SIGNAL–THEORETIC CHARACTERIZATION OF WAVEGUIDE MESH GEOMETRIES FOR MEMBRANE SIMULATION

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ABSTRACT

Physical modeling of two-dimensional acoustic systems can be performed using discrete–time structures of delay units commonly known as Waveguide Meshes. In this paper, signal-theoretic concepts are used to compare the most common mesh geometries in terms of propagation error, space aliasing, and computational complexity.

1. INTRODUCTION

Discrete–time waveguide models of uniform and isotropic multidimensional media are made of lattices of wires and junction nodes. For the simulation of 2–D media, such as membranes, a few kinds of waveguide meshes have been proposed: rectangular (square), hexagonal, triangular [VanDuyne and Smith 93, VanDuyne and Smith 95].

In order to choose a mesh geometry, we felt the need to conduct an analysis of the aliasing error due to sampling in time and space of the signal on the mesh ; this criterion completes former performance analysis based on the evaluation of the dispersion error.

2. 2-D SAMPLING SCHEMES AND WAVEGUIDE MESHES

Models of membranes capable to preserve isotropy as much as possible lead to three mesh geometries, all corresponding to a tiling of the surface into regular poligons:

- the hexagonal mesh (three branches per node) ;
- the rectangular (square) mesh (four branches per node) ;
- the triangular mesh (six branches per node).

In the simulation of a multidimensional medium, the simple non-aliasing condition, holding when a signal is bandlimited to half the sampling rate F_s, is no longer sufficient for accurate reconstruction. Since we are forcing multidimensional wave propagation to fit a prescribed (small) number of directions, it is unavoidable to introduce numerical artifacts even when the signals are bandlimited. However, space aliasing can still affect the accuracy of simulations, and it is not only dependent on the sampling rate, but also on the particular geometry we are using. As we will show, the choice of the mesh geometry is critical for ensuring that the non-aliasing condition is satisfied with the minimum density of sampling points. We will conduct our analysis by using some signal-theoretic tools borrowed from [Dudgeon and Mersereau 84, Cariolaro 91].

An M-dimensional sampling lattice is a group $L = \left\{\mathbf{L}\mathbf{h} \,|\, \mathbf{h} \in \mathcal{Z}^M,\, \mathbf{L} \in \mathcal{R}^{M\times M}\right\}$, $\mathbf{L}$ being a matrix with rank equal to M, called the *basis* of the lattice. Such a basis is not unique, but any basis can be obtained from $\mathbf{L}$ by multiplication by a unimodular integer matrix.

The reciprocal lattice of L is $L^* = \left\{\mathbf{L}^{-T}\mathbf{h} \,|\, \mathbf{h} \in \mathcal{Z}^M,\, \mathbf{L} \in \mathcal{R}^{M\times M}\right\}$, where $^{-T}$ denotes inversion followed by transposition.

The Fourier transform $V(\xi)$ of a signal $v(\mathbf{x}), \mathbf{x} \in L$, is defined over $\mathcal{R}^M$ and periodic with images centered on L^*.

Let us do a space sampling, on an ideal membrane of infinite extension, of a space bandlimited 2–D signal $v(x, y, t)$ at a fixed time t. Minimum space sampling frequencies depend on the extension of the spatial Fourier transform $V(\xi_x, \xi_y, t)$,

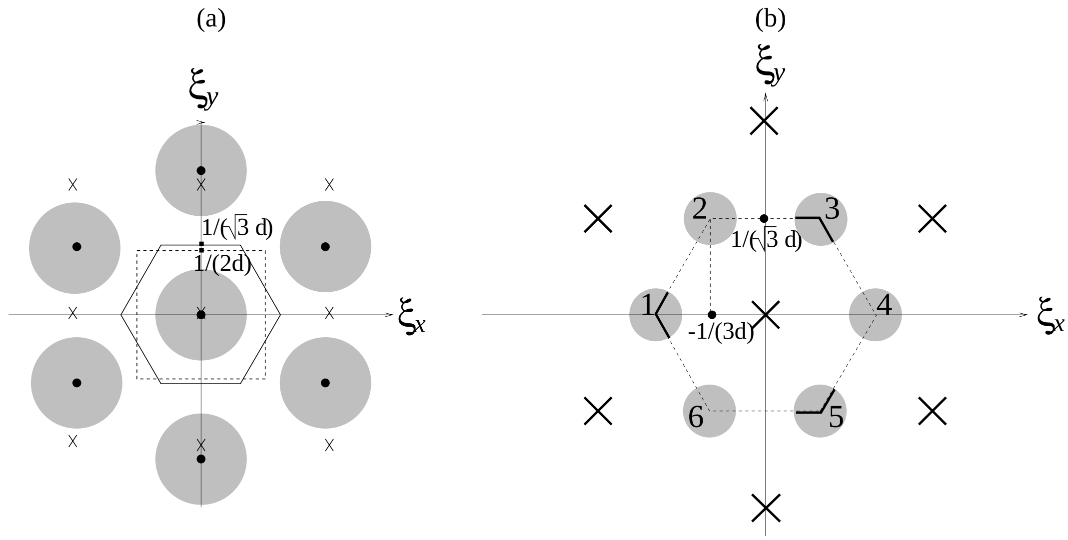

Figure 1: (a) *Allowed extensions for the spatial Fourier transforms in the square (dashed line) and in the triangular (solid line) geometry.* (b) *Spectral support (grey areas) of a bandlimited signal sampled on a hexagonal mesh. Image centers caused by sampling on L_t are also depicted ($\times$).*

