H-Interpretation and H-Satisfiability

Summary

- H-Interpretaton and H-Satisfiability [Chang-Lee Ch. 4.3]
- Semantic Trees [Chang-Lee Ch. 4.4]

Interpretations over the Herbrand Universe

Interpretations and the Herbrand Universe

Let us consider Interpretations over the Herbrand universe.
Given a set of clauses S an interpretation must provide:

- assignment for costants to element of the domain
- an assignment for function symbols to element of the domain
- an assignment for predicate symbols to $\top, \bot$

Where the domain is the Herbrand Universe for S

H Interpretations

Definition (H Interpretation)

Let S be a set of clauses, H the Herbrand Universe of S and $I = \langle D, A \rangle$ an Interpretation of S. I is an H-Interpretation of S if the following holds:

- $D = H$
- $t^A = t$ for all terms t

In more detail

- Let c be a costant symbol $c^A = c$.
- Let f be a n-ary function symbol f^A maps $(h_1, \cdots, h_n) \in H^n$ to $f(h_1, \cdots, h_n) \in H$

H Interpretations: Predicate symbols

H-Interpretations: Predicate

No restrictions for predicate symbols

Given S, let $A = \{A_1, \cdots A_n, \cdots\}$ be the Herbrand base (or atom set) of S, an H-Interpretation can be represented as:

$$I = \{m_1, \cdots m_n, \cdots\}$$

where $m_j = A_j$ or $m_j = \neg A_j$ for $j = 1, \cdots, n, \cdots$

Example of H-Interpretation

Example

Consider the set $S = \{P(x) \vee Q(x), R(f(y))\}$

Example of H-Interpretation

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Consider the set $S = \{P(x) \vee Q(x), R(f(y))\}$

H-Interpretation

- $H = \{a, f(a), f(f(a)), \cdots\}$
- $A = \{P(a), Q(a), R(a), P(f(a)), Q(f(a)), R(f(a)), \cdots\}$
- Possible H-Interpretations:
  - $I_1 = \{P(a), Q(a), R(a), P(f(a)), Q(f(a)), R(f(a)), \cdots\}$
  - $I_2 = \{\neg P(a), Q(a), R(a), \neg P(f(a)), Q(f(a)), R(f(a)), \cdots\}$
  - $I_3 = \{\neg P(a), \neg Q(a), \neg R(a), \neg P(f(a)), \neg Q(f(a)), \neg R(f(a)), \cdots\}$

Example of not H-Interpretation

Example (not H-Interpretation)

Consider the set $S = \{P(x) \vee Q(x), R(f(y))\}$.
$NHI = \langle D, A \rangle$

- $D = \{1, 2\}$
- $f^A(1) = 1, f^A(2) = 2$
- $\{P(1), \neg P(2), Q(1), \neg Q(2), R(1), \neg R(2)\}$

Mapping among Interpretations

mapping to H-Interpretations

Given an Interpretation I we can always find a **corresponding** I^* H-Interpretation

Example of not H-Interpretation II

Example

Consider the set $S = \{P(x), Q(y, f(y, a))\}$.
$I = \langle D, A \rangle$

- $D = \{1, 2\}$
- $a^A = 2$
- $f^A(1,1) = 1, f^A(1,2) = 2, f^A(2,1) = 2, f^A(2,2) = 1$
- $\{P(1), \neg P(2), \neg Q(1,1), Q(1,2), \neg Q(2,1), Q(2,2)\}$

Example of mapping between H-Interpretation

Example

Given $S = \{P(x), Q(y, f(y, a))\}$ and I we can define I^* as follows:

1. $H = \{a, f(a, a), f(a, f(a, a)), f(f(a, a), a), f(f(a, a), f(a, a)), \cdots\}$
2. $A = \{P(a), Q(a, a), Q(a, f(a, a)), Q(f(a, a), a), P(f(a, a)), Q(f(a, a), f(a, a)), \cdots\}$
3. $I^* = \{\neg P(a), Q(a, a), P(f(a, a)), \neg Q(a, f(a, a)), \cdots\}$
    - $P(a) = P(2) = \perp$
    - $Q(a, a) = Q(2, 2) = \top$
    - $P(f(a, a)) = P(1) = \top$
    - $Q(a, f(a, a)) = Q(2, f(2, 2)) = Q(2, 1) = \perp$
    - $\cdots$

Multiplicity of H-Interpretation mapping

Multiple H-Interpretions

Consider an Interpretation I

- If there is no constant appearing in S then the added costant a in the Herbrand Universe can be mapped to any element in D.
- Therefore there are more that one H-Interpretation I^* corresponding to I depending on values given to a

