The Resolution Principle

Summary

- Introduction [Chang-Lee Ch. 5.1]
- Resolution Principle for Propositional Logic [Chang-Lee Ch. 5.2]

Herbrand's Theorem and refutation procedures

Satisfiability procedures

- We can build refutation procedures building on Herbrand's Theorem.
- For example Gilmore's method using DPLL for checking satisfiability.
- This requires the generation of sets S'_0, S'_1, $\cdots$ of ground clauses.
- Computaton issue: for most cases this sequence grows exponentially.

Computational issue

Exponential grow of sequence

- Consider $S = \{P(x, g(x), y, h(x, y), z, k(x, y, z)), \neg P(u, v, e(v), w, f(v, w), x)\}$
- $H_0 = \{a\}$ $H_1 = \{a, g(a), h(a, a), k(a, a, a), e(a), f(a, a)\}$
- $|S'_0| = 2$, $|S'_1| = 1512$
- Earliest unsatisfiable set is S'_5 which has approximately 10^{256} elements!

The Resolution Principle

Robinson 1965

- Aim: directly test unsatisfiability of a set of clauses S without generating all possible associated ground clauses.
- Basic idea: test whether S contains the empty clause $\square$
  - If $\square \in S$ then S is unsatisfiable
  - Otherwise need to check whether $S \models \square$

Connection with Sematic trees

Res. Principle and Sem. Trees

- Recall: by Herbrand's Theorem (version I) S is unsatisfiable iff there is a finite closed semantic tree T for S.
- S contains $\Box$ iff the corresponsing closed semantic tree T contains only the root node.
- If S does not contain $\Box$ then T must contain more than one node.
- If we can reduce the number of nodes in T then we can force $\Box$ to appear.

Inference Rules

Inference Rules

- The resolution principle is an Inference Rule
- Inference Rule: a rule that generates new clauses which are a logical consequence of some of the existing clauses
- New clauses can be used to turn some of the nodes in T to failure nodes.
- Thus number of nodes in T are reduced and $\square$ will eventually appear.

Example (Resolution Principle and Sem. Trees)

The semantic tree for $S = \{\neg P \vee Q, P, \neg Q\}$ can be reduced to $\square$ by adding $\{\neg P\}$ to S.

Resolution principle for Propositional Logic

Resolution and One-Literal rule

- Extension of One-Literal rule of DPLL to any pair of clauses
- Focus on a unit clause containing a literal L and look for the complement of L in another clause. Obtain a new clause deleting the One-Literal clause, and the complement literal from the other clause.

Example (One-Literal and resolution)

$C_1 = P, \; C_2 = \neg P \vee Q$
Applying the One-Literal rule of DPLL to $\{C_1, C_2\}$ we obtain $C_3 = Q$

The Resolution Principle

Resolution Principle

For any two clauses C_1 and C_2 if there is a literal L_1 in C_1 that is complementary to a literal L_2 in C_2 then delete L_1 and L_2 from C_1 and C_2 and generate a new clause C_3 as the disjunction of the remaining clauses.
C_3 is a resolvent for C_1 and C_2.

Resolution Principle: Inference rule

$$\frac{L_1 \vee C_1' \qquad\qquad \neg L_1 \vee C_2'}{C_1' \vee C_2'}$$

Example

Example (Resolution Principle)

Consider the following clauses $C_1 = P \vee R$ and $C_2 = \neg P \vee Q$

$$\frac{P \vee R \qquad\qquad \neg P \vee Q}{}$$

Example

Example (Resolution Principle)

Consider the following clauses $C_1 = P \vee R$ and $C_2 = \neg P \vee Q$

$$\frac{P \vee R \qquad\qquad \neg P \vee Q}{R \vee Q}$$

Example

Example (Resolution Principle)

Consider the following clauses $C_1 = P \vee R$ and $C_2 = \neg P \vee Q$

$$\frac{P \vee R \qquad\qquad \neg P \vee Q}{R \vee Q}$$

$C_3 = R \vee Q$ is the resolvent for C_1 and C_2.

Example II

Example (Resolution Principle)

Consider the following clauses $C_1 = \neg P \vee Q \vee R$ and $C_2 = \neg Q \vee S$

$$\frac{\neg P \vee Q \vee R \qquad\qquad \neg Q \vee S}{}$$

Example II

Example (Resolution Principle)

Consider the following clauses $C_1 = \neg P \vee Q \vee R$ and $C_2 = \neg Q \vee S$

$$\frac{\neg P \vee Q \vee R \qquad\qquad \neg Q \vee S}{\neg P \vee R \vee S}$$

Example II

Example (Resolution Principle)

Consider the following clauses $C_1 = \neg P \vee Q \vee R$ and $C_2 = \neg Q \vee S$

$$\frac{\neg P \vee Q \vee R \qquad\qquad \neg Q \vee S}{\neg P \vee R \vee S}$$

$C_3 = \neg P \vee R \vee S$ is the resolvent for C_1 and C_2.

