_2, \ldots, x_n)))
\end{align*}
\]

\( bf(t)(x_2, \ldots, x_n) \)

where \( bo(+) = b \) and \( bo(–) = –b, b \in \{0, 1\} \)

The size of TDGs, although very sensible to the variable order, is often reasonable but can be exponential in the number of variables.
Operations on Typed Decision Graphs (TDG)

- Since the representation of a Boolean function by a TDG is unique, equality of Boolean functions can be represented by the equality (of the physical addresses) of the representations.

- Negation just inverts the signs at the leaves:

\[
\neg t(x_1, \ldots, x_n) = \text{match } t \text{ with } \\
\quad \text{case } n \geq 1 \\
\quad \langle x_1, (s_1, 1), (s_2, 1) \rangle \rightarrow \langle x_1, (\neg s_1, 1), (\neg s_2, 1) \rangle \\
\quad \langle x_1, (s_1, 1), (s_2, t_2) \rangle \rightarrow \langle x_1, (\neg s_1, 1), (s_2, \neg t_2) \rangle \\
\quad \langle x_1, (s_1, t_1), (s_2, 1) \rangle \rightarrow \langle x_1, (s_1, \neg t_1), (\neg s_2, 1) \rangle \\
\quad \langle x_1, (s_1, t_1), (s_2, t_2) \rangle \rightarrow \langle x_1, (s_1, \neg t_1), (s_2, \neg t_2) \rangle
\]

where \(\neg(+) = -\) and \(\neg(-) = +\).

- Other operations use the Shannon decomposition (as well as memoization by a hash table to avoid identical recursive calls).

Encoding of complete join morphisms with join irreducibles

Join irreducible elements of a poset

- Let \(\langle P, \leq \rangle\) be a poset. An element \(x \in P\) is join irreducible iff

1. \(x\) is not the infimum of \(P\)
2. if \(x = a \lor b\) then \(x = a\) or \(x = b\), for all \(a, b \in P\)

- Examples:

\[
\text{Join irreducible elements of a poset}
\]
- Counter-examples:
  The lattice of open subsets of \( R \) (that is subsets which are unions of open intervals \( [a, b) \)) has no join-irreducible element.

- When the second condition is generalized to arbitrary joins
  \( \bigvee_{i \in \Delta} a_i, x \) is called **completely join-irreducible**

- In a lattice the second condition is equivalence to:
  
  \[ \forall a, b \in P : (x < a \wedge x < b) \implies (a \vee b < x)^6 \]

- The **meet irreducible elements** are defined dually

- We let \( \mathcal{J}(P) \) and \( \mathcal{M}(P) \) be the set of join-irreducible and meet-irreducible elements of \( P \)

---

6 Assume \( x \) is join irreducible. We have \( (x < a \wedge x < b) \implies (a \vee b < x) \implies (a \vee b < x) \), since \( a \wedge b < x \) implies \( (a \vee b < x) \). Hence \( x \) is join irreducible since \( a \wedge b < x \).

Reciprocally, if \( (a \wedge b < x) \), then \( (x < a \wedge x < b) \). Since \( x \) is join irreducible, there exists \( c < x \) such that \( a \wedge b < c \). Hence \( x \) is join irreducible.

\[ \forall a, b \in L : (a \wedge b < x) \implies (a \vee b < x) \implies (a \vee b < x) \implies (a \vee b < x) \]

\[ \forall a, b \in L : (a \wedge b < x) \implies (a \vee b < x) \implies (a \vee b < x) \]

\[ \forall a, b \in L : (a \wedge b < x) \implies (a \vee b < x) \implies (a \vee b < x) \]

\[ \forall a, b \in L : (a \wedge b < x) \implies (a \vee b < x) \implies (a \vee b < x) \]

\[ \forall a, b \in L : (a \wedge b < x) \implies (a \vee b < x) \implies (a \vee b < x) \]

---

Decomposition of elements of a lattice satisfying the descending chain condition (DCC) into join irreducibles

**Theorem.** Let \( \langle L, \leq, \lor \rangle \) be a lattice satisfying the DCC.

\[ \forall a \in L : \{ x \in \mathcal{J}(L) \mid x \leq a \} = a \]  

**Proof.** (i) \( \forall a, b \in L : (a \not\leq b) \implies (\exists x \in \mathcal{J}(L) : x \leq a \land x \not\leq b) \)

Assume \( a \not\leq b \). Let \( S = \{ x \in L \mid x \leq a \land x \not\leq b \} \). The set \( S \) is not empty since \( a \in S \). Since \( L \) satisfies the DCC, there exists a minimal element \( x \) of \( S \). This element is join irreducible, since \( x = c \lor d \) with \( c < x \) and \( d < x \), by the minimality of \( x \) that \( c \not\in S \) and \( d \not\in S \). We have \( c < x \leq a \) so \( c \leq a \) and similarly \( d < a \). Therefore \( c, d \not\in S \) implies \( c \leq b \) and \( d \leq b \). But then \( x = c \lor d \leq b \), a contradiction. Thus \( x \in \mathcal{J}(L) \cap S \), which proves (i).