Example of Multiple H-Interpretations

Example

Consider the set $S = \{P(x), Q(y, f(y, z))\}$.
$I = \langle D, A \rangle$

- $D = \{1, 2\}$
- $f^A(1,1) = 1, f^A(1,2) = 2, f^A(2,1) = 2, f^A(2,2) = 1$
- $\{P(1), \neg P(2), \neg Q(1,1), Q(1,2), \neg Q(2,1), Q(2,2)\}$

Example

Corresponding H-Interpretations

- $I_1^* = \{\neg P(a), Q(a,a), P(f(a,a)), \neg Q(a, f(a,a)), \cdots\}$ if $a = 2$
- $I_2^* = \{P(a), \neg Q(a,a), P(f(a,a)), \neg Q(a, f(a,a)), \cdots\}$ if $a = 1$

Example of Multiple H-Interpretations II

Example

Given $S = \{P(x) \vee Q(x), R(f(y))\}$ and $NHI = \langle D, A \rangle$

- $D = \{1, 2\}$
- $f^A(1) = 1, f^A(2) = 2$
- $\{P(1), \neg P(2), Q(1), \neg Q(2), R(1), \neg R(2)\}$

we can define I_1^* as follows:

1. $H = \{a, f(a), f(f(a)), \cdots\}$
2. $A = \{P(a), Q(a), R(a), P(f(a)), Q(f(a)), R(f(a)), \cdots\}$
3. $a^A = 1$
4. $\{P(a) = P(1) = \top, Q(a) = Q(1) = \top, R(a) = R(1) = \top, P(f(a)) = P(1) = \top \cdots\}$

Example of Multiple H-Interpretations II

Example (cont. from previous example)

we can also define I_2^* as follows:

1. $H = \{a, f(a), f(f(a)), \cdots\}$
2. $A = \{P(a), Q(a), R(a), P(f(a)), Q(f(a)), R(f(a)), \cdots\}$
3. $a^A = 2$
4. $\{P(a) = P(2) = \perp, Q(a) = Q(1) = \perp, R(a) = R(1) = \perp, P(f(a)) = P(1) = \perp \cdots\}$

Mapping to H-Interpretation

Definition (Mapping to H-Interpretation)

Given $I = \langle D, A \rangle$ interpretation over D, an H-interpretation $I^* \langle H, A^* \rangle$ corresponding to I is an H-interpretation that satisfies the following condition:

- Let $h_1, \cdots, h_n$ be elements of H and let $m : H \to D$ be a mapping from H to D, then $P^{A^*}(h_1, \cdots, h_n) = P^A(m(h1), \cdots, m(h_n))$

Preserving Satisfiability

Lemma

If an interpretation I over a domain D satisfies a set of clauses S, then any of the H-Interpretation I^* corresponding to I satisfies S.

Sketch of proof.

Suppose $I \models S$ but $I^* \not\models S$.

- Since $I^* \not\models S$ then $\exists C^*$ ground that is not satisfied by I^*
- Since I^* is an H-Interpretation corresponding to I, for each element in I^* we can find an element in I with the same truth value.
- Therefore we have a ground clause C corresponding to C^* that is not satisfied by I, which contradicts the hypothesis

$\square$

Preserving Satisfaibility Example

Example

Consider the set of clauses $S = \{P(x, f(x))\}$.
Consider the interpretation I:

- $D = 1, 2$
- $f(1) = 1, f(2) = 2$
- $P(1,1) = \top, P(1,2) = \bot, P(2,1) = \bot, P(2,2) = \top,$

$I \models S$ because all ground clauses $\{P(1,1), P(2,2)\}$ are satisfied by I. Assume I^* is the H-Interpretation corresponding to I with $a = 1$.

- $H_0 = \{a\}, H_1 = \{a, f(a)\}, H_3 = \{a, f(a), f(f(a)), \cdots\}$
- $A = \{P(a,a), P(a, f(a)), P(f(a), a), P(f(a), f(a)), \cdots\}$
- $P(a,a) = P(1,1) = \top, P(a, f(a)) = P(1,1) = \top, P(f(a), a) = P(1,1) = \top P(f(a), f(a)) = P(1,1) = \top$

$I^* \models S$ as well.

H-Satisfiability

Theorem (H-Satisfiability)

A set S of clauses is unsatisfiable iff S is false under all the H-Interpretations

Proof.

