Propositional and First Order Logic

Background Knowledge

Summary

- Propositional Logic [Chang-Lee Ch. 2]
- First Order Logic [Chang-Lee Ch. 3]

Propositional Logic

Summary

- Syntax
- Semantics
- Normal Forms
- Deduction and Refutation

Basic Concepts

Propositional logic is the simplest logic—illustrates basic ideas using propositions

- $P_1 \triangleq$ Snow is whyte
- $P_2 \triangleq$ Today it is raining
- $P_3 \triangleq$ This automated reasoning course is boring

P_i is an atom or atomic formula

Each P_i can be either **true** or **false** but **never both**

The values **true** or **false** assigned to each proposition is called **truth value** of the proposition

Syntax

Recursive definition of **well-formed formulas**

1. An atom is a formula
2. If S is a formula, $\neg S$ is a formula (**negation**)
3. If S_1 and S_2 are formulas, $S_1 \wedge S_2$ is a formula (**conjunction**)
4. If S_1 and S_2 are formulas, $S_1 \vee S_2$ is a formula (**disjunction**)
5. All well-formed formulas are generated by applying above rules

Shortcuts:

- $S_1 \rightarrow S_2$ can be written as $\neg S1 \vee S2$
- $S_1 \leftrightarrow S_2$ can be written as $(S_1 \rightarrow S_2) \wedge (S_2 \rightarrow S_1)$

Semantics

Relationships between truth values of atoms and truth values of formulas

$\neg S$	is true iff	S	is false		
$S_1 \wedge S_2$	is true iff	S_1	is true **and**	S_2	is true
$S_1 \vee S_2$	is true iff	S_1	is true **or**	S_2	is true
$S_1 \rightarrow S_2$	is true iff	S_1	is false **or**	S_2	is true
i.e.,	is false iff	S_1	is true **and**	S_2	is false
$S_1 \leftrightarrow S_2$	is true iff	$S_1 \rightarrow S_2$	is true **and**	$S_2 \rightarrow S_1$	is true

Semantics: Example

Example (Truth Tables for main logical connectives)

P_1	P_2	$\neg P_1$	$P_1 \wedge P_2$	$P_1 \vee P_2$	$P_1 \rightarrow P_2$	$P_1 \leftrightarrow P_2$
T	T	F	T	T	T	T
T	F	F	F	T	F	F
F	T	T	F	T	T	F
F	F	T	F	F	T	T

Propositional logic: Evaluation of Formula

Recursive Evaluation

Consider the formula $G \triangleq \neg P_1 \wedge (P_2 \vee P_3)$
Suppose we know that $P_1 = F,\ P_2 = F,\ P_3 = T$
Then we have

$$\neg P_1 \wedge (P_2 \vee P_3) = true \wedge (false \vee true) = true \wedge true = true$$

Note

We evaluate $\neg P_1$ before $P_1 \wedge P_2$, this is because the following decreasing rank for connectives operator holds:

$$\leftrightarrow \quad \rightarrow \quad \vee \quad \wedge \quad \neg$$

Exercise: Truth Tables

Example (XOR)

Write the truth table for the formula:

$$G \triangleq (P \vee Q) \wedge \neg(P \wedge Q)$$

Exercise: Truth Tables

Example (XOR)

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Sol.

P	Q	$P \vee Q$	$P \wedge Q$	$\neg(P \wedge Q)$	G
T	T	T	T	F	F
T	F	T	F	T	T
F	T	T	F	T	T
F	F	F	F	T	F

Interpretation

Definition

Interpretation: Given a propositional formula G, let $\{A_1, \cdots, A_n\}$ be the set of atoms which occur in the formula, an **Interpretation** I of G is an assignment of truth values to $\{A_1, \cdots, A_n\}$.

Example

Consider the formula: $G \triangleq (P \vee Q) \wedge \neg(P \wedge Q)$

Set of atoms: $\{P, Q\}$

Interpretation for G: $I = \{P = \mathbf{T}, Q = \mathbf{F}\}$

Interpretation contd.