---

Encoding of complete join morphisms on lattices satisfying the descending chain condition (DCC) by the image of join irreducibles

**Theorem.** Let \( \langle L, \leq, \lor \rangle \) be a lattice satisfying the DCC.

Let \( f \in L \to L \) be a complete join morphism. Define \( g \overset{\text{def}}{=} f \upharpoonright \mathcal{J}(L) \), that is \( g \) coincide with \( f \) on join irreducibles. Define \( f'(a) = \bigvee \{ g(x) \mid x \in \mathcal{J}(L) \land x \leq a \} \)

Then \( f' = f \).

**Proof.**

\[ f(a) = f(\bigvee \{ x \in \mathcal{J}(L) \mid x \leq a \}) \]  

\( \forall x \in L : x \leq a \)
Atoms and join irreducibles in Boolean lattices

**Theorem.** Let \( \langle L, \leq, \bot, \lor \rangle \) be a lattice with infimum \( \bot \). Then

(i) \( \bot \prec x \in L \implies x \in J(L) \)

(ii) If \( L \) is a boolean lattice then \( J(L) \subseteq A(L) \)

**Proof.**

(i) Assume \( \bot \prec x \) and \( x = a \lor b \) with \( a < x \) and \( b < x \). Since \( \bot \prec x \), we have \( a = b = \bot \) whence \( x = \bot \), a contradiction proving that \( x \in J(L) \).

(ii) Let \( L \) be a Boolean lattice and \( x \in J(L) \). Assume \( \bot \leq y < x \). We have:

\[
x = x \lor y = (x \lor y) \land (\neg y \lor y) = (x \land \neg y) \lor y
\]

Since \( x \in J(L) \) and \( y < x \), we must have \( x = x \land \neg y \) whence \( x \leq \neg y \). But then \( y = x \land y \leq \neg y \lor y = \bot \) so \( y = \bot \). This proves \( \bot \prec x \) so \( x \in A(L) \) whence \( J(L) \subseteq A(L) \). □

So in Boolean lattices it suffices to know complete join morphisms on the atoms.

---

**Atoms**

- Let \( \langle P, \leq, \bot \rangle \) be a poset with an infimum \( \bot \). An atom of \( P \) is \( a \in P \) such that \( \bot \prec a \) in \( P \) (i.e. \( \bot < a \) and \( \not\exists b \in P : \bot < b < a \)).

- The set of atoms of \( \langle P, \leq, \bot \rangle \) is denoted \( A(P) \).
Encoding of complete join morphisms on Boolean lattices satisfying the DCC by the image of atoms

- Atoms may no exist in infinite lattices (for example in \( \langle \mathbb{R}^+, \leq \rangle \)). However if they exist, they can replace join irreducible to encode complete join morphisms.
- Example:

\[
\begin{array}{c}
\text{Atoms:} \\
\text{Complete Join Morphism:}
\end{array}
\]

**Theorem.** Let \( \langle L, \leq, \bot, \lor \rangle \) be a Boolean lattice satisfying the DCC. Let \( f \in L \xrightarrow{\text{def}} L \) be a complete join morphism. Define \( g \overset{\text{def}}{=} f \restriction A(L) \), that is \( g \) coincide with \( f \) on atoms. Then \( f = \lambda a. \lor \{ g(x) \mid x \in A(L) \land x \leq a \} \).

**Proof.** Immediate consequence of the previous two theorems. \( \square \)

**Closure Operators**

Kazimierz Kuratowski
Definition of an upper closure operator

- An operator on a set \( P \) is a map of \( P \) into \( P \)
- An upper closure operator \( \rho \) on a poset \( \langle P, \leq \rangle \) is
  - extensive: \( \forall x \in P : x \leq \rho(x) \)
  - monotone: \( \forall x, y \in P : (x \leq y) \implies (\rho(x) \leq \rho(y)) \)
  - idempotent: \( \rho(\rho(x)) = \rho(x) \)

Example:

No other possibility!

Example of upper closure operator:

- Let \( \Sigma \) be a set and \( t \subseteq (\Sigma \times \Sigma) \) be a relation on \( \Sigma \)
  - \( t^0 \overset{\text{def}}{=} 1_{\Sigma}, \quad t^{n+1} \overset{\text{def}}{=} t^n \circ t \)

Topological closure operator

- A topological closure operator \( \rho \) on a poset \( \langle P, \leq, \bot, \lor \rangle \) with infimum \( \bot \) and lub \( \lor \), if any, satisfies
  - strict: \( \rho(\bot) = \bot \)
  - extensive: \( \forall x \in P : x \leq \rho(x) \)
  - join morphism: \( \forall x, y \in P : (\rho(x) \lor y) = (\rho(x) \lor \rho(y)) \)
  - idempotent: \( \rho(\rho(x)) = \rho(x) \)

\(^7\) This is the original definition given by K. Kuratowski on \( \langle \rho(S), \subseteq \rangle \) to characterize a unique topology on \( S \): Let \( \rho \) be a topological closure operator on \( S \). Let \( T = \{ S \setminus A \mid A \subseteq S \land \rho(A) = A \} \). Then \( T \) is a topology on \( S \) and \( \rho(A) \) is the \( T \)-closure of \( A \) for each subset \( A \) of \( S \).