Sketch of proof

- $\Rightarrow$ If unsatisfiable then must be false under all interpretations and thus specifically under all H-Interpretations
- $\Leftarrow$ Assume S is false under all H-Interpretations but S is satisfiable. Then there exists $I \models S$. Then for the above lemma there exists an H-Interpretation I^* corresponding to I such that $I^* \models S$ which contraddicts the hypothesis

$\square$

Importance of H-Satisfiability

H-Interpretations are all we need

- Given the above theorem to proove unsatisfiability of S we need only to consider $H - Interpretation$
- We can thus restrict our attention only to the Herbrand universe
- From now onwards we consider only H-Interpretations.

Observations on Satisfiability I

Observation I

A ground instance C' fo a clause C is satisfied by an H-Interpretation I iff there is at least one literal $L' \in C'$ such that $L' \in I$, which is $C' \cap I \neq \{\,\}$.

Example

Given $C \triangleq \neg P(x) \vee Q(f(x))$ and $C' \triangleq \neg P(a) \vee Q(f(a))$ a ground instance, and
$I = \{P(a), \neg Q(a), P(f(a)), Q(f(a)), \neg Q(f(f(a))) \cdots\}$. Does $I \models C'$?

Observations on Satisfiability I

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Example

Given $C \triangleq \neg P(x) \vee Q(f(x))$ and $C' \triangleq \neg P(a) \vee Q(f(a))$ a ground instance, and $I = \{P(a), \neg Q(a), P(f(a)), Q(f(a)), \neg Q(f(f(a))) \cdots\}$. Does $I \models C'$?

Sol.

$I \cap C' = Q(f(a)) \neq \{\,\}$ therefore $I \models C'$

Observation on Satisfiability II

Observation II

Given a clause C and an H-Interpretation I, $I \models C$ iff for every C' ground instance $I \models C'$

Observation III

A clause C is falsified by an H-Interpretation I iff there is at least one C' ground instance such that $I \not\models C'$

Example

Given $C \triangleq \neg P(x) \vee Q(f(x))$, and $I = \{P(a), \neg Q(a), P(f(a)), Q(f(a)), \neg Q(f(f(a))) \cdots\}$. Does $I \models C$?

Observation on Satisfiability II

Observation II

Given a clause C and an H-Interpretation I, $I \models C$ iff for every C' ground instance $I \models C'$

Observation III

A clause C is falsified by an H-Interpretation I iff there is at least one C' ground instance such that $I \not\models C'$

Example

Given $C \triangleq \neg P(x) \vee Q(f(x))$, and $I = \{P(a), \neg Q(a), P(f(a)), Q(f(a)), \neg Q(f(f(a))) \cdots\}$. Does $I \models C$?

Sol.

$C'' = \neg P(f(a)) \vee Q(f(f(a)))$ $I \cap C'' = \{\}$ therefore $I \not\models C$

Observations on Satisfiability III

Observation IV

A set of clause S is unsatisfiable iff for every H-Interpretation I there is at least one C' ground clause of some $C \in S$ such that $I \not\models C'$

Example

Given $S \triangleq \{\neg P(x), P(a)\}$ is S unsatisfiable ?

Observations on Satisfiability III

Observation IV

A set of clause S is unsatisfiable iff for every H-Interpretation I there is at least one C' ground clause of some $C \in S$ such that $I \not\models C'$

Example

Given $S \triangleq \{\neg P(x), P(a)\}$ is S unsatisfiable ?

Sol.

- $H = \{a\}$, $A = \{P(a)\}$
- Only two H-Interpretations $I_1 = \{P(a)\}$ and $I_2 = \{\neg P(a)\}$
- $I_1 \not\models S$: $C' = \neg P(a)$ ground instance of $C = \neg P(x)$ and $I_1 \not\models C'$
- $I_2 \not\models S$: $C'' = P(a)$ ground instance of $C = P(a)$ and $I_2 \not\models C''$
- Therefore S is unsatisfiable.

Example on Satisfiability

Example

Consider the clause $C = \neg P(x) \vee Q(f(x))$.
$H = \{a, f(a), f(f(a)), \cdots\}$ and
$A = \{P(a), Q(a), P(f(a)), Q(f(a)), \cdots\}$

- $I_1 = \{\neg P(a), \neg Q(a), \neg P(f(a)), \neg Q(f(a)), \cdots\}$
- $I_2 = \{P(a), Q(a), P(f(a)), Q(f(a)), \cdots\}$
- $I_3 = \{P(a), \neg Q(a), P(f(a)), \neg Q(f(a)), \cdots\}$

Then $I_1 \models C$, $I_2 \models C$ but $I_3 \not\models C$.