- Each atom A_i can be assigned either **True** or **False** but never both.
- Given an interpretation I a formula G is said to be true in I iff G is evaluated to **True** in the interpretation
- Given a formula G with n distinct atoms there will be 2^n distinct interpretations for the atoms in G.
- Convention: $\{P, \neg Q, \neg R, S\}$ represents an interpretation $I : \{P = T, Q = F, R = F, S = T\}$.
- Given a formula G and an interpretation I, if G is true under I we say that I is a model for G.and we can write $I \models G$

Validity

Definition

Valid Formula: A formula F is **valid** iff it is true in all its interpretation

- A valid formula can be also called a **Tautology**
- A formula which is not valid is **invalid**
- If F is valid we can write $\models F$

Example (de Morgan's Law)

$(\neg(P \wedge Q) \leftrightarrow (\neg P \vee \neg Q))$ is a valid formula

P	Q	$\neg(P \wedge Q)$	$\neg P \vee \neg Q$	$(\neg(P \wedge Q) \leftrightarrow (\neg P \vee \neg Q))$
T	T	F	F	T
T	F	T	T	T
F	T	T	T	T
F	F	T	T	T

Inconsistency

Definition

Inconsistent Formula: A formula F is **inconsistent** iff it is false in all its interpretation

- An inconsistent formula is said to be **unsatisfiable**
- A formula which is not inconsistent is **consistent** or **satisfiable**
- Invalid is different from Inconsistent

Example

$\neg((\neg(P \wedge Q) \leftrightarrow (\neg P \vee \neg Q)))$ is inconsistent

P	Q	$(\neg(P \wedge Q) \leftrightarrow (\neg P \vee \neg Q))$	$\neg(\neg(P \wedge Q) \leftrightarrow (\neg P \vee \neg Q))$
T	T	T	F
T	F	T	F
F	T	T	F
F	F	T	F

Inconsistency and Validity

- A formula is valid iff its negation is inconsistent (and vice versa)
- A formula is invalid (consistent) iff there is at least an interpretation in which the formula is false (true)
- An inconsistent formula is invalid but the opposite does not hold
- A valid formula is consistent but the opposite does not hold

Example

The formula $G \triangleq P \vee Q$ is invalid (e.g., it is false when P and Q are false) but is not inconsistent because it is true in all other cases. Moreover, G is consistent (e.g., it is true whenever P or Q are false) but is not valid because it is false when both P and Q are false.

Decidability

Property

Propositional Logic is decidable: there is a terminating method to decide whether a formula is **valid**.

- To decide whether a formula is valid:
  1. we can enumerate all possible interpretations
  2. for each interpretation evaluate the formula
- Number of interpretations for a formula are finite (2^n)
- Decidability is a very strong and desirable property for a Logical System
- Trade off between representational power and decidability

Logical Equivalence

Definition

Logical Equivalence: Two formulas F and G are logically equivalent $F \equiv G$ iff the truth values of F and G are the same under every interpretation of F and G.

Useful equivalence rules

$(P \wedge Q)$	$\equiv$	$(Q \wedge P)$	commutativity of $\wedge$
$(P \vee Q)$	$\equiv$	$(Q \vee P)$	commutativity of $\vee$
$((P \wedge Q) \wedge R)$	$\equiv$	$(P \wedge (Q \wedge R))$	associativity of $\wedge$
$((P \vee Q) \vee R)$	$\equiv$	$(P \vee (Q \vee R))$	associativity of $\vee$
$\neg(\neg P)$	$\equiv$	P	double-negation elimination
$(P \rightarrow Q)$	$\equiv$	$(\neg Q \rightarrow \neg P)$	contraposition
$(P \rightarrow Q)$	$\equiv$	$(\neg P \vee Q)$	implication elimination
$(P \leftrightarrow Q)$	$\equiv$	$((P \rightarrow Q) \wedge (Q \rightarrow P))$	biconditional elimination
$\neg(P \wedge Q)$	$\equiv$	$(\neg P \vee \neg Q)$	de Morgan
$\neg(P \vee Q)$	$\equiv$	$(\neg P \wedge \neg Q)$	de Morgan
$(P \wedge (Q \vee R))$	$\equiv$	$((P \wedge Q) \vee (P \wedge R))$	distributivity of $\wedge$ over $\vee$
$(P \vee (Q \wedge R))$	$\equiv$	$((P \vee Q) \wedge (P \vee R))$	distributivity of $\vee$ over $\wedge$

Normal Forms

Standard ways of writing formulas

Two main normal forms:

- Conjunctive Normal Form (CNF)
- Disjunctive Normal Form (DNF)

Definition

Literal: a literal is an atom or the negation of an atom

Definition

Negation Normal Form: A formula is in Negation Normal Form (NNF) iff negations appears only in front of atoms

CNF

Definition

Conjunctive Normal Form: A formula F is in Conjunctive Normal Form (CNF) iff it is in Negation Normal Form and it has the form $F \triangleq F_1 \wedge F_2 \wedge \cdots \wedge F_n$, where each F_i is a disjunction of literals.