\(^8\) This implies that \( \rho \) is monotonic.
**Morgado Theorem (on upper closure operators)**

**THEOREM.** An operator \( \rho \) on a poset \( \langle P, \leq \rangle \) is an upper closure operator if and only if

\[ \forall x, y \in P : x \leq \rho(y) \iff \rho(x) \leq \rho(y) \]

**Proof.** — Let \( \rho \) be an upper closure operator

\[ x \leq \rho(y) \]

\[ \implies \rho(x) \leq \rho(\rho(y)) \quad \{ \text{monotony} \} \]

\[ \implies \rho(x) \leq \rho(y) \quad \{ \text{idempotence} \} \]

\[ \implies x \leq \rho(x) \leq \rho(y) \quad \{ \text{extensive} \} \]

\[ \implies x \leq \rho(y) \quad \{ \text{transitivity} \} \]

— Conversely, let \( \rho \) satisfying the above condition.

— Exchanging the roles of \( \rho_1 \) and \( \rho_2 \), we get \( \rho_1(y) \leq \rho_2(y) \) in the same way.

— By antisymmetry, we conclude that \( \rho_1(y) = \rho_2(y) \).

— By duality, a lower closure operator is uniquely determined by its fixpoints.

**Fixpoints of a closure operator**

The set of **fixpoints** of an operator \( f \in P \mapsto P \) on a set \( P \) is \( \{ x \mid f(x) = x \} \).

**THEOREM.** A closure operator is uniquely defined by its fixpoints

**Proof.** Let \( \rho_1 \) and \( \rho_2 \) be two upper closure operators on a poset \( \langle P, \leq \rangle \) with identical fixpoints:

\[ \forall x \in P : \rho_1(x) = x \iff \rho_2(x) = x \]

We prove that \( \rho_1 = \rho_2 \).

— \( \forall x \in P : z \leq \rho_1(z) \) so \( \rho_2(z) \leq \rho_2(\rho_1(z)) \) by extensivity of \( \rho_1 \) and monotony of \( \rho_2 \).

— \( \rho_1(\rho_1(z)) = \rho_1(z) \) by idempotence so \( \rho_2(\rho_1(z)) = \rho_1(z) \) since \( \rho_1 \) and \( \rho_2 \) have the same fixpoints.

— It follows that \( \rho_2(z) \leq \rho_2(\rho_1(z)) = \rho_1(z) \)

— By antisymmetry, we conclude that \( \rho_1(z) = \rho_2(z) \).

— By duality, a lower closure operator is uniquely determined by its fixpoints.
Definition of a Galois connection

- Let \( \langle P, \leq \rangle \) and \( \langle Q, \sqsubseteq \rangle \) be posets. A pair \( \langle \alpha, \gamma \rangle \) of maps \( \alpha \in P \mapsto Q \) and \( \gamma \in Q \mapsto P \) is a Galois connection if and only if
  \[ \forall x \in P : \forall y \in Q : \alpha(x) \sqsubseteq y \iff x \leq \gamma(y) \]
which is written:
  \[ \langle P, \leq \rangle \xleftarrow{\alpha} \langle Q, \sqsubseteq \rangle \]

- \( \alpha \) is the lower adjoint
- \( \gamma \) is the upper adjoint

- Example:
Example of Galois connection: bijection

Let $P$ and $Q$ be two sets and $b \in P \mapsto Q$ be a one-to-one map of $p$ onto $q$ with inverse $b^{-1}$. Then
\[
\langle P, = \rangle \xrightarrow{b^{-1}} \langle Q, = \rangle
\]
(where $\langle P, = \rangle$ is $P$ ordered by equality)

**Proof.**
\[
b(x) = y \iff x = b^{-1}
\]
(by def. bijection)

---

Example of Galois connection: functional abstraction

Let $C$ and $A$ be sets an $f \in C \mapsto A$. Define
\[
\alpha(X) \overset{\text{def}}{=} \{ f(x) \mid x \in X \}
\]
\[
\gamma(Y) \overset{\text{def}}{=} \{ x \mid f(x) \in Y \}
\]
then
\[
\langle \wp(C), \subseteq \rangle \xrightarrow{\alpha} \langle \wp(A), \subseteq \rangle
\]

---

Example of Galois connections with Pre and Post

Recall that given a set $\Sigma$ and $t \subseteq \Sigma \times \Sigma$, we have defined
\[
\text{post}[t]X \overset{\text{def}}{=} \{ x' \mid \exists x \in X : \langle x, x' \rangle \in t \}
\]
\[
\text{pre}[t]X \overset{\text{def}}{=} \text{post}[t^{-1}]X
\]
\[
\text{post} \circ [t] \overset{\text{def}}{=} \{ x \mid \exists x' \in X : \langle x, x' \rangle \in t \}
\]
\[
\text{pre} \circ [t] \overset{\text{def}}{=} \{ x \mid \forall x' : \langle x, x' \rangle \in t \implies x \in X \}
\]

