Skolem Standard Form

Background Knowledge

Summary

- Introduction and a bit of History [Chang-Lee Ch.4.1]
- Skolem Standard Form [Chang-Lee Ch. 4.2]
- Properties of Skolem Standard Form [Chang-Lee 4.2]

Intro

Solving problem by proving theorem

- formalise a situation using a logic formula (e.g. FOL)
- prove validity
- e.g. Quacks and Doctor
- Similar techniques applied to other scenarios (e.g. is a software module bug free ?)

Problems with Logic formalisation

FOL is not decidable

- Prop Logic not flexible enough
- FOL is not decidable Negative Result!
- Church Turing: FOL is semidecidable: there is no finite procedure to check whether a formula is valid
- Can verify that a formula is valid in finite steps but for invalid ones, the procedure might not terminate

Herbrand and FOL interpretation

Herbrand contribution

- We want to find a procedure to verify valid (inconsistent) formulas at least!
- Herbrand showed we can focus on only one interpretation
  Positive Result!
- We can then construct a semi-decidable automatic procedure, and this is the best we can do
- Further developments: build more efficient procedures

Automatic proof procedures: Gilmore

Gilmore 1960

- Following Herbrand's idea
- First implemented proof procedure
- Given a formula F, check inconsistency of $\neg F$
- Check inconsistency of a series of propositional formula
- If $\neg F$ is inconsistent, program will terminate detecting this.
- Worked on simple known valid formulas.
- But could not proove many FOL valid formulas.
- Main bottleneck: checking satisfiability of propositional formulas.

Automatic proof procedures: using DPLL

Davis and Putnam 1960

- Improvement on Gilmore's work
- More efficient method to check satisfiability
- Consistent improvement, but still not satisfactory

Resolution proof procedures

Robinson 1965

- very efficient
- many refinements:
  - Semantic Resolution
  - Lock Resolution
  - Unit Resolution
  - Set of Support Strategy
  - Ordered Resolution

Refutational Procedures

Proving inconsistency

- Given a formula G we want to proove that G is valid
- But we know that $\neg G$ is inconsistent iff G is valid
- We try to refute $\neg G$
- We use refutation just for convenience

Standard Form

Davis Putnam 1960

Given a FOL formula F we can:

1. obtain a prenex normal form $Q_1 x_1 \cdots Q_n x_n M$
2. reduce M to CNF $C_1 \wedge \cdots \wedge C_m$ where $C_i = L_1 \vee \cdots \vee L_k$.
3. remove all existential quantifiers from the prefix

We know how to do 1 and 2, we will see how 3 can be done.

Eliminating Existential Quantifiers

Procedure to eliminate existential quantifiers

Given a formula $G \triangleq Q_1 x_1 \cdots Q_n x_n M$.

- Let's focus on one specific existential quantifier Q_r with $1 \leq r \leq n$
  1. if no universal quantifiers appear before Q_r then we can remove $Q_r x_r$ from the prefix and replace every occurrence of x_r with a **new** constant c
  2. Suppose $Q_{s_1}, \cdots, Q_{s_m}$ are all universal quantifiers appearing before Q_r, then we can remove $Q_r x_r$ from the prefix and replace every occurrence of x_r with a **new** m-placed function sysmbol $f(x_{s_1}, \cdots, x_{s_m})$

Skolem Form, Constant and Functions

- When the above procedures has been applied to remove all exisestential quantifier, the formula is in Skolem Standard Form
- The new introduced constants are called Skolem constants
- The new introduced functions are called Skolem functions

Intuition

Intuitive explanation

- $F \triangleq \exists x E(s(0), x)$ $F^{Sko} \triangleq E(s(0), a)$
- $G \triangleq \forall x \exists y E(s(x), y)$ $G^{Sko} \triangleq \forall x E(s(x), g(x))$
- If we consider that $E(x, y)$ x equals y, $s(x)$ successor of x.
- F = there exists a number which is the successor of zero
  - F^{sko} = we make that number a costant and call it a
- G = for every number there exists a successor of that number
  - G^{sko} = we make up a new function that gives us the successor of x and call it $g(x)$