Note

We are assuming a pattern on the Interpretations otherwise we could not decide on satisfiability

Exercises

Exercise

- Consider the following clause $C : P(x) \vee Q(x, f(x))$
  $I : \{\neg P(a), \neg P(f(a)), \neg P(f(f(a))), \cdots$
  $\neg Q(a, a), Q(a, f(a)), \neg Q(a, f(f(a))), \cdots$
  $\neg Q(f(a), a), Q(f(a), f(a)), \neg Q(f(a), f(f(a))), \cdots\}$ Does $I \models C$? [Chang-Lee 8 page 68]
- Consider the following set of clauses $S : \{P(x), Q(f(y))\}$
  $I : \{P(a), P(f(a)), P(f(f(a))), \cdots$
  $Q(a), \neg Q(f(a)), Q(f(f(a))), \cdots\}$ Does $I \models S$?
  [Chang-Lee 9 page 68]
- Consider the following set of clauses $S : \{P(x), \neg P(f(y))\}$
  1. Give H_0, H_1, H_2 and H_3.
  2. Is it possible to find an interpretation that satisfies S ? If yes provide one. If no explain why [Chang-Lee 10 page 68].

Semantic Trees

Basic Concept

- Tree representation of a set of clauses
- Provides information on the satisfiability of the set of clauses

Example

Simple Example for Propositional Logic

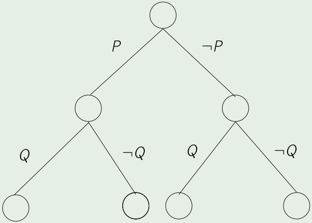

Figure: Semantic tree for $S = P \vee Q$

Semantic Trees: Definition

Definition (Semantic Tree)

Given a set of Clauses S let A be the Herbrand base (or atom set) of S a Semantic Tree for S is a tree T, where each link of the tree is annotated with a set of atoms or negation of atoms from A such that

1. property I For each node N there are only finitely many immediate links $\{L_1, \cdots, L_m\}$ from N. Let Q_i be the conjunction of all literals attached to the link L_i, then $Q_1 \vee Q_2 \vee \cdots \vee Q_n$ is a valid propositional formula.
2. property II For each node N let $I(N)$ be the union of all sets attached to the links of the branch connecting N up to the root and including N. Then $I(N)$ does not contain any complementary pair.

Complementary Pair: Definition

Definition (Complementary Pair)

If A is an atom then the two literals A and $\neg A$ are said to be each other's complement and the set $\{A, \neg A\}$ is said to be a complementary pair.

Note

A Clause that contains a complementary pair is a tautology

Example

$C = P(x) \vee Q(y, f(y)) \vee \neg R(z) \vee \neg P(x)$ is a tautology as $\{P(x), \neg P(x)\}$ is a complementary pair

Example I

Example

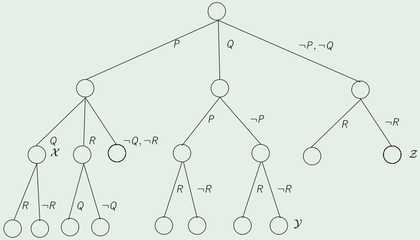

Figure: Semantic tree for the atom set $A = P, Q, R$

$I(\mathcal{X}) = \{Q, P\}$ $I(\mathcal{Y}) = \{\neg R, \neg P, Q\}$ $I(\mathcal{Z}) = \{\neg R, \neg P, \neg Q\}$
Note that for the root node we have $Q_1 = \{P\}$ and $Q_2 = Q$ and $Q_3 = \{\neg P, \neg Q\}$ therefore $Q_1 \vee Q_2 \vee Q_3$ is a valid formula.

Example II

Example

Consider the set of clauses $S = \{P(x), P(a)\}$. The atom set for this set of clauses is $A = P(a)$

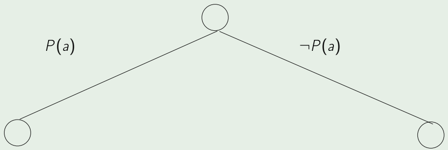

Figure: Semantic tree the atom set $A = P(a)$

Complete Semantic tree: Definition

Definition

Complete Semantic Tree Given an atom set $A = A_1, \cdots, A_k, \cdots$
A semantic tree is complete iff for every leaf node N, $I(N)$ contains A_i or $\neg A_i$ for $i = 1, 2, \cdots$

Note

All previous semantic trees were complete

Example

Example

Consider a set of cluses $S = P(f(x))$, the Herbrand Base for S is $A = \{P(a), P(f(a)), \cdots\}$ The following Semantic Tree represents S and is not complete