---
We have
\[
\langle \varrho(\Sigma), \subseteq \rangle \xleftarrow{\text{post}[t]} \preceq \xrightarrow{\text{pre}[t]} \langle \varrho(\Sigma), \subseteq \rangle
\]
By letting \( t' = t^{-1} \), we get in the same way
\[
\langle \varrho(\Sigma), \subseteq \rangle \xleftarrow{\text{pre}[t]} \preceq \xrightarrow{\text{post}[t]} \langle \varrho(\Sigma), \subseteq \rangle
\]

**Example of Galois connections induced by upper closure operators**

Recall Morgado's theorem for an upper closure operator on a poset \( (P, \subseteq) \)
\[
\forall x, y \in P : x \leq \varrho(y) \iff \varrho(x) \leq \varrho(y)
\]
Let \( \rho(P) = \{ \varrho(x) | x \in P \} \). This can be written as follows (with \( z = \varrho(y) \))
\[
\forall x \in P : \forall z \in \rho(P) : x \leq 1_P(z) \iff \varrho(x) \leq z
\]
which by definition of a Galois connection implies that
\[
(P, \subseteq) \xleftarrow{1_P \rho} \langle \rho(P), \subseteq \rangle
\]
Reciprocally, this implies that
\[
\forall x \in P : \forall z \in \rho(P) : \varrho(x) \leq z \iff x \leq 1_P(z)
\]

**Theorem.** \( \rho \) is an upper closure of \( (P, \subseteq) \) if and only if
\[
(P, \subseteq) \xleftarrow{1_P \rho} \langle \rho(P), \subseteq \rangle
\]
Unique adjoints

**Theorem.** In a Galois connection
\[
\langle P, \leq \rangle \leq \rho \alpha \quad \langle Q, \sqsupseteq \rangle
\]
one adjoint uniquely determines the other, in that
\[
\alpha(x) = \bigcap \{y \mid x \leq \gamma(y)\} \quad \gamma(y) = \bigvee \{x \mid \alpha(x) \subseteq y\}
\]

**Proof.** - The set \{y \mid \alpha(x) \subseteq y\} has a glb which is precisely \(\alpha(x)\) so \(\alpha(x) = \bigcap \{y \mid \alpha(x) \subseteq y\}\) since \(\alpha(x) \subseteq y \iff x \leq \gamma(y)\).
- The set \(\{x \mid x \leq \gamma(y)\}\) has a lub which is precisely \(\gamma(y)\) so \(\gamma(y) = \bigvee \{x \mid x \leq \gamma(y)\}\) since \(\alpha(x) \subseteq y \iff x \leq \gamma(y)\).

\[
\alpha \circ \gamma \circ \alpha = \alpha \quad \text{and} \quad \gamma \circ \alpha \circ \gamma = \gamma
\]

**Proof.** - \(\alpha \circ \gamma (x) \subseteq x\) so \(\alpha \circ \gamma \circ (y) \subseteq \alpha(y)\) when \(x = \alpha(y)\). \(1_P \subseteq \gamma \circ \alpha\) so \(\alpha \subseteq \alpha \circ \gamma \circ \alpha\) by monotony, concluding \(\alpha \circ \gamma \circ \alpha = \alpha\) by antisymmetry.
- \(x \leq \gamma \circ \alpha(x)\) so \(\gamma(y) \leq \gamma \circ \alpha \circ \gamma(y)\) for \(x = \gamma(y)\) so \(\alpha \circ \gamma \circ \gamma(y) \subseteq y\) so \(\gamma \circ \alpha \circ \gamma(y) \subseteq \gamma(y)\) by monotony, concluding \(\gamma \circ \alpha \circ \gamma = \gamma\) by antisymmetry.

\[
\alpha \circ \gamma \text{ is a lower closure operator on } \langle P, \leq \rangle
\]
\[
\gamma \circ \alpha \text{ is a upper closure operator on } \langle Q, \sqsupseteq \rangle
\]
Example:

The upper adjoint of a Galois connection preserves existing lubs

**Theorem.** Let \( \langle P, \leq \rangle \xrightarrow{\gamma} \langle Q, \sqsubseteq \rangle \) be a Galois connection and \( X \subseteq P \) such that its lub \( \bigvee X \) does exists in \( P \). Then \( \alpha(\bigvee X) \) is the lub of \( \{\alpha(x) \mid x \in X\} \) in \( Q \), that is \( \alpha(\bigvee X) = \bigsqcup \alpha(X) \).

**Proof.**

1. \( \forall x \in X : x \leq \bigvee X \) by existence of the lub \( \bigvee X \) so \( \forall x \in X : \alpha(x) \sqsubseteq \alpha(\bigvee X) \) by monotony of \( \alpha \) proving that \( \alpha(\bigvee X) \) is the least of the upper bounds of \( \{\alpha(x) \mid x \in X\} \).
2. If \( \bigvee X \) exists in \( \langle P, \leq \rangle \) then \( \bigvee \alpha(X) \) does exists in \( \langle Q, \sqsubseteq \rangle \) and \( \alpha(\bigvee X) = \bigsqcup \alpha(X) \).