- If F is in CNF Each F_i is called a **clause**
- CNF is also refered to as Clausal Form

Example

The formula $G \triangleq (\neg P \vee Q) \wedge (\neg P \vee R)$ is in CNF. We can write G as a set of clauses $\{C_1, C_2\}$ where $C_1 = \neg P \vee Q$ and $C_2 = \neg P \vee R$.
The formula $G \triangleq \neg(P \vee Q) \wedge (\neg P \vee R)$ is not in CNF because negation appears in front of a formula and not only in front of atoms.

DNF

Definition

Disjunctive Normal Form: A formula F is in Disjunctive Normal Form (DNF) iff it is in Negation Normal Form and it has the form $F \triangleq F_1 \vee F_2 \vee \cdots \vee F_n$, where each F_i is a conjunction of literals.

Example

The formula $G \triangleq (\neg P \wedge R) \vee (Q \wedge \neg P) \vee (Q \wedge P)$ is in DNF.

Any formula can be transformed into a normal form by using the equivalence rules given above.

Transforming Formulas

Example (Formula transformations)

Prove that the following logical equivalences hold by transforming formulas:

$P \vee Q \wedge \neg(P \wedge Q) \leftrightarrow (P \vee Q) \wedge (\neg P \vee \neg Q) \leftrightarrow (\neg P \wedge Q) \vee (P \wedge \neg Q)$

Transforming Formulas

Example (Formula transformations)

Prove that the following logical equivalences hold by transforming formulas:
$P \vee Q \wedge \neg(P \wedge Q) \leftrightarrow (P \vee Q) \wedge (\neg P \vee \neg Q) \leftrightarrow (\neg P \wedge Q) \vee (P \wedge \neg Q)$

Sol.

Given $P \vee Q \wedge \neg(P \wedge Q)$ apply de Morgan's law on the second part and directly obtain $(P \vee Q) \wedge (\neg P \vee \neg Q)$
For more examples see Examples 2.8, 2.9 [Chang and Lee Ch. 2]
Try to prove the other equivalence

Logical Consequence

Definition

Given a set of formulas $\{F_1, \cdots, F_n\}$ and a formula G, G is said to be a logical consequence of $F_1, \cdots, F_n$ iff for any interpretation I in which $F_1 \wedge \cdots \wedge F_n$ is true G is also true.

- If G is a logical consequence of $\{F_1, \cdots, F_n\}$ we write $F_1 \wedge \cdots \wedge F_n \models G$.
- $F_1, \cdots, F_n$ are called axioms or premises for G.
- $F \equiv Q$ iff $F \models Q$ and $Q \models F$

Example

$S \to C, C \to F, S$ are premises for F

Deduction Theorem

Theorem

Given a set of formulas $\{F_1, \cdots, F_n\}$ *and a formula* G*,* $(F_1 \wedge \cdots \wedge F_n) \models G$ *if and only if* $\models (F_1 \wedge \cdots \wedge F_n) \to G$*.*

Sketch of proof.

- $\Rightarrow$ For each interpretation I in which $F_1 \wedge \cdots \wedge F_n$ is true G is true, $I \models (F_1 \wedge \cdots \wedge F_n) \to G$, however for every interpretation I' in which $F_1 \wedge \cdots \wedge F_n$ is false then $(F_1 \wedge \cdots \wedge F_n \to G)$ is true, thus $I' \models (F_1 \wedge \cdots \wedge F_n) \to G$. Therefore, $\models (F_1 \wedge \cdots \wedge F_n) \to G$.
- $\Leftarrow$ for every interpretation we have that when $F_1 \wedge \cdots \wedge F_n$ is true G is true therefore $(F_1 \wedge \cdots \wedge F_n) \models G$.

□

Proof by Refutation

Theorem

Given a set of formulas $\{F_1, \cdots, F_n\}$ *and a formula* G*,* $(F_1 \wedge \cdots \wedge F_n) \models G$ *if and only if* $F_1 \wedge \cdots \wedge F_n \wedge \neg G$ *is inconsistent.*

Sketch of proof.

