Written test

1 Exercise 1

Decide, using DPLL, whether the following set of clauses are satisfiable, and if they are, provide a model

1. $\neg a \vee \neg b \vee x,\ \neg x \vee a,\ \neg x \vee b,\ \neg x \vee z,\ \neg b \vee z,\ \neg z \vee x \vee b$
2. $p \vee q,\ \neg p \vee \neg q,\ p \vee s,\ \neg p \vee \neg s,\ q \vee s,\ \neg q \vee \neg s$

Solution Exercise 1

question 1

$\neg a \vee \neg b \vee x,\ \neg x \vee a,\ \neg x \vee b,\ \neg x \vee z,\ \neg b \vee z,\ \neg z \vee x \vee b$
Split on b
Consider the branch were $b = \top$ first
$\{b\}\ \neg a \vee x,\ \neg x \vee a,\ \neg x \vee z,\ z$
One literal on z
$\{b, z\}\ \neg a \vee x,\ \neg x \vee a$
Split on a
Consider the branch were $a = \top$ first
$\{b, z, a\}\ x$
One literal on x
$\{b, z, a, x\}\ \{\}$
The set of clauses is satisfiable and $\{b, z, a, x\}$ is a model.

question 2

$p \vee q,\ \neg p \vee \neg q,\ p \vee s,\ \neg p \vee \neg s,\ q \vee s,\ \neg q \vee \neg s$ Split on p
Consider the branch were $p = \top$ first
$\{p\}\ \neg q,\ \neg s,\ q \vee s,\ \neg q \vee \neg s$ One literal on $\{\neg q\}$
$\{p, \neg q\}\ \neg s,\ s$ One literal on $\{\neg s\}$
$\square$
Consider the branch were $p = \bot$
$\{\neg p\}\ q,\ s,\ q \vee s,\ \neg q \vee \neg s$
One literal on $\{q\}$
$\{p, q\}\ s,\ \neg s$
One literal on $\{s\}$
$\square$
The set of clauses is unsatisfiable.

2 Exercise 2

Prove the unsatisfiability of the following set of ordered clauses by Linear Ordered Resolution

1. $\neg D(x) \vee P(x)$
2. $\neg L(x, y) \vee \neg C(y) \vee D(x)$
3. $L(x, y) \vee D(x)$
4. $C(a)$
5. $\neg P(x)$

Use the last clause as top clause.

Solution Exercise 2

$\neg P(x)$

1) Resolve with $\neg D(z) \vee P(z)$ with $\sigma = \{x/z\}$ and obtain:

$\underline{\neg P(x)} \vee \neg D(x)$

We have two branches:

first branch:

2a) Resolve with $\neg L(z, k) \vee \neg C(k) \vee D(z)$ with $\sigma = \{x/z\}$ and obtain:

$\underline{\neg P(x)} \vee \underline{\neg D(x)} \vee \neg L(x, k) \vee \neg C(k)$

3a) Resolve with $C(a)$ with $\sigma = \{a/k\}$ and obtain:

$\underline{\neg P(x)} \vee \underline{\neg D(x)} \vee \neg L(x, a)$

4a) Resolve with $L(z, k) \vee D(z)$ with $\sigma = \{x/z, a/k\}$ and obtain:

$\underline{\neg P(x)} \vee \underline{\neg D(x)} \vee \underline{\neg L(x, a)} \vee D(x)$

Reduce this by observing that $D(x)$ resolve with $\underline{\neg D(x)}$ and obtain:

$\underline{\neg P(x)} \vee \underline{\neg D(x)} \vee \underline{\neg L(x, a)}$

Removing underlined rightmost literals obatin:

$\square$

second branch:

2b) Resolve with $L(z, k) \vee D(z)$ with $\sigma = \{x/z\}$ and obtain:

$\underline{\neg P(x)} \vee \underline{\neg D(x)} \vee L(x, k)$

3b) Resolve with $\neg L(z, y) \vee \neg C(y) \vee D(z)$ with $\sigma = \{x/z, k/y\}$ and obtain:

$\underline{\neg P(x)} \vee \underline{\neg D(x)} \vee \underline{L(x, k)} \vee \neg C(k) \vee D(x)$

Reduce this by observing that $D(x)$ resolve with $\underline{\neg D(x)}$ and obtain:

$\underline{\neg P(x)} \vee \underline{\neg D(x)} \vee \underline{L(x, k)} \vee \neg C(k)$

4b) Resolve with $C(a)$ with $\sigma = \{a/k\}$ and obtain:

$\underline{\neg P(x)} \vee \underline{\neg D