Galois connection induced by lub preserving maps

**Theorem.** Let \( \alpha \in P \xrightarrow{\gamma} Q \) be a complete join preserving map between posets \( \langle P, \leq \rangle \) and \( \langle Q, \sqsubseteq \rangle \). Define:

\[
\gamma = \lambda y. \bigvee \{z \mid \alpha(z) \sqsubseteq y\}
\]

If \( \gamma \) is well-defined then

\[
\langle P, \leq \rangle \xrightarrow{\gamma} \langle Q, \sqsubseteq \rangle
\]
Proof. – Assume that for all $y \in Q$, $\bigvee \{ z \mid \alpha(z) \sqsubseteq y \}$ does exist. A counterexample is $\alpha$ is the identity on $P = \omega$. Then $\omega \in \omega + 1 = Q$. $\{ z \mid \alpha(z) \sqsubseteq \omega \} = \omega$ but $\bigvee \{ z \mid \alpha(z) \sqsubseteq y \} = \bigvee \{0, 1, 2, \ldots \}$ does not exist in $\omega$!

The proof that $\langle \alpha, \gamma \rangle$ is a Galois connection proceeds as follows:

\begin{align*}
\alpha(x) \sqsubseteq y & \quad \Rightarrow x \in \{ z \mid \alpha(z) \sqsubseteq y \} \\
& \quad \Rightarrow x \leq \bigvee \{ z \mid \alpha(z) \sqsubseteq y \} \quad \text{[lub assumed to exist!]} \\
& \quad \Rightarrow x \leq \gamma(y) \\
& \quad \Rightarrow \alpha(x) \sqsubseteq \alpha(\bigvee \{ z \mid \alpha(z) \sqsubseteq y \}) \quad \text{[def. $\gamma$ and $\alpha$ monotone]} \\
& \quad \Rightarrow \alpha(x) \sqsubseteq \bigsqcup \{ \alpha(z) \mid \alpha(z) \sqsubseteq y \} \quad \text{[def. lub]} \\
& \quad \Rightarrow \alpha(x) \sqsubseteq y \quad \text{[def. lub]} \\
\end{align*}

Similarly, if $\gamma$ preserves glbs and $\alpha = \lambda x. \bigsqcap \{ y \mid x \leq \gamma(y) \}$ is well-defined then $\langle P, \leq \rangle \xleftrightharpoons{\alpha} \langle Q, \sqsubseteq \rangle$.

\[ \text{Examples:} \]

- The dual of “$\alpha$ preserves existing lubs” is “$\gamma$ preserves existing glbs”

- The dual of $\alpha(x) = \bigsqcap \{ y \mid x \leq \gamma(y) \}$ is $\gamma(y) = \bigvee \{ y \mid x \sqsupseteq \alpha(y) \}$ that is $\gamma(y) = \bigvee \{ x \mid \alpha(x) \sqsubseteq y \}$

- The dual of $\alpha \circ \gamma \circ \alpha = \alpha$ is $\gamma \circ \alpha \circ \gamma = \gamma$

\[ \text{Duality principle for Galois connections} \]

Theorem. We have $\langle P, \leq \rangle \xleftrightharpoons{\gamma} \langle Q, \sqsubseteq \rangle$ if and only if $\langle Q, \sqsupseteq \rangle \xleftrightharpoons{\alpha} \langle P, \geq \rangle$ whence the dual of a Galois connection $\langle \alpha, \gamma \rangle$ is $\langle \gamma, \alpha \rangle$ (exchange of adjoints).

Proof.

\[ \langle P, \leq \rangle \xleftrightharpoons{\gamma} \langle Q, \sqsubseteq \rangle \]

\[ \text{def} \]

\[ \forall x \in P : \forall y \in Q : \alpha(x) \sqsubseteq y \iff x \leq \gamma(y) \]

\[ \iff \forall y \in Q : \forall x \in P : \gamma(y) \geq x \iff y \sqsupseteq \alpha(x) \]

\[ \text{def} \]

\[ \langle Q, \sqsupseteq \rangle \xleftrightharpoons{\alpha} \langle P, \geq \rangle \]

\[ \square \]
Composition of Galois connections

**Theorem.** The composition of Galois connections is a Galois connection: if
\[ P \leq Q \quad \text{and} \quad Q \leq R \]
then
\[ P \leq R \]

**Proof.** Assume
\[ P \leq Q \quad \text{and} \quad Q \leq R \]
then \( \forall x \in P : \forall z \in R: \)
\[ \alpha_2 \circ \alpha_1(x) \leq z \]
\[ \iff \alpha_1(x) \sqsubseteq \gamma_2(z) \]
\[ \iff x \leq \gamma_1 \circ \gamma_2(z) \]

---

**Example:**

- So \( \alpha \) is antitone: \( x \leq y \implies \alpha(x) \supseteq \alpha(y) \)
- Hence when composing \( \alpha_2 \circ \alpha_1 \) is monotonic, hence not a Galois correspondence
- This justifies the introduction of Galois connections in [3] (by taking semi-dual Galois correspondences).