$(F_1 \wedge \cdots \wedge F_n) \models G$ holds iff for every interpretation under which $F_1 \wedge \cdots \wedge F_n$ is true also G is true. This holds iff there is no interpretation for which $F_1 \wedge \cdots \wedge F_n$ is true and G is false, but this happens precisely when $F_1 \wedge \cdots \wedge F_n \wedge \neg G$ is false for every interpretation, i.e. when $F_1 \wedge \cdots \wedge F_n \wedge \neg G$ is inconsistent. □

Discussion

- Previous theorems show that:
  - We can prove logical consequence by proving validity of a formula
  - We can prove logical consequence by refuting a given formula, i.e. by proving a given formula is inconsistent
  - Notice that we did not use any specific properties of propositional logic

Logical consequences are usually referred to as theorems, and G is the **conclusion** of the theorem.

Logical Consequence: Example

Example

We want to show that $(P \rightarrow Q) \wedge P \models Q$

Using definition

We show that for each interpretation in which $(P \rightarrow Q) \wedge P$ is true, also Q is true. We can do that by writing the truth table of the formulas.

Logical Consequence: Example contd.

Using deduction theorem

We know from the deduction theorem that $(P \to Q) \land P \models Q$ iff $\models ((P \to Q) \land P) \to Q$. Therefore we need to show that $((P \to Q) \land P) \to Q$ is valid, we can do that by writing the truth table of the formula and verifying that the formula is evaluated true for all its possible interpretation.

Using Refutation

We know that $(P \to Q) \land P \models Q$ iff $(P \to Q) \land P \land \neg Q$ is inconsistent. Therefore we need to show that $(P \to Q) \land P \land \neg Q$ is inconsistent, we can do that by writing the truth table of the formula and verifying that the formula is evaluated false for all its possible interpretation.

Exercises

Exercise

- Consider the following formulas: $F_1 \triangleq (P \to Q)$, $F_2 \triangleq \neg Q$, $G \triangleq \neg P$. Show that $F_1 \wedge F_2 \models G$ using all three approaches [Chang-Lee example 2.11]
- Given that if the congress refuses to enact new laws, then the strike will not be over unless it lasts for more than a year or the president of the firm resigns, will the strike be over if the congress refuses to act and the strike just started ? [Chang-Lee example 2.12]

First Order Logic

Summary

- Motivation
- Syntax
- Semantics
- Prenex Normal Form

Characteristics of Propositional Logic

- Propositional logic is **declarative**: pieces of syntax correspond to facts
- Propositional logic is **decidable**: We can always decide through a terminating process whether a formula is valid.
- Propositional logics does not represent structure of atoms

Lack of structure in Prop. Logic

Example

$P \triangleq$ Every man is mortal
$S \triangleq$ Socrate is a man
$Q \triangleq$ Socrate is mortal
In propositional logic Q is not a logic consequence of P and S, but we would like to express this relationship.

Examples of expressions in FOL

Example

Every man is mortal	$\forall x(man(x) \rightarrow mortal(x))$
Socrate is a man	$man(Socrate)$
Socrate is mortal	$mortal(Socrate)$

Components of First Order Logic

Objects, Relations, Functions

Whereas propositional logic assumes world contains facts, first-order logic (like natural language) assumes the world contains: **Objects, Relations, Functions**.

- Objects: people, houses, numbers, theories, colors, football games, wars, centuries $\cdots$
- Relations: red, round, multistoried $\cdots$, brother of, bigger than, inside, part of, has color, occurred after, owns, comes between, $\cdots$
- Functions: father of, best friend, second half of, one more than, beginning of $\cdots$

The Language: Logical Symbols

Logical Symbols

A first order language $\mathcal{L}$ is built upon the following sets of symbols:

- propositional connectives: $\neg, \wedge, \vee$ (plus the shortcuts $\rightarrow$ and $\leftrightarrow$);
- propositional constants $\top$ and $\bot$ (represent **True** and **False** respectively);
- equality $=$ (not always included);
- a denumerable set of individual variable symbols: $x_1, x_2, \cdots$;
- universal quantification $\forall$;
- existentional quantification $\exists$;

The Language: Parameters

Parameters

- A denumerable set of predicate symbols, each associated with a positive integer n, arity. A predicate with arity n is called n-ary;
- A denumerable set of function symbols, each associated with a positive integer n, arity. A function with arity n is called n-ary;
- A denumerable set of constant symbols.

Note

The parameters characterise different first order languages, while logical symbols are always the same.
Therefore parameters are often called the Signature of a First Order Language.

Example I

The language of pure predicates

n-ary predicate symbols: $P_1^n, P_2^n, \cdots$;

constant symbols: $c_1, c_2, \cdots$;

no function symbols, no equality.

Example

The Book is on the table:

- OnTable(Book)
- On(Table,Book)

Example II

The language of set theory

- Equality;
- predicate symbols: only the binary predicate $\in$;
- constant symbols: $\{\,\}$;
- no function symbols.

