DPLL method

Summary

- DPLL procedure
- DPLL example

DPLL procedure: Intro

Basic facts

- Pobably the most efficient procedure for checking satisfiability of ground clauses
- Input: a set of ground clauses
- Output: decides whether the set of clauses is unsatisfiable or not
  - if the set of clauses is satisfiable the algorithm outputs an assignment that satisfies the set of clauses

DPLL procedure: history

History

- DPLL = Davis-Putnam-Logemann-Loveland.
- Introduced in two article in 1960.
- Impressive engineering work to make it efficient over the years.

The SAT problem

SAT

- Given a propositional formula find an assignment that satisfies the formula or show that such an assignment does not exists
- SAT is decidable, but computationally untractable: SAT is NP-Complete
  - NP-Complete: any NP problems can be reduced to SAT in polynomial time: e.g. Travelling Salesman Problem.
- Since SAT is NP-Complete it is very important to find efficient procedures
- Many applications to practical problems: e.g. hardware verification.

DPLL procedure: general concepts

general concepts

- Start from a ground CNF formula
- Try to build an assignment that verifies the formula
- The assignment is built using a backtracking mechanism

DPLL procedure: simple sketch

Simple sketch

- A tree of possible assignments is used to guide the procedure
  - Each node is a set of clauses S_i
  - At each node one of the Literal is assigned a truth value
  - Truth values are propagated to reduce the number of future assignments

DPLL procedure: Algorithm

Algorithm

- Input: $S = C_0 = \{C_1, \cdots, C_k\}$ where $C_i = L_1 \vee L_2 \vee \cdots \vee L_{r_i}$.
- Set C_0 as the root of the tree.
- Apply (inference) rules to leaves, expanding the tree.
- A branch of the tree is no longer expanded if $S_i = \{\,\}$ or $\square \in S_i$ where $\square$ is the empty clause.
- If $S_i = \{\,\}$ then S is satisfiable and we can stop the procedure.
- If $\square \in S_i$ for all branches then the set is unsatisfiable.

DPLL procedure: Applying rules

Applying Rules

- We apply a given set of rules that preserve satisfiability.
- When we apply a rule, we build at the same time a partial interpretation for S.
- Rules:
  1. Tautology elimination
  2. One-Literal
  3. Pure Literal
  4. Splitting

Tautology Elimination

Tautology Elimination

Delete all the ground clauses from S that are tautologies. The remaining set S' is unsatisfiable iff S is unsatisfiable.

Example (Tautology elimination)

$S = \{(\neg P \vee Q \vee P \vee \neg R) \wedge Q \wedge R\}$
$S' = \{Q \wedge R\}$

One-Literal

One-Literal (Unit clause)

If there is a unit ground clause $L \in S$, obtain S' from S by deleting those ground clauses in S containing L. If $S' = \{\,\}$ then S is satisfiable Otherwise obtain a set S'' from S' by deleting $\neg L$ from all clauses. S is unsatisfiable iff S'' is.unsatisfiable. When we apply this rule we fix $L = \top$ in the partial assignment.

Example (One-Literal)

$S = \{P \vee Q \vee \neg R, P \vee \neg Q, \neg P, R, U\}$
Apply One-Literal rule with $L = \neg P$:
$S' = \{P \vee Q \vee \neg R, P \vee \neg Q, R, U\}$
Remove $\neg L = P$ from cluases in S' $S'' = \{Q \vee \neg R, \neg Q, R, U\}$
We fix $L = \top$ therefore $P = \bot$

Pure-Literal

Definition (Pure Literal)

$L \in S$ is a pure literal iff $\neg L \notin S$

Pure-Literal Rule

If there is a pure literal $L \in S$, obtain S' from S by deleting all clauses where L appears. S' is unsatisfiable iff S' is unsatisfiable. When we apply this rule we fix $L = \top$ in the partial assignment.

Example (Pure-Literal)

$S = \{P \vee Q, P \vee \neg Q, R \vee Q, R \vee \neg Q\}$
Apply Pure-Literal rule with $L = P$:
$S' = \{R \vee Q, R \vee \neg Q\}$
We fix $L = \top$ therefore $P = \top$

Splitting

Splitting Rule

If we can write S in the following form:

$$S = (C_1 \vee L) \wedge \cdots \wedge (C_m \vee L) \wedge (D_1 \vee \neg L) \wedge \cdots \wedge (D_m \vee \neg L) \wedge S_r$$

Where C_i and D_i are clauses in which L and $\neg L$ do not appear, and where S_r is a set of clauses where L and $\neg L$ do not appear. Then we can obtain two sets $S' = C_1 \wedge \cdots \wedge C_m \wedge S_r$ and $S'' = D_1 \wedge \cdots \wedge D_m \wedge S_r$. The set S is unsatisfiable iff S' and S'' are unsatisfiable. When we apply this rule we split the tree and we fix $L = \top$ for the branch of S'' and $L = \bot$ in the branch of S'.