---

**Reference**

Galois surjections (insertions)

**Theorem.** If \( \langle P, \leq \rangle \xrightarrow{\alpha} \langle Q, \sqsubseteq \rangle \) then

\[
\begin{align*}
\iff & \gamma \text{ is one-to-one} \\
\iff & \alpha \circ \gamma = 1_Q
\end{align*}
\]

**Proof.** – Assume that \( \alpha \) is onto (\( \forall y \in Q : \exists x \in P : \alpha(x) = y \))
– Assume \( \gamma(x) = \gamma(y) \). \( \exists x', y' \in P : \alpha(x') = y \text{ and } \alpha(y') = y \), and so
  \[
  \gamma(\alpha(x')) = \gamma(\alpha(y'))
  \]
  \[
  \implies x' \leq \gamma(\alpha(y')) \quad \{ \text{since } x' \leq \gamma \circ \alpha(x') \}\}
  \implies \alpha(x) \sqsubseteq \alpha(y') \quad \{ \text{by def. Galois connection} \}
\]

Exchanging the roles of \( x \) and \( y \), we get \( y \sqsubseteq x \) so \( x = y \) by antisymmetry, proving that \( x \neq y \implies \gamma(x) \neq \gamma(y) \), by composition.
– \( \alpha \circ \gamma(y) = \alpha \circ \gamma \circ \alpha(y') \) where \( \alpha(y') = y \). So \( \alpha \circ \gamma(y) = \alpha(y) = y \) so \( \alpha \circ \gamma = 1_Q \)
– Assume \( \alpha \circ \gamma = 1_Q \). Then given \( y \in Q \), we have \( \alpha \circ \gamma(y) = y \) proving that
  \( \exists x = \gamma(y) : \alpha(x) = y, \alpha \) is onto.

**Example of Galois surjection:**

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Galois injections

**Theorem.** By duality, if \( \langle P, \leq \rangle \xleftarrow{\gamma} \langle Q, \sqsubseteq \rangle \) then

\[
\begin{align*}
\iff & \gamma \text{ is onto} \\
\iff & \alpha \text{ is one-to-one} \\
\iff & \gamma \circ \alpha = 1_P
\end{align*}
\]

---

Notations:

– \( \langle P, \leq \rangle \xrightarrow{\gamma} \langle Q, \sqsubseteq \rangle \defeq \langle P, \leq \rangle \xleftarrow{\gamma} \langle Q, \sqsubseteq \rangle \land \alpha \) is onto
– \( \langle P, \leq \rangle \xleftarrow{\gamma} \langle Q, \sqsubseteq \rangle \defeq \langle P, \leq \rangle \xrightarrow{\gamma} \langle Q, \sqsubseteq \rangle \land \alpha \) is one-to-one
– \( \langle P, \leq \rangle \xrightarrow{\gamma} \langle Q, \sqsubseteq \rangle \defeq \langle P, \leq \rangle \xleftarrow{\gamma} \langle Q, \sqsubseteq \rangle \land \alpha \) is bijective
Conjugate Galois connections in a Boolean algebra

**Theorem.** Let \( \langle P, \leq, 0, 1, \lor, \land, \neg \rangle \) and \( \langle Q, \subseteq, \bot, \top, \lor, \land, \neg \rangle \) be Boolean algebras and the Galois connection

\[
\langle P, \leq \rangle \xleftarrow{\gamma} \langle Q, \subseteq \rangle
\]

Define the conjugates \( \alpha = \neg \alpha(\neg x) \) and \( \gamma = \neg \gamma(\neg x) \). Then

\[
\langle P, \geq \rangle \xrightarrow{\gamma} \langle Q, \supseteq \rangle
\]

**Proof.**

\[
\alpha(x) \leq y
\]

11 This is also called the dual, but this may cause confusion with lattice duality.

---

Example of dual Galois connections in a Boolean algebra: Pre, Post and their duals

We have

\[
\langle \wp(\Sigma), \subseteq \rangle \xleftrightarrow{\text{post}[t]} \langle \wp(\Sigma), \subseteq \rangle
\]

By conjugate/complement duality, we get

\[
\langle \wp(\Sigma), \supseteq \rangle \xleftrightarrow{\text{pre}[t]} \langle \wp(\Sigma), \supseteq \rangle
\]

since \( \neg \text{pre} = \text{pre} \), hence by order duality

\[
\langle \wp(\Sigma), \subseteq \rangle \xleftrightarrow{\text{pre}[t]} \langle \wp(\Sigma), \subseteq \rangle
\]

---

Example of reduction of a Galois connection

- Assume a Galois connection is not a surjection, for example:

\[
\langle P, \leq \rangle \xleftarrow{\gamma} \langle Q, \subseteq \rangle
\]