Example

There exists no set such that all other sets are its element

$$\neg \exists x \forall y (y \in x)$$

Example III

The language of elementary number theory

Equality;

predicate symbols: only the binary predicate $<$;

constant symbols: 0;

function symbols: a unary function symbol s, successor function, and the binary function symbols $+$ and $\times$, addition and multiplication

Example

There exists no number greater than all others

$$\neg \exists x \forall y (y < x)$$

Definition of FOL Formulas

FOL formulas

Inductive definition of basic components

1. Terms
2. Atomic Formulas

Terms

FOL terms

The set Term of the terms of $\mathcal{L}$ is inductively defined as follows:

1. Every constant is a term;
2. Every variable symbol is a term
3. If $t_1 \dots t_n$ are terms and f is a n-ary function symbol, $f(t_1, \dots, t_n)$ is a term (functional term).
4. All terms are generated by applying the above rules

Example (Terms for FOL)

$c, \quad x, \quad f(x, y), \quad f(g(c), y), \quad plus(plus(x, 1), 3), \dots$

Atomic Formulas

Atoms

The set Atom of the atomic formulae is inductively defined as follows:

1. $\bot$ and $\top$ are atoms;
2. If t_1 and t_2 are terms then $t_1 = t_2$ is an atom;
3. If $t_1, \cdots, t_n$ are terms and P is a n-ary predicate symbol $P(t_1, \cdots, t_n)$ is an atom;
4. All atomic formulas are defined by applying the above rules

Example (Atoms in FOL)

$P(x), \quad Q(x, c), \quad R(x, f(x, y + c)), \cdots$

Scope of Quantifiers

Definition (Scope of quantifiers)

The scope of a quantifier occurring in a formula is the formula to which the quantifier applies

Example (Scope of quantifiers)

$\forall x(Q(x) \rightarrow R(x))$ the scope of $\forall$ is $(Q(x) \rightarrow R(x))$
$\forall x(Q(x) \rightarrow \exists y\ R(y))$ the scope of $\forall$ is $(Q(x) \rightarrow \exists y\ R(y))$ and the scope of $\exists$ is $R(y)$

Free and bounded variables

Definition (Free occurence of a variable)

An occurrence of a variable in a formula is free if the variable is not in the scope of any quantifier. An occurence of a variable which is not free is bound

Definition (Free variable)

A variable in a formula is free if at least one occurrence of the variable is free. A variable is bound if at least one occurrence is bound.

Examples of free and bound variables

Example (Free occurences and free variables)

$\forall x(Q(x,y) \rightarrow R(x,y))$ the occurence of y is free while the occurence of x is bound, therefore y is free while x is bound
$\forall x(Q(x,y) \rightarrow \exists y\ R(x,y))$ the occurrence of y in Q is free while the occurence of y in R is bound, the occurrences of x in both formulas are bound. Therefore, the variable x is bound while the variable y is both free and bound

First Order Formulas

Well-Formed Formulas

The set of formulae of $\mathcal{L}$ is inductively defined as follows:

- Every atom is a formula;
- If A is a formula $\neg A$ is a formula;
- If $\circ$ is a binary operator, A and B are formulas, then $A \circ B$ is a formula;
- If A is a formula, x is a free variable in A then $\forall x A$ and $\exists x A$ are formulas
- All formulas are generated by a finite number of applications of the above rules.

Example (FOL Formulas)

$P(x), \quad \exists x Q(x, c), \quad \forall x R(x, f(x, y + c)), \cdots$

Operator Precedence

Operator Precedence

Precedence among logical operators is defined as follows:

$$\forall, \exists, \neg, \wedge, \vee, \rightarrow, \leftrightarrow$$

convention: all operators are right associative (as in propositional logic).

Example

$$\forall x P(x) \rightarrow \exists y \exists z Q(y, z) \wedge \neg \exists x R(x)$$

$$(\forall x P(x)) \rightarrow \exists y (\exists z (Q(y, z) \wedge \neg (\exists x (R(x))).$$

Note

The inner occurrence of x is bound to the innermost existential quantifier

Ground and Closed Formulas

Definition (Ground Formula)

A formula F is ground if it does not contain variables

Definition (Closed Formula)

A formula F is closed if it does not contain free variables

Example (Ground and Closed Formulas)

$Boring(GrandeFratello)$	(ground)
$\forall x(Reality(x) \rightarrow Boring(x))$	(closed, not ground)
$\forall x(Reality(x) \rightarrow BetterProgram(y, x))$	(not closed, not ground)

Example of FOL formalisation

Example (Basic axioms of natural language)

- A_1: for every number there is one and only one immediate successor
- A_2: there is no number for which 0 is the immediate successor
- A_3: for every number other than 0 there is one and only one immediate predecessor

Assume:

- $s(x)$ is function for immediate successor
- $p(x)$ is function for immediate predecessor
- $E(x, y)$ is true iff x is equal to y

Example contd.