Example III

Example (Resolution Principle)

Consider the following clauses $C_1 = \neg P \vee Q$ and $C_2 = \neg P \vee S$
There is no resolvent in this case as no complementary pair can be found in the clauses.

Property of Resolution

Logical consequence

Given two clauses C_1 and C_2, and their resolvent C, $C_1 \wedge C_2 \models C$ (C is a logical consequence of C_1 and C_2).

Proof.

Let $C_1 = L \vee C_1'$, $C_2 = \neg L \vee C_2'$, $C = C_1' \vee C_2'$ where C_1' and C_2' are disjunctions of literals. Suppose $I \models C_1 \wedge C_2$, we want to show that $I \models C$.

- Note that either $I \models L$ or $I \models \neg L$.
- Assume $I \models \neg L$
- Then since $I \models C_1$, $C_1' \neq \square$ and $I \models C_1'$.
- Therefore since $C = C_1' \vee C_2'$ we have that $I \models C$.
- Similar considerations hold for $I \models L$.

$\square$

Derivation of the empty clause

Resolution and satisfiability

- If C_1 and C_2 are unit clauses then, if there is resolvent, that resolvent will necessary be $\Box$.
- If we can derive the empty clause from S, then S is unsatisfiable (correctness)
- If S is unsatisfiable using resolution we can always derive the empty clause (completeness)

Deduction

Definition (Deduction)

Given a set of clauses S a (resolution) deduction of C from S is a finite sequence $C_1, C_2, \cdots, C_k$ of clauses such that each C_i is either a clause in S or a resolvent of clauses preceding C_i, and $C_k = C$.

Example I: Deduction

Example (deduction)

Consider $S = \{C_1, C_2, C_3\}$, where $C_1 = \neg P \vee Q$ $C_2 = P$ and $C_3 = \neg Q$. Applying resolution to C_1 and C_2 we have:

$$\frac{\neg P \vee Q, \qquad P}{Q}$$

Then applying

$$\frac{\neg Q, Q}{\square}$$

Deducing the empty clause

Empty clause, Deduction and Unsatisfiability

- Given S, suppose we derive $\Box$ using resolution;
- $\Rightarrow \Box$ is a logical consequence of S;
- Since $S \models \Box$ then $\forall I$ if $I \models S$ then $I \models \Box$;
- But there is no I that can verify $\Box$;
- $\Rightarrow$ if we derive $\Box$ from S using refutation then S is unsatisfiable.
- Later we will show that if S is unsatisfiable then we can always derive $\Box$ using resolution.

Definition (Refutation)

A deduction of $\Box$ is called a refutation (or a proof) of S

Example II: Deduction

Example (deduction)

Given $S = \{C_1, C_2, C_3, C_4\}$ and $C_1 = \{P \vee Q\}$, $C_2 = \{\neg P \vee Q\}$, $C_3 = \{P \vee \neg Q\}$ and $C_4 = \{\neg P \vee \neg Q\}$. Apply resolution to C_1 and C_2 and obtain $C' = \{Q\}$.
Apply resolution to C_3 and C_4 and obtain $C'' = \{\neg Q\}$.
Apply resolution to C' and C'' and obtain $\square$.
Hence S is unsat.

Example II: Deduction Tree

Example (deduction)

Consider S from previous example and the associated deduction steps.

The deduction tree is:

$$\cfrac{\cfrac{P \vee Q, \qquad \neg P \vee Q}{Q} \qquad \cfrac{P \vee \neg Q, \qquad \neg P \vee \neg Q}{\neg Q}}{\square}$$

Exercise

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Prove that the following formulas are unsat. using the resolution principle

1. $\neg q \vee p,\ \neg p \vee \neg q,\ q \vee r,\ \neg q \vee \neg r,\ \neg p \vee \neg r,\ p \vee \neg r$
2. $P(a), \neg D(a) \vee L(a,a), \neg P(a) \vee \neg Q(a) \vee \neg L(a,a), D(a), Q(a)$

Exercise II

Exercise

1. $F_1 \triangleq P \rightarrow (\neg Q \vee (R \wedge S))$
2. $F_2 \triangleq P$
3. $F_3 \triangleq \neg S$
4. $G = \neg Q$

Prove using the resolution principle that $F_1 \wedge F_2 \wedge F_3 \models G$