- It is always possible to reduce \( Q \) by identifying elements with the same \( \gamma \)-image

\[
x \equiv y \overset{\text{def}}{=} \gamma(x) = \gamma(y)
\]
and to reduce $Q$ to the quotient $Q/\equiv$, in which case $\alpha$ becomes surjective:

\[
\alpha\equiv(x) = [\alpha(x)]\equiv
\gamma\equiv([y]\equiv) = \gamma(y)
\begin{align*}
[x]\equiv & \subseteq [y]\equiv \overset{\text{def.}}{=} x \subseteq y \text{ on } Q/\equiv
\end{align*}
\]

Reduction of a Galois connection

**Theorem.** If $\langle P, \leq \rangle \overset{\gamma}{\leftrightarrow} \langle Q, \subseteq \rangle$, $x \equiv y \overset{\text{def.}}{=} \gamma(x) = \gamma(y)$, $\alpha\equiv(x) = [\alpha(x)]\equiv$ and $\gamma\equiv([y]\equiv) = \gamma(y)$, then $\langle P, \leq \rangle \overset{\alpha}{\leftrightarrow} \langle Q/\equiv, \subseteq \rangle$

where $[x]\equiv \subseteq [y]\equiv \overset{\text{def.}}{=} x \subseteq y \text{ on } Q/\equiv$

**Proof.** $\equiv$ is an equivalence relation. We let $[x]\equiv$ be the equivalence class of $x \in Q$ in the quotient $Q/\equiv$.

- We have a Galois connection $\langle P, \leq \rangle \overset{\gamma}{\leftrightarrow} \langle Q, \subseteq \rangle$ as follows:

\[
\alpha(x) \subseteq [y]\equiv
\begin{align*}
\iff & [\alpha(x)]\equiv \subseteq [y]\equiv \overset{\text{def. } \alpha\equiv(x)}{=} \\
\iff & \alpha(x) \subseteq y \overset{\text{def. } \subseteq }{=}
\end{align*}
\]

Linear Sum of Galois connections

**Theorem.** Let $\langle P_1, \leq_1 \rangle \overset{\gamma_1}{\leftrightarrow} \langle Q_1, \subseteq_1 \rangle$ and $\langle P_2, \leq_2 \rangle \overset{\gamma_2}{\leftrightarrow} \langle Q_2, \subseteq_2 \rangle$ be Galois connections. Define the linear (ordinal) sums of posets $\langle P, \leq \rangle \overset{\text{def.}}{=} \langle P_1, \leq_1 \rangle \oplus \langle P_2, \leq_2 \rangle$ and $\langle Q, \subseteq \rangle \overset{\text{def.}}{=} \langle Q_1, \subseteq_1 \rangle \oplus \langle Q_2, \subseteq_2 \rangle$ as well as $\alpha = \alpha_1 \oplus \alpha_2$ and $\gamma = \gamma_1 \oplus \gamma_2$ as follows:

\[
\begin{align*}
\alpha(0, x) & \overset{\text{def.}}{=} 0, \alpha_1(x) \quad \gamma(0, x) \overset{\text{def.}}{=} 0, \gamma_1(x) \\
\alpha(1, x) & \overset{\text{def.}}{=} 1, \alpha_2(x) \quad \gamma(1, x) \overset{\text{def.}}{=} 1, \gamma_2(x)
\end{align*}
\]

then

\[
\begin{align*}
\langle P, \leq \rangle & \overset{\gamma}{\leftrightarrow} \langle Q, \subseteq \rangle
\end{align*}
\]
Disjoint sum of Galois connections

**Theorem.** Let $\langle P_1, \leq_1 \rangle \xleftarrow{\gamma_1} \langle Q_1, \sqsubseteq_1 \rangle$ and $\langle P_2, \leq_2 \rangle \xleftarrow{\gamma_2} \langle Q_2, \sqsubseteq_2 \rangle$ be Galois connections. Define the disjoint sums of posets $\langle P, \leq \rangle \overset{\text{def}}{=} \langle P_1, \leq_1 \rangle + \langle P_2, \leq_2 \rangle$ and $\langle Q, \sqsubseteq \rangle \overset{\text{def}}{=} \langle Q_1, \sqsubseteq_1 \rangle + \langle Q_2, \sqsubseteq_2 \rangle$ as well as $\alpha = \alpha_1 + \alpha_2$ and $\gamma = \gamma_1 + \gamma_2$ as follows:

\[
\alpha(\langle 0, x \rangle) \overset{\text{def}}{=} \langle 0, \alpha_1(x) \rangle \quad \gamma(\langle 0, x \rangle) \overset{\text{def}}{=} \langle 0, \gamma_1(x) \rangle
\]

\[
\alpha(\langle 1, x \rangle) \overset{\text{def}}{=} \langle 1, \alpha_2(x) \rangle \quad \gamma(\langle 1, x \rangle) \overset{\text{def}}{=} \langle 1, \gamma_2(x) \rangle
\]

then

\[
\langle P, \leq \rangle \xleftarrow{\gamma} \langle Q, \sqsubseteq \rangle
\]

Similar results hold for the smashed disjoint sum.