- $A_1 \triangleq \forall x \exists y (E(s(x), y) \wedge (\forall z)(E(s(x), z) \rightarrow E(z, y)))$
- $A_2 \triangleq \neg((\exists x) E(s(x), 0))$
- $A_3 \triangleq \forall x (\neg E(x, 0) \rightarrow \exists y (E(p(x), y) \wedge (\forall z)(E(p(x), z) \rightarrow E(z, y)))$

Interpretations in FOL

- In Prop. Logic an Interpretation for a formula G is an assignment of truth values to each atoms occuring in the formula
- In FOL we have to do more than that:
  1. Specify a domain of interest (e.g., real numbers)
  2. An assignment to constants, function symbols and predicate symbols

Example (Interpretation)

Consider the set of formulas: $\{\forall x P(x), \exists x Q(x)\}$;
An interpretation will need to specify a domain, e.g. $D = \{1, 2\}$ and an assignment for all predicate symbol from D to the set $\{T, F\}$, for example $\{P(1) = T, P(2) = F\}$ and $\{Q(1) = F, Q(2) = T\}$.

Interpretation: Formal Definition

Definition of Interpretation

An Interpretation for the language $\mathcal{L}$ is a pair $I = \langle D, A \rangle$ where:

- D is a non empty set called domain of I;
- A is a function that maps:
  - every constant symbol c into an element $c^A \in D$;
  - every n-ary function symbol f into a function $f^A : D^n \rightarrow D$;
  - every n-ary predicate symbol P into a n-ary relation $P^A : D^n \rightarrow \{\top, \bot\}$.

Interpretation: Example

Example (Interpretation)

$$\forall x \exists y P(x, y)$$

- D, the set of human beings
  $P^A(a, b) = true$ iff b is father of a
  All human beings have a father
- D, the set of human beings
  $P^{A'}(a, b) = true$ iff b is mother of a
  All human beings have a mother
- D the set of natural numbers
  $P^{A''}(a, b) = true$ iff $a < b$
  For every nat number there is a greater one

Evaluation of FOL formulas

Given an interpretation $I = \langle D, A \rangle$, FOL formulas are evaluated to **true** or **false** according to the following rules:

- If S is an atomic formula and $S \triangleq P(t_1, \cdots, t_n)$, S is **true** iff $P^A(t_1^A, \cdots, t_n^A) = \top$
- If S is an atomic formula and $S \triangleq t_1 = t_2$, S is **true** iff $t_1^A = t_2^A$.
- If S is a formula evaluated to **true** then $\neg S$ is **false**.
- If S and T are two formulas then $S \wedge T$ is **true** iff A and T are **true**.
- If S and T are two formulas then $S \vee T$ is **true** iff A or T are **true**
- If $S \triangleq \forall x G$ is **true** iff G is true for every element $d \in D$.
- If $S \triangleq \exists x G$ is **true** iff G is true for at least one element $d \in D$.

Evaluation of FOL formulas contd.

Note

- According to this evaluation procedure formulas containing free variables can not be evaluated.
- The logical operators $\rightarrow$ and $\leftrightarrow$ are evaluated using the usual shortcuts:
  - $A \rightarrow B \equiv \neg A \vee B$
  - $A \leftrightarrow B \equiv A \rightarrow B \wedge B \rightarrow A$

Examples of FOL formula evaluation

Example (Example of evaluation)

$$G \triangleq \forall x \exists y P(x, y)$$

Interpretation I for G

$D = 1, 2$ and $P^A(x, y) = true$ iff $x < y$

To evaluate G we have to evaluate for each element $d \in D$ the formula $H \triangleq \exists y P(d, y)$.

- $x = 1$ we have to check whether there is at least one element $d' \in D$ such that $P^A(1, d')$ hols, i.e. such that $1 < d'$. We observe that $1 < 2$ holds, thus for $x = 1$ the formula H is **true**.
- $x = 2$ however H is **false**.