Skolem Standard Form: Example I

Example (Skolem standard form)

Obtain standard Normal Form for the formula:
$\exists x \forall y \forall z \exists u \forall v \exists w P(x, y, z, u, v, w)$

- x is not preceded by any $\forall$ thus $x \to c$
- u is preceded by $\forall y$ and $\forall z$ thus $u \to f(y, z)$
- w is preceded by $\forall y$ and $\forall z$ and $\forall v$ thus $w \to g(y, z, v)$
- we have $\forall y \forall z \forall v P(c, y, z, f(y, z), g(y, z, v))$

Skolem Standard Form: Example II

Example (Skolem standard form)

Obtain standard Normal Form for the formula:
$\forall x \exists y \exists z ((\neg P(x, y) \wedge Q(x, z)) \vee R(x, y, z))$

- transform the matrix in CNF:
  $\forall x \exists y \exists z ((\neg P(x, y) \vee R(x, y, z)) \wedge (Q(x, z) \vee R(x, y, z))$
- y is preceded by $\forall x$ thus $y \rightarrow f(x)$
- z is preceded by $\forall x$ thus $z \rightarrow g(x)$
- we have $\forall x ((\neg P(x, f(x)) \vee R(x, f(x), g(x))) \wedge (Q(x, g(x)) \vee R(x, f(x), g(x)))$

Exercise: Skolem Standard Form

Example (Skolem Standard Form)

Given the formula $G \triangleq \forall x \forall y \exists z \forall w \exists u (P(x, y) \wedge (Q(z, w) \vee R(u)))$ find its Skolem Normal Form G^{sko}

Exercise: Skolem Standard Form

Example (Skolem Standard Form)

Given the formula $G \triangleq \forall x \forall y \exists z \forall w \exists u (P(x, y) \wedge (Q(z, w) \vee R(u)))$ find its Skolem Normal Form G^{sko}

Sol.

- z is preceded by $\forall x \forall y$ thus $z \to f(x, y)$
- u is preceded by $\forall x \forall y \forall w$ thus $u \to g(x, y, w)$
- thus we have $G^{sko} \triangleq \forall x \forall y \forall w (P(x, y) \wedge (Q(f(x, y), w) \vee R(g(x, y, w))))$

Exercise: Skolem Standard Form

Exercise

Give the skolem standard form for each of the formulas below:

1. $F_1 \triangleq \forall(x)\forall(y)((P(x,y) \wedge L(x)) \rightarrow C(y))$
2. $F_2 \triangleq \forall(x)(L(x) \rightarrow \exists(y)(P(y,x) \wedge L(y)))$
3. $F_3 \triangleq \exists y \forall(x)(C(x) \rightarrow V(x,y))$

Notation and terminology

Notation and Terminology for Clauses

- Given a Clause $C = L_1 \vee \cdots \vee L_k$ we can write $C = \{L_1, \cdots, L_k\}$
- Example: $C \triangleq \neg Q \vee S \vee P = \{\neg Q, S, P\}$
- A clause with only one literal is a unit clause
- The empty clause will be often represented with $\square$
- The empty clause is considered to be always false
- We assume that every variable in a set of clauses is always universally quantified

Example

Example (No quantifiers)

With: $S \triangleq \{((\neg P(x, f(x)) \vee R(x, f(x), g(x))), (Q(x, g(x)) \vee R(x, f(x), g(x)))\}$
we represent: $\forall x \exists y \exists z ((\neg P(x, y) \wedge Q(x, z)) \vee R(x, y, z))$

Preserving Inconsistency

Theorem

Inconsistency Preservation of Skolem Form Let F be a formula and S be a set of clauses in Skolem Standard Form that represent F, then F is inconsistent iff S is inconsistent

Sketch of proof.