being (ξ_x, ξ_y) the spatial frequencies, and on the geometry of the sampling scheme. In order to respect non–aliasing conditions for a given 2–D bandlimited signal, any sampling scheme exhibits its own efficiency measurable in terms of *density* of samples, equal to the value $1/\det(\mathbf{L})$.

The relation, holding at the time sample n in a Waveguide Mesh, between the velocity signal at a junction j and the wave signals at its N Waveguide branches, is [VanDuyne and Smith 93]

$$v_j(nT) = v_{i-}(nT) + v_{i+}(nT) = \frac{2}{N}\sum_{k=1}^{N} v_{k+}(nT) \quad , \quad i = 1, \ldots, N$$

being T the unit time increment, and v_{i-} and v_{i+} the outgoing and incoming velocity waves respectively. Hence, it can be noticed that all the information needed to reconstruct the signal are local to the junction.

The geometry of the Waveguide Mesh and the distance between adjacent junctions establishes the sampling scheme. In particular, the square mesh and the triangular mesh correspond respectively to an orthogonal and an hexagonal (non–orthogonal) sampling scheme, as they are defined in classical multidimensional signal processing.

2.1. Triangles Vs. Squares

Velocity signals on an infinitely extended membrane, under the assumption of isotropic propagation and excitation, are real and have a spectrum which is hermitian symmetric around the origin.

Figure 1(a) shows, in grey areas, the spatial spectral extension of a circularly bandlimited signal, sampled on the junctions of a triangular group L_t having waveguide length $d_t = d$. In order to reconstruct the original signal without space aliasing, its spectral extension cannot cross the exhagon in solid line.

Bounds for the spectral extension of the signal to be sampled on a rectangular group L_r, having waveguide length $d_r = d$, are drawn in Figure 1(a), in dashed line. Since $\det(\mathbf{L}_t)/\det(\mathbf{L}_r) = \sqrt{3}/2$, the triangular mesh is 15.5% denser than the square mesh; at the same time, a circular bandshape fits better an hexagonal than a square area.

The performance of the two geometries may be equated, in terms of space aliasing, by stretching the waveguide length in the triangular scheme by a factor $2/\sqrt{3}$: $d_t = (2/\sqrt{3})d_r = (2/\sqrt{3})d$. After stretching, the density ratio becomes $\det(\mathbf{L}_t)/\det(\mathbf{L}_r) = 2/\sqrt{3}$.

Summarizing, a membrane which is modeleded without space aliasing by a square mesh can also be modeled without aliasing by a triangular mesh having 13.4 % less junctions per unit area.

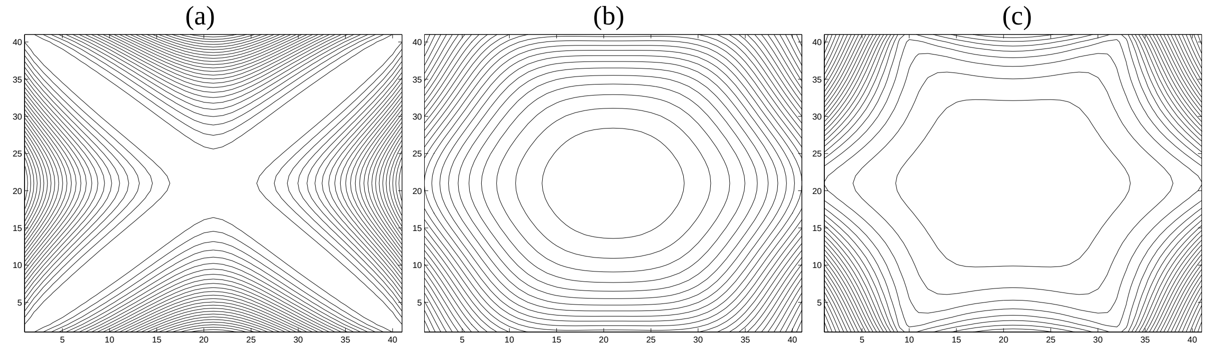

Figure 2: *Relative propagation speeds for a square mesh* (a)*, a triangular mesh* (b) *and an hexagonal mesh* (c)*.*

2.2. Triangles Vs. Hexagons

An hexagonal mesh can be seen as a set L_h obtained from L_t, in which the nodes belonging to a sparser triangular mesh, defined by the group L_T, have been removed: $L_h = L_t \setminus L_T$. Let a 2–D signal $v \in \mathcal{R}^2$ be sampled on L_t and L_h, obtaining respectively the signals v_t and v_h ; let $\tilde{v}_h$ be obtained upsampling v_h over L_t.

Figure 1(b) depicts, in grey areas, the spectral support of $\tilde{V}_h$. It can be noticed that the original signal v can be reconstructed from $\tilde{v}_h$ only if these areas — images of $V(\xi)$ — do not intersect, and this happens only if v can be sampled on L_T without space aliasing.

These considerations show that, from a signal–theoretic viewpoint, there is no advantage in modeling circularly bandlimited signals using an hexagonal mesh. Since $\det(\mathbf{L}_T)/\det(\mathbf{L}_t) = 3$, the mesh defined over the lattice L_T is 3 times as sparse as the mesh defined over L_t, and 2 times as sparse as the hexagonal mesh. Hence, the number of nodes per unit area of the hexagonal mesh is twice as high as that of the triangular mesh defined over L_T, with no benefits on the accuracy of sampling.

3. SIGNAL TIME EVOLUTION

Let us now consider the time evolution of signals traveling with wave velocity c along the ideal membrane. It can be shown that, in order to reconstruct without aliasing a signal $v(\mathbf{x}, t)$ from its sampled version, a time sampling frequency

$$F_s \geq 2c \max_{\xi | V(\xi,t) \neq 0} \{ \sqrt{\xi_x^2 + \xi_y^2} \}$$

is required.

The absolute wave velocity $\tilde{c}$ in a Digital Waveguide Mesh is a function of the spatial frequencies (this phenomenon is called *dispersion* error) and depends on F_s and d [VanDuyne and Smith 93, Fontana and Rocchesso 98]. Different geometries may be compared, in their dynamic behavior, calculating the relative wave velocity $\tilde{c}/c$ — again a function of spatial frequencies — for a given value of the critical bandwidth, thus for a given critical F_s. Figure 2 shows three contour plots representing relative wave velocities versus normalized spatial frequencies, respectively for the square (a), triangular (b) and hexagonal (c) meshes.

As the spatial frequencies get closer to DC, the relative wave velocities approach their maxima, equal to the values $1/\sqrt{2}$, $\sqrt{2/3}$, and $\sqrt{2}/3$ for the rectangular, triangular and hexagonal waveguide meshes, respectively[1].

3.1. Choosing the lowest sampling frequency

Simulations of membranes may require an accurate setting of the ratio $\tilde{c}/c$, in order to get the proper position of modes. This can be done either by stretching the waveguides or, since $\tilde{c}$ is proportional to F_s at low spatial frequencies, by increasing F_s. The former solution is not advisable because it affects space aliasing. The latter solution can be pursued without affecting space aliasing, and the lowest values of sampling rate needed in order to set $\tilde{c} = c$ at low spatial frequencies turn out to be

[1] In fact, the values are equal to the ratios d_r/d, d_t/d and d_h/d normalized by the nominal propagation speed of the signal on a Finite Difference Scheme (FDS), equal to $1/\sqrt{2}$ waveguides per time sample [Chaigne 92].

$$F_{s,r} = \sqrt{2}\frac{c}{d_r} = \sqrt{2}\frac{c}{d} \quad , \quad F_{s,t} = \sqrt{2}\frac{c}{d_t} = \sqrt{\frac{3}{2}}\frac{c}{d} \quad , \quad F_{s,h} = \sqrt{2}\frac{c}{d_h} = \frac{3}{\sqrt{2}}\frac{c}{d}$$

respectively for the rectangular, triangular and hexagonal geometries. It is worth noticing that the lowest sampling rate is required by the triangular mesh.

4. PERFORMANCE

During their "scattering pass", the waveguide meshes compute wave signals as they come out of lossless junctions; then signals are stored in memory cells located over the branches.

The three geometries can be compared by setting the number of junctions per unit area and the sampling rate in such a way that equal non–aliasing conditions hold in time and space, and the relative wave velocity is unitary at low frequencies. Table 1 summarizes the performances of the three geometries, when the number of junctions per unit area and the sampling rate for the rectangular mesh are set to a nominal value equal to 1.

geometry	sq	tr	hex
operations per junction	8	12	6
memory occupation (branches) per junction	4	6	3
junctions per unit area	1	0.866	1.732
memory occupation per unit area	4	5.2	5.2
sample rate	1	0.866	1.5
operations per unit time and unit area	8	9	15.588

Table 1: *Performance of the geometries (*sq*: square;* tr*: triangular;* hex*: hexagonal) in terms of number of junctions, operations and memory occupation.*

5. SUMMARY

We have considered the square, triangular and hexagonal Waveguide Meshes and we have shown that, in order to sample on them a signal with a given space aliasing error, the triangular mesh uses the least number of junctions per unit area, while the hexagonal one uses the most. Then we have noticed that the geometry capable of modeling signals propagating on membranes with the lowest sampling rate is, again, the triangular one, while the hexagonal one must use the highest rate. Finally, we have shown that the square mesh requires the least number of operations and memory occupation per unit time and unit area, even though the triangular and the hexagonal meshes exhibit a more uniform dispersion error.

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