Thus G is **false** under I

Models, validity, satisfiability

Given the notion of interpretion, the concepts of model, validity and satisfiability can be defined as for propositional logic.

Definition (Model)

An interpretation I is a model for G iff G is evaluated to true under I. We write $I \models G$.

Definition (Validity)

A formula G is valid iff it is evaluated to **true** under all its interpretations. We write $\models G$

Definition (Inconsistency)

A formula G is inconsistent iff it is evaluated to **false** under all its interpretations

Logical consequence

Definition (Logical consequence)

A formula G is a logical consequence of formulas $\{F_1, \cdots, F_n\}$ iff for every interpretation I if $I \models F_1 \wedge \cdots \wedge F_n$ we have that $I \models G$.

The following theorems hold also for First Order Logic

Theorem (Deduction Theorem)

Given a set of formulas $\{F_1, \cdots, F_n\}$ and a formula G,
$F_1 \wedge \cdots \wedge F_n \models G$ *iff* $\models F_1 \wedge \cdots \wedge F_n \rightarrow G$

Theorem (Proof by Refutation)

Given a set of formulas $\{F_1, \cdots, F_n\}$ and a formula G,
$\models F_1 \wedge \cdots \wedge F_n \rightarrow G$ *iff* $F_1 \wedge \cdots \wedge F_n \wedge \neg G$ *is inconsistent.*

Example of logical consequence

Example (Logical consequence)

$$\forall x P(x) \rightarrow Q(x) \wedge P(a) \models Q(a)$$

Using deduction theorem

Deduction theorem: $\forall x P(x) \rightarrow Q(x) \wedge P(a) \models Q(a)$ iff $\models (\forall x P(x) \rightarrow Q(x) \wedge P(a)) \rightarrow Q(a)$

Suppose I falsifies the formula then

1. $Q(a)$ is false under I
2. $I \models \forall x P(x) \rightarrow Q(x) \wedge P(a)$

If 2 then $I \models \forall x P(x) \rightarrow Q(x)$ and $I \models P(a)$

Then $I \models Q(a)$ which gives us a contradiction with 1

First Order Logic and decidability

- FOL is not decidable
- To prove that a formula is valid in FOL we can not simply enumerate all its possible interpretations
  - possible interpretations of a formula can be infinitely many: we can have an infinite number of domains.
- We need an automated mechanism to verify inconsistent formulas

Prenex Normal Forms

Definition (Prenex normal form)

A formula F is in *prenex normal form* iff it is in the form of

$$Q_1 x_1 \cdots Q_n x_n M$$

Where $Q_i x_i$ are quantifiers (i.e. either $\forall$ or $\exists$) and M is a quantifier free formula.

- $Q_1 x_1 \cdots Q_n x_n$ is called the prefix of the formula;
- M is called the matrix of the formula.

Prenex Normal Forms: Examples

Example (Prenex Normal Form)

- $\forall x \exists y P(x) \rightarrow Q(y)$
- $\forall x \exists y \forall z Q(x) \rightarrow R(z, y)$

Example (Not Prenex Normal Form)

- $\forall x P(x) \rightarrow \exists y Q(y)$
- $\forall x \exists y Q(x) \rightarrow \forall z R(z, y)$
- $\forall x Q(x, y) \rightarrow \forall y R(y)$

Logical Equivalence

Definition (Logical Equivalence)

Two formulas F and G are logically equivalent iff $F \models G$ and $G \models F$ and we write $F \equiv G$.

- F and G are equivalent iff the truth values of F and G are the same under every possible interpretations.
- Same as in in prop. Logic
- all logical equivalences defined for prop. logic still hold in FOL
- additional rules for formulas containing quantifiers

Equivalences for Quantifiers

Formulas are logically equivalent if they differ in

- the name of variables in the scope of quantifiers
  $\forall x P(x) \equiv \forall y P(y)$
- the order of quantifiers of the same kind
  $\forall x \forall y P(x, y) \equiv \forall y \forall x P(x, y) \equiv \forall x, y P(x, y)$
- addition or elimination of quantifiers whose variable does not occurr in their scope
  $\forall x P(y) \equiv P(y)$

Additional Equivalence Rules

Negation

$$\neg(\forall x F[x]) \equiv \exists x \neg F[x] \quad (1)$$

$$\neg(\exists x F[x]) \equiv \forall x \neg F[x] \quad (2)$$

And, Or

$$QxF[x] \vee G \equiv Qx(F[x] \vee G) \quad (3)$$

$$QxF[x] \wedge G \equiv Qx(F[x] \wedge G) \quad (4)$$

$$Q_1xF[x] \vee Q_2xH[x] \equiv Q_1xQ_2y(F[x] \vee H[y]) \quad (5)$$

$$Q_2xF[x] \wedge Q_2xH[x] \equiv Q_1xQ_2y(F[x] \wedge H[y]) \quad (6)$$

Note

We assume that y does not appear in F

Additional Equivalence Rules contd.