We assume $F \triangleq (Q_1 x_1) \cdots (Q_n x_n) M[x_1, \cdots, x_n]$
Let Q_r be the first existential quantifier and let
$F_1 \triangleq (Q_1 x_1) \cdots (Q_r x_r)(Q_{r+1} x_{r+1}) \cdots (Q_n x_n)$
$M[x_1, \cdots, x_{r-1}, f(x_1, \cdots, x_{r-1}), x_{r+1}, \cdots, x_n]$ with $1 \leq r \leq n$
we want to show that F is inconsistent iff F_1 is. $\square$

Preserving Inconsistency: sketch of proof I

$\Rightarrow$.

Suppose F is inconsistent and F_1 is consistent
Then there exists $I \models F_1$, therefore
$\forall x_1, \cdots, x_{r-1} \exists f(x_1, \cdots, x_{r-1})$ such that
$I \models Q_{r+1} x_{r+1}, \cdots, Q_n x_n$
$M[x_1, \cdots, x_{r-1}, f(x_1, \cdots, x_{r-1}), x_{r+1}, \cdots, x_n]$ but this means that $I \models F$ which contradicts the assumption. $\square$

Preserving Inconsistency: sketch of proof II

$\Leftarrow$.

Suppose F_1 is inconsistent and F is consistent
Then there exists $I \models F$, therefore $\forall x_1, \cdots, x_{r-1} \exists e$ such that $I \models Q_{r+1} x_{r+1}, \cdots, Q_n x_n$
$M[x_1, \cdots, x_{r-1}, e, x_{r+1}, \cdots, x_n]$ We can then extend the interpretation I including a function symbol $f(.)$ such that $\forall x_1, \cdots, \forall x_{r-1} f(\forall x_1, \cdots, \forall x_{r-1}) = e$. Then $I' \models F_1$ which contradicts the assumption. $\square$

Preserving Inconsistency: sketch of proof III

Generalising to m existential quantifiers.

Suppose we have m existential quantifiers in F. Let's denote $F_0 = F$ and let F_k be obtained from F_{k-1} by replacing the first existential quantifier appearing in F_{k-1} with $k = 1, \cdots, m$. and $S = F_m$. Then we can clearly show that F_{k-1} is consistent iff F_k is. for all k. Therefore we conclude that F is inconsistent iff S is. $\square$

Logical Equivalence

- Given a formula F and a set of clauses S that represents a standard normal form for S
- Can we write $S \equiv F$?

Example (Equivalence and Skolem Form)

$F \triangleq \exists x P(x)$ and $S \triangleq P(a)$
$I = \langle D, A \rangle$ assume $D = \{1, 2\}$, $a^A = 1$ and $\{P^A(1) = \perp, P^A(2) = \top\}$ Then $I \models F$ but $I \not\models S$ therefore in general $S \not\leftrightarrow F$

Skolem Form Does not Preserve Logical Equivalence

- Given a formula F and a set of clauses S that represent a standard normal form for S
- We have that $S \Leftrightarrow F$ iff F is inconsistent
- If F is not inconsistent then S is not a logically equivalent to F:
  - we can find an interpretation I such that $I \models S$ but $I \not\models F$

Skolem Form is not Unique

Given a formula F we can have more than one set of clauses that represents a skolem standard form for F

Example

Consider this formula $F \triangleq \forall x P(x) \wedge \exists y Q(y)$

- $\Phi_1 \triangleq \forall x \exists y (P(x) \wedge Q(y))$
- $\Phi_2 \triangleq \exists y \forall x (P(x) \wedge Q(y))$

Both are prenex normal forms.