**Proof.**

(i) if $i = j = 0$ then

\[
\iff \quad \alpha_1 \leq_1 y \\
\iff \quad x \sqsubseteq_1 \gamma_1(y) \\
\iff \quad \langle 0, x \rangle \subseteq \langle 0, \gamma_1(y) \rangle \\
\iff \quad \langle 1, x \rangle \subseteq \gamma(\langle 0, y \rangle)
\]

(ii) if $i = 0$, $j = 1$ then $\langle i, x \rangle = \langle 0, x \rangle \subseteq \langle 1, \gamma_2(y) \rangle = \gamma(\langle 1, y \rangle) = \gamma(\langle j, y \rangle)$

(iii) if $i = j = 1$

\[
\iff \quad \alpha_2 \leq_2 y \\
\iff \quad x \sqsubseteq_2 \gamma_2(y) \\
\iff \quad \langle 1, x \rangle \subseteq \langle 1, \gamma_2(y) \rangle
\]
Product of Galois connections

**Theorem.** Let \( \langle P_1, \leq_1 \rangle \xleftarrow{\gamma_1} \langle Q_1, \sqsubseteq_1 \rangle \) and \( \langle P_2, \leq_2 \rangle \xleftarrow{\gamma_2} \langle Q_2, \sqsubseteq_2 \rangle \) be Galois connections. Define the cartesian product of posets \( \langle P, \leq \rangle \xleftarrow{\alpha} \langle Q, \sqsubseteq \rangle \) as well as \( \alpha = \gamma_1 \times \gamma_2 \) and \( \gamma = \gamma_1 \times \gamma_2 \) as follows:

\[
\alpha((x, y)) = \langle \alpha_1(x), \alpha_2(y) \rangle \quad \gamma((x, y)) = \langle \gamma_1(x), \gamma_2(y) \rangle
\]

then \( \langle P, \leq \rangle \xleftarrow{\gamma} \langle Q, \sqsubseteq \rangle \).

**Proof.**

\[
\begin{align*}
\alpha((x', y')) & \subseteq (x', y') \\
\iff & \alpha_1(x') \sqsubseteq_1 x' \land \alpha_2(y') \sqsubseteq_1 y' \\
& \iff x \leq_1 \gamma_1(x') \land y \leq_2 \gamma_2(y') \\
& \iff (x, y) \subseteq \gamma((x', y'))
\end{align*}
\]

This can be generalized to \( \langle P, \leq \rangle \xleftarrow{\gamma} \langle Q, \sqsubseteq \rangle \) implies \( \langle P^n, \leq^n \rangle \xleftarrow{\gamma^n} \langle Q^n, \sqsubseteq^n \rangle \) where

\[
\begin{align*}
\alpha^n((x_1, \ldots, x_n)) & = \langle \alpha(x_1), \ldots, \alpha(x_n) \rangle \\
\gamma^n((y_1, \ldots, y_n)) & = \langle \gamma(y_1), \ldots, \gamma(y_n) \rangle
\end{align*}
\]

Power of Galois connections

**Theorem.** Let \( \langle P_1, \leq_1 \rangle \xleftarrow{\gamma_1} \langle Q_1, \sqsubseteq_1 \rangle \) and \( \langle P_2, \leq_2 \rangle \xleftarrow{\gamma_2} \langle Q_2, \sqsubseteq_2 \rangle \) be Galois connections and \( \langle P_1 \Rightarrow P_2, \leq_2 \rangle \) as well as \( \langle Q_1 \Leftarrow Q_2, \sqsubseteq_2 \rangle \) be sets of monotone maps with the pointwise ordering. Then

\[
\langle P_1 \Rightarrow P_2, \leq_2 \rangle \xleftarrow{\lambda \cdot \gamma \circ g \circ \alpha} \langle Q_1 \Leftarrow Q_2, \sqsubseteq_2 \rangle
\]

**Proof.**

\[
\begin{align*}
\alpha(f) & \subseteq_2 g \\
\iff & \alpha_2 \circ f \circ \gamma_1 \subseteq_2 g \\
\iff & \forall x : \alpha_2(f(\gamma_1(x))) \subseteq_2 g(x) \\
\iff & \forall x : f(\gamma_1(x)) \subseteq_2 \gamma_2(g(x)) \\
\iff & \forall y : f(y) \subseteq_2 \gamma_2(g(y)) \quad \text{(since } y \leq_2 \gamma_2(g(y)) \text{)} \\
\iff & f \subseteq_2 \gamma_2 \circ \gamma_1 \\
\iff & f \subseteq_2 \gamma(g) \\
\iff & f \subseteq_2 \gamma_2 \circ g \circ \alpha_1 \quad \text{(def. } \subseteq_2 \text{ and } \circ \text{)}
\end{align*}
\]
\[ \Rightarrow \alpha_2 \circ f \circ \gamma_1 \preceq g \]  \qquad \{ \text{since } \alpha_2 \circ \gamma_2 \text{ reductive} \}
\[ \Rightarrow \alpha(f) \preceq g \]  \qquad \{ \text{def. } \alpha \}

and so \( \alpha(f) \preceq g \iff f \preceq \gamma(g) \). \hfill \Box

\textbf{THE END}