More specific rules for and, or

$$\forall x F[x] \wedge \forall x G[x] \equiv \forall x (F[x] \wedge G[x]) \quad (7)$$

$$\exists x F[x] \vee \exists G[x] \equiv \exists x (F[x] \vee G[x]) \quad (8)$$

Note

For rules 5 and 6 we renamed the variable in H because otherwise the rule could not be applied. e.g.
$\forall x A[x] \vee \forall x B[x] \not\equiv \forall x (A[x] \vee B[x])$

Example of Prenex normal Form Transformation I

Example

$$\forall x P(x) \rightarrow \exists x Q(x)$$

1. $\neg(\forall x P(x)) \lor \exists x Q(x)$ (elimination of implication)
2. $\exists x \neg P(x) \lor \exists x Q(x)$ (rule 1)
3. $\exists x(\neg P(x) \lor Q(x))$ (rule 8)

Example of Prenex normal Form Transformation II

Example

$$\forall x(P(x) \rightarrow \exists y Q(x,y))$$

1. $\forall x(\neg P(x) \vee \exists y Q(x,y))$ (elimination of implication)
2. $\forall x \exists y(\neg P(x) \vee Q(x,y))$ (rule 3)

Example of FOL application

Example (Doctors and Quacks)

Assume the following sentences are true: Some patients like all doctors, No patient likes any quack. Show that we can conclude that no doctor is a quack.

Formalisation

$F_1 \triangleq$ Some patients like all doctors:
$(\exists x)(Patient(x) \wedge (\forall y)(Doctor(y) \rightarrow Likes(x, y)))$
$F_2 \triangleq$ No patient likes any quack:
$(\forall x)(Patient(x) \rightarrow (\forall y)(Quack(y) \rightarrow \neg Likes(x, y)))$
$F_3 \triangleq$ No doctor is a quack:
$(\forall x)(Doctor(x) \rightarrow \neg Quack(x))$

Doctors and Quacks contd.

Logical Equivalence

We want to show that $(F_1 \wedge F_2) \models F_3$. Suppose $I \models F_1 \wedge F_2$ we want to show that $I \models F_3$

- If I models F_1 then for $e \in D$ we have $Patient(e) \wedge (\forall y)(Doctor(y) \rightarrow Likes(e, y))$ is true.
- Since I models F_2 we also have that $Patient(e) \rightarrow (\forall y)(Quack(y) \rightarrow \neg Likes(e, y))$ is true.
- From F_1 being true we have that $(\forall y)(Doctor(y) \rightarrow Likes(e, y))$ must be true.

Doctors and Quacks contd.

Logical Equivalence contd.

- From F_1 being true we have that
  - $Patient(e)$ is true in I
  - and thus from F_2 we have that $(\forall y)(Quack(y) \rightarrow \neg Likes(e, y))$ must be true in I
- Therefore we have that $(\forall y)((Doctor(y) \rightarrow Likes(e, y)) \wedge (Quack(y) \rightarrow \neg Likes(e, y)))$ must be true in I
- From this we can conclude that $(\forall y)(Doctor(y) \rightarrow \neg Quack(y))$ must be true in I

What if we modify F_1 as follows ?

$$F_1 \triangleq (\exists x)(Patient(x) \rightarrow (\forall y)(Doctor(y) \rightarrow Likes(x, y)))$$

Exercises

Exercise

- $A \triangleq (\exists x)P(x) \to (\forall x)P(x)$ [Ex. 6 page 42 Chang-Lee]
  1. Prove that A is valid for any domain D which contains only one element
  2. Let $D = \{a, b\}$ find one interpretation I such that $I \neg \models A$
- Transorm the following formulas into prenex normal form [Ex. 9 page 43 Chang-Lee]
  1. $(\forall x)(P(x) \to (\exists y)Q(x, y))$
  2. $(\exists x)(\neg((\exists y)P(x, y)) \to ((\exists z)Q(z) \to R(z)))$
  3. $(\forall x)(\forall y)((\exists z)P(x, y, z) \wedge ((\exists u)Q(x, u) \to (\exists v)Q(y, v)))$