Splitting:Example

Example (Splitting)

$S = \{P \vee \neg Q \vee R, \neg P \vee Q, Q \vee \neg R, \neg Q \vee \neg R\}$

Apply Splitting on P

$S' = \{\neg Q \vee R, Q \vee \neg R, \neg Q \vee \neg R\},\ P = \bot$

$S'' = \{Q, Q \vee \neg R, \neg Q \vee \neg R\},\ P = \top$

DPLL Example I

Example (Example I)

$$S = (P \vee Q \vee \neg R) \wedge (P \vee \neg Q) \wedge \neg P \wedge R \wedge U$$

$\{\}$	$(P \vee Q \vee \neg R) \wedge (P \wedge \neg Q) \wedge \neg P \wedge R \wedge U$	One-Literal on $\neg P$
$\{\neg P\}$	$(Q \vee \neg R) \wedge \neg Q \wedge R \wedge U$	One-Literal on $\neg Q$
$\{\neg P, \neg Q\}$	$\neg R \wedge R \wedge U$	One-Literal on R
$\{\neg P, \neg Q\}$	$\neg R \wedge R \wedge U$	One-Literal on R
$\{\neg P, \neg Q, R\}$	$\Box \wedge U$	unsatisfiable

DPLL Example II

Example (Example II)

$$S = (P \vee Q) \wedge \neg Q \wedge (\neg P \vee Q \vee \neg R)$$

$\{\}$	$S = (P \vee Q) \wedge \neg Q \wedge (\neg P \vee Q \vee \neg R)$	One-Literal on $\neg Q$
$\{\neg Q\}$	$P \wedge (\neg P \vee \neg R)$	One-Literal on P
$\{\neg Q, P\}$	$\neg R$	One-Literal on $\neg R$
$\{\neg Q, P, \neg R\}$	$\{\}$	Satisfiable

Soundness of DPLL rules

Soundness

- DPLL rules must be sound
- If the original set S is unsatisfiable then the remaining set S' after applying the rule is still unsatisfiable and viceversa.

Soundness of Tautology Elimination

Need to show that S' is unsatisfiable iff S is unsatisfiable. Since a tautology is satisfied by every interpretation, S' is unsatisfiable iff S is unsatisfiable.

Soundness of One-Literal I

Soundness of One-Literal I

Recall that:

- S' obtained by removing all clauses containing L from S.
- S'' obtained by removing $\neg L$ from clauses in S'.

Need to show that:

- If $S' = \{\,\}$ then S is satisfiable.
- If S' is empty then any clause in S contains L therefore any interpretation containing L satisfies S

Soundness of One-Literal II

Soundness of One-Literal II

Recall that:

- S' obtained by removing all clauses containing L from S.
- S'' obtained by removing $\neg L$ from S'.

Need to show that:

- S'' is unstaisfiable iff S is unsatisfiable.
- $\Rightarrow$ Suppose S'' is unsatisfiable and S is satisfiable, then there is I such that $L \in I$ and $I \models S$. I must satisfy all clauses which are not in S''; since I falsifies $\neg L$ then I must satisfy all clauses in S that contains $\neg L$. Then $I \models S''$.
- $\Leftarrow$ Suppose S is unsatisfiable and S'' is satisfiable, Then there is $I \models S''$. Since S'' does not contain L and $\neg L$, we can build I' by adding L to I, $I' \models S$.

Soundness of Pure-Literal

Soundness of Pure Literal

Recall that:

- S' obtained by removing all clauses containing L from S.

Need to show that:

- S' unsatisfiable iff S is unsatisfiable.
- $\Rightarrow$ suppose S' unsat. and S sat. Then there is $I \models S$, since $S' \subseteq S$ then $I \models S'$
- $\Leftarrow$ suppose S unsat and S' sat. Then there is $I \models S'$, since $L \notin S'$ and $\neg L \notin S'$ we can build I' adding L to I, and $I' \models S$.