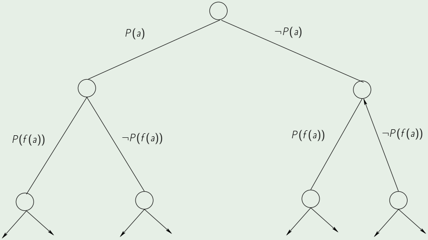

Figure: Not Complete Semantic tree

Observations on Semantic Tree

- Given a semantic tree T representing a set of clause S for each node N, $I(N)$ is a subset of an interpretation for S
- $I(N)$ is therefore a partial interpretation of S
- Given S, if A infinite then any complete semantic tree T for S is infinite

Observations on Semantic Trees and Satisfiability

We can use semantic trees to check satisfiability of S

- Given a set of clause S any complete semantic tree for S contains all possible interpretations of S.
- When expanding the semantic tree, we can stop expanding as soon as a partial interpretation falsifies S.
- If $I(N)$ falsifies S we can stop at node N.

Definition: Failure Node

Definition (Failure Node)

Given a set of clauses S and a semantic tree for S, a node N is called a failure node iff $I(N)$ falsifies some ground instances of a clause in S, but $I(N')$ does not falsify any ground instance of a clause in S for every ancestor N' of N.

Example

Consider the clause $S = \{P \vee Q, Q\}$ build a semantic tree and check which node is a failure node.

Definition: Closed Tree

Definition (Closed Semantic Tree)

A semantic tree T is said to be closed iff every branch of T terminates at a failure node.

Definition (Inference Node)

A node N of a closed semantic tree is called an inference node if all its immediate descendant nodes are failure nodes.

Closed Semantic Tree: Example I

Example

Consider the formula $S = \{P, Q \vee R, \neg P \vee \neg Q, \neg P \vee \neg R\}$
$A = \{P, Q, R\}$

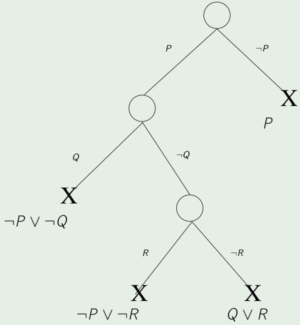

Figure: Closed Semantic tree

Closed Semantic Tree: Example II

Example

Consider the formula $S = \{\neg P(x) \vee Q(x), P(a), \neg Q(z)\}$,
$H = \{a\}$ $A = \{P(a), Q(a)\}$

Closed Semantic Tree: Example II

Example

Consider the formula $S = \{\neg P(x) \vee Q(x), P(a), \neg Q(z)\}$,
$H = \{a\}$ $A = \{P(a), Q(a)\}$

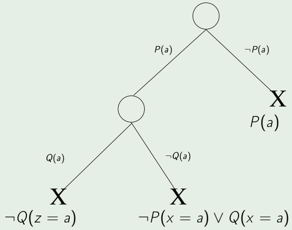

Figure: Closed Semantic tree

Closed Semantic Tree: Example III

Example

Consider the formula $S = \{\neg P(x) \vee Q(x), P(f(a)), \neg Q(z)\}$
$H = \{a, f(a), f(f(a)), \cdots\}$ $A = \{P(a), Q(a), P(f(a)), Q(f(a)), \cdots\}$

Closed Semantic Tree: Example III

Example

Consider the formula $S = \{\neg P(x) \vee Q(x), P(f(a)), \neg Q(z)\}$
$H = \{a, f(a), f(f(a)), \cdots\}$ $A = \{P(a), Q(a), P(f(a)), Q(f(a)), \cdots\}$

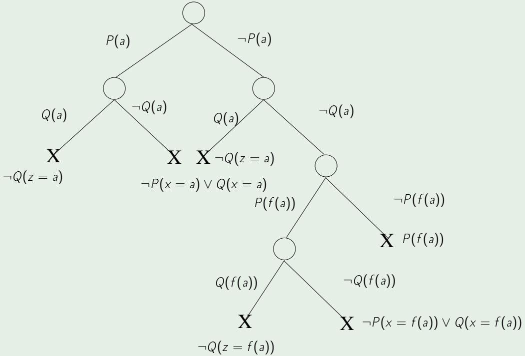

Figure: Closed Semantic tree

Exercise: Semantic Tree

Exercise

1. $S = \{P, \neg P \vee Q, \neg Q\}$ Give a closed Semantic Tree of S [Chang-Lee Ex 11, page 68]
2. $S = \{P(x), \neg P(x) \vee Q(x, a), \neg Q(y, a)\}$ [Chang-Lee Ex 12, page 68]
   - Give the atom set of S
   - Give a complete Semantic Tree of S
   - Givw a closed Semantic Tree of S