"Better" Skolem Forms

We want to find skolem forms which are as simple as possible

Example

- $\Phi_1 \Rightarrow^{sko} S_1 \triangleq \forall x(P(x) \wedge Q(f(a)))$
- $\Phi_2 \Rightarrow^{sko} S_2 \triangleq \forall x(P(x) \wedge Q(a))$

The less arguments we have in the skolem functions the better
$\Rightarrow$ Move existential quantifiers to the left as much as possible.

Skolemisation for Clauses

If we have $F = F_1 \wedge \cdots \wedge F_n$ We can:

- Obtain skolem a form S_i for each F_i
- $S \triangleq S_1 \cup \cdots \cup S_n$
- F is inconsistent iff S is.

Example

$S \triangleq \{S_1, S_2\} = \{P(x), Q(a)\}$

Exercise

Example (Validity and prenex normal form)

Suppose $S \triangleq \exists x \forall y M[x, y]$ is prenex normal form of F. Prove that F is valid $\leftrightarrow$ $S' \triangleq \exists x M[x, f(x)]$ is valid.

Exercise

Example (Validity and prenex normal form)

Suppose $S \triangleq \exists x \forall y M[x, y]$ is prenex normal form of F. Prove that F is valid $\leftrightarrow$ $S' \triangleq \exists x M[x, f(x)]$ is valid.

Sol.

1. F valid $\leftrightarrow \neg F$ inc. $\leftrightarrow \neg S$ inc.
2. $\neg S \equiv \neg \exists x \forall y M[x, y] \equiv \forall x \neg \forall y M[x, y] \equiv \forall x \exists y \neg M[x, y] \triangleq \phi$
3. ϕ inc. $\leftrightarrow \phi^{sko}$ inc.
4. $\phi^{sko} \equiv \forall x \neg M[x, f(x)] \equiv \neg(\exists x M[x, f(x)]) \equiv \neg S'$
5. $\neg S'$ inc. $\leftrightarrow S'$ valid.

Complete example

Example (Using Skolemisation)

We want to show that

$F \triangleq \forall x P(x) \to Q(x) \wedge \exists y P(y) \models G \triangleq \exists z Q(z)$

Complete example

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Sol. : Proof by refutation

1. We know that $F \models G$ iff $F \wedge \neg G$ is inconsistent
2. $\phi \triangleq \forall x (P(x) \to Q(x)) \wedge \exists y P(y) \wedge \neg \exists z Q(z)$
3. $\phi \equiv \phi_1 \wedge \phi_2 \wedge \phi_3$
4. $\phi^{sko} \equiv \phi_1^{sko} \cup \phi_2^{sko} \cup \phi_3^{sko}$
5. ϕ inc. iff ϕ^{sko} inc.
6. $\phi_1^{sko} = \neg P(x) \vee Q(x)$ $\phi_2^{sko} = P(a)$ $\phi_3^{sko} = \neg Q(z)$
7. $\phi^{sko} \equiv \neg P(x) \vee Q(x), P(a), \neg Q(z)$

We need to find a way to show that ϕ^{sko} is inconsistent.

Exercises

Exercise

- Find a standard form for each of the following formulas [Chang-Lee 1 pag 67]
  1. $\neg((\forall x)P(x) \rightarrow (\exists y)(\forall z)Q(y,z))$
  2. $(\forall x)(\neg E(x,0) \rightarrow ((\exists y)(E(y,g(x)) \wedge (\forall z)(E(z,g(x)) \rightarrow E(y,z)))))$
  3. $\neg((\forall x)P(x) \rightarrow (\exists y)P(y))$
- Given the following formulas:
  1. $F_1 \triangleq (\forall x)(\forall y)(S(x,y) \wedge M(y) \rightarrow (\exists z)(I(z) \wedge E(x,z)))$
  2. $F_2 \triangleq ((\neg \exists I(x)) \rightarrow \neg(\exists x)(\forall y)(M(y) \wedge S(x,y)))$

  find the skolem standard form of $F_1 \wedge \neg F_2$ [Chang-Lee 4 pag 67 (partial)]