Soundness of Splitting

Soundness of Splitting

Recall that:

- $S = (C_1 \vee L) \wedge \cdots \wedge (C_m \vee L) \wedge (D_1 \vee \neg L) \wedge \cdots \wedge (D_m \vee \neg L) \wedge S_r$
- $S' = C_1 \wedge \cdots \wedge C_m \wedge S_r$
- $S'' = D_1 \wedge \cdots \wedge D_m \wedge S_r$

Need to show that:

- S is unsatisfiable iff S' and S'' are unsatisfiable.
- $\Rightarrow$ suppose S unsat and S' or S'' are sat. If $S'(S'')$ has a model I then for any interpretation $I' = \neg L(L) \cup I$ we have $I' \models S$.
- $\Leftarrow$ suppose S' and S'' are unsat. and S is sat. Then there is $I \models S$. If I contains $\neg L(L)$ then $I \models S'$ $(I \models S'')$ therefore either S' or S'' is satisfiable.

DPLL Example III

Example (Example III)

$$S = (P \vee Q) \wedge (P \vee \neg Q) \wedge (\neg P \vee Q) \wedge (\neg P \vee \neg R)$$

$\{\,\}$	$(P \vee Q) \wedge (P \wedge \neg Q) \wedge (\neg P \vee Q) \wedge (\neg P \vee \neg R)$	Split on P

DPLL Example III

Example (Example III)

$$S = (P \vee Q) \wedge (P \vee \neg Q) \wedge (\neg P \vee Q) \wedge (\neg P \vee \neg R)$$

$\{\}$	$(P \vee Q) \wedge (P \wedge \neg Q) \wedge (\neg P \vee Q) \wedge (\neg P \vee \neg R)$	Split on P
S' $\{\neg P\}$	$Q \wedge \neg Q$	One-Literal on Q

DPLL Example III

Example (Example III)

$$S = (P \vee Q) \wedge (P \vee \neg Q) \wedge (\neg P \vee Q) \wedge (\neg P \vee \neg R)$$

$\{\}$	$(P \vee Q) \wedge (P \wedge \neg Q) \wedge (\neg P \vee Q) \wedge (\neg P \vee \neg R)$	Split on P
S' $\{\neg P\}$	$Q \wedge \neg Q$	One-Literal on Q
S' $\{\neg P, Q\}$	$\Box$	S' Unsat

DPLL Example III

Example (Example III)

$$S = (P \vee Q) \wedge (P \vee \neg Q) \wedge (\neg P \vee Q) \wedge (\neg P \vee \neg R)$$

$\{\}$	$(P \vee Q) \wedge (P \wedge \neg Q) \wedge (\neg P \vee Q) \wedge (\neg P \vee \neg R)$	Split on P
S' $\{\neg P\}$	$Q \wedge \neg Q$	One-Literal on Q
S' $\{\neg P, Q\}$	$\square$	S' Unsat
	Backtrack	

DPLL Example III

Example (Example III)

$$S = (P \vee Q) \wedge (P \vee \neg Q) \wedge (\neg P \vee Q) \wedge (\neg P \vee \neg R)$$

$\{\,\}$	$(P \vee Q) \wedge (P \wedge \neg Q) \wedge (\neg P \vee Q) \wedge (\neg P \vee \neg R)$	Split on P
S' $\{\neg P\}$	$Q \wedge \neg Q$	One-Literal on Q
S' $\{\neg P, Q\}$	$\square$	S' Unsat

DPLL Example III

Example (Example III)

$S = (P \vee Q) \wedge (P \vee \neg Q) \wedge (\neg P \vee Q) \wedge (\neg P \vee \neg R)$

$\{\}$	$(P \vee Q) \wedge (P \wedge \neg Q) \wedge (\neg P \vee Q) \wedge (\neg P \vee \neg R)$	Split on P
S' $\{\neg P\}$	$Q \wedge \neg Q$	One-Literal on Q

DPLL Example III

Example (Example III)

$$S = (P \vee Q) \wedge (P \vee \neg Q) \wedge (\neg P \vee Q) \wedge (\neg P \vee \neg R)$$

$\{\}$	$(P \vee Q) \wedge (P \wedge \neg Q) \wedge (\neg P \vee Q) \wedge (\neg P \vee \neg R)$	Split on P

DPLL Example III

Example (Example III)

$$S = (P \lor Q) \land (P \lor \neg Q) \land (\neg P \lor Q) \land (\neg P \lor \neg R)$$

$\{\}$	$(P \lor Q) \land (P \land \neg Q) \land (\neg P \lor Q) \land (\neg P \lor \neg R)$	Split on P

DPLL Example III

Example (Example III)

$$S = (P \vee Q) \wedge (P \vee \neg Q) \wedge (\neg P \vee Q) \wedge (\neg P \vee \neg R)$$

$\{\}$	$(P \vee Q) \wedge (P \wedge \neg Q) \wedge (\neg P \vee Q) \wedge (\neg P \vee \neg R)$	Split on P
S'' $\{P\}$	$Q \wedge \neg R$	One-Literal on Q

DPLL Example III

Example (Example III)

$$S = (P \vee Q) \wedge (P \vee \neg Q) \wedge (\neg P \vee Q) \wedge (\neg P \vee \neg R)$$

$\{\,\}$	$(P \vee Q) \wedge (P \wedge \neg Q) \wedge (\neg P \vee Q) \wedge (\neg P \vee \neg R)$	Split on P
S'' $\{P\}$	$Q \wedge \neg R$	One-Literal on Q
S'' $\{P, Q\}$	$\neg R$	One-Literal on $\neg R$

DPLL Example III

Example (Example III)

$$S = (P \vee Q) \wedge (P \vee \neg Q) \wedge (\neg P \vee Q) \wedge (\neg P \vee \neg R)$$

$\{\}$	$(P \vee Q) \wedge (P \wedge \neg Q) \wedge (\neg P \vee Q) \wedge (\neg P \vee \neg R)$	Split on P
S'' $\{P\}$	$Q \wedge \neg R$	One-Literal on Q
S'' $\{P, Q\}$	$\neg R$	One-Literal on $\neg R$
S'' $\{P, Q, \neg R\}$	$\{\}$	Satisfiable!

DPLL Example IV: using splitting first

Example (Example IV: using splitting first)

$S = (P \vee \neg Q) \wedge (\neg P \vee R) \wedge Q$

$\{\}$	$(P \vee \neg Q) \wedge (\neg P \vee R) \wedge Q$	Split on P

DPLL Example IV: using splitting first

Example (Example IV: using splitting first)

$S = (P \vee \neg Q) \wedge (\neg P \vee R) \wedge Q$

$\{\}$	$(P \vee \neg Q) \wedge (\neg P \vee R) \wedge Q$	Split on P
S' $\{\neg P\}$	$\neg Q \wedge Q$	One-Literal on Q

DPLL Example IV: using splitting first

Example (Example IV: using splitting first)

$S = (P \vee \neg Q) \wedge (\neg P \vee R) \wedge Q$

$\{\}$	$(P \vee \neg Q) \wedge (\neg P \vee R) \wedge Q$	Split on P
S' $\{\neg P\}$	$\neg Q \wedge Q$	One-Literal on Q
S' $\{\neg P, Q\}$	$\square$	S' Unsat

DPLL Example IV: using splitting first

Example (Example IV: using splitting first)

$S = (P \vee \neg Q) \wedge (\neg P \vee R) \wedge Q$

$\{\}$	$(P \vee \neg Q) \wedge (\neg P \vee R) \wedge Q$	Split on P
S' $\{\neg P\}$	$\neg Q \wedge Q$	One-Literal on Q
S' $\{\neg P, Q\}$	$\square$	S' Unsat
	Backtrack	

DPLL Example IV: using splitting first

Example (Example IV: using splitting first)

$S = (P \vee \neg Q) \wedge (\neg P \vee R) \wedge Q$

$\{\}$	$(P \vee \neg Q) \wedge (\neg P \vee R) \wedge Q$	Split on P
S' $\{\neg P\}$	$\neg Q \wedge Q$	One-Literal on Q
S' $\{\neg P, Q\}$	$\square$	S' Unsat

DPLL Example IV: using splitting first

Example (Example IV: using splitting first)

$S = (P \vee \neg Q) \wedge (\neg P \vee R) \wedge Q$

$\{\}$	$(P \vee \neg Q) \wedge (\neg P \vee R) \wedge Q$	Split on P
S' $\{\neg P\}$	$\neg Q \wedge Q$	One-Literal on Q

DPLL Example IV: using splitting first

Example (Example IV: using splitting first)

$S = (P \vee \neg Q) \wedge (\neg P \vee R) \wedge Q$

$\{\}$	$(P \vee \neg Q) \wedge (\neg P \vee R) \wedge Q$	Split on P

DPLL Example IV: using splitting first

Example (Example IV: using splitting first)

$S = (P \vee \neg Q) \wedge (\neg P \vee R) \wedge Q$

$\{\}$	$(P \vee \neg Q) \wedge (\neg P \vee R) \wedge Q$	Split on P

DPLL Example IV: using splitting first

Example (Example IV: using splitting first)

$S = (P \vee \neg Q) \wedge (\neg P \vee R) \wedge Q$

$\{\}$	$(P \vee \neg Q) \wedge (\neg P \vee R) \wedge Q$	Split on P
S'' $\{P\}$	$R \wedge Q$	One-Literal on Q

DPLL Example IV: using splitting first

Example (Example IV: using splitting first)

$S = (P \vee \neg Q) \wedge (\neg P \vee R) \wedge Q$

$\{\}$	$(P \vee \neg Q) \wedge (\neg P \vee R) \wedge Q$	Split on P
S'' $\{P\}$	$R \wedge Q$	One-Literal on Q
S'' $\{P, Q\}$	R	One-Literal on R

DPLL Example IV: using splitting first

Example (Example IV: using splitting first)

$S = (P \vee \neg Q) \wedge (\neg P \vee R) \wedge Q$

$\{\}$	$(P \vee \neg Q) \wedge (\neg P \vee R) \wedge Q$	Split on P
S'' $\{P\}$	$R \wedge Q$	One-Literal on Q
S'' $\{P, Q\}$	R	One-Literal on R
S'' $\{P, Q, R\}$	$\{\}$	Satisfiable!

DPLL Example IV: using splitting last

Example (Example IV: usin splitting last)

$S = (P \vee \neg Q) \wedge (\neg P \vee R) \wedge Q$

$\{\}$	$(P \vee \neg Q) \wedge (\neg P \vee R) \wedge Q$	One-Literal on Q

DPLL Example IV: using splitting last

Example (Example IV: usin splitting last)

$S = (P \vee \neg Q) \wedge (\neg P \vee R) \wedge Q$

$\{\}$	$(P \vee \neg Q) \wedge (\neg P \vee R) \wedge Q$	One-Literal on Q
$\{Q\}$	$P \wedge (\neg P \vee R)$	Pure-Literal on R

DPLL Example IV: using splitting last

Example (Example IV: usin splitting last)

$S = (P \vee \neg Q) \wedge (\neg P \vee R) \wedge Q$

$\{\,\}$	$(P \vee \neg Q) \wedge (\neg P \vee R) \wedge Q$	One-Literal on Q
$\{Q\}$	$P \wedge (\neg P \vee R)$	Pure-Literal on R
$\{Q, R, \}$	P	One-Literal on P

DPLL Example IV: using splitting last

Example (Example IV: usin splitting last)

$S = (P \vee \neg Q) \wedge (\neg P \vee R) \wedge Q$

$\{\}$	$(P \vee \neg Q) \wedge (\neg P \vee R) \wedge Q$	One-Literal on Q
$\{Q\}$	$P \wedge (\neg P \vee R)$	Pure-Literal on R
$\{Q, R, \}$	P	One-Literal on P
$\{Q, R, P\}$	$\{\}$	Satisfiable!

Exercise on DPLL I

Exercise

- Decide whether the following set of clauses are satisfiable using DPLL
  1. $p \vee q, \neg p \vee q, \neg r \vee \neg q, r \vee \neg q$
  2. $p \vee q \vee r,\ \neg p \vee \neg q \vee \neg r,\ \neg p \vee q \vee r,\ \neg q \vee r,\ q \vee \neg r$
  3. $\neg q \vee p,\ \neg p \vee \neg q,\ q \vee r,\ \neg q \vee \neg r,\ \neg p \vee \neg r,\ p \vee \neg r$

Exercise on DPLL II

Exercise

- Solve the Quack and Doctors problem using Gilmore + DPLL
  1. $F_1 \triangleq$ Some patients like all doctors
     - $(\exists x)(P(x) \wedge (\forall y)(D(y) \rightarrow L(x,y)))$
  2. $F_2 \triangleq$ No patient likes any quack:
     - $(\forall x)(P(x) \rightarrow (\forall y)(Q(y) \rightarrow \neg L(x,y)))$
  3. $F_3 \triangleq$ No doctor is a quack:
     - $(\forall x)(D(x) \rightarrow \neg Q(x))$
  4. $F_1 \wedge F_2 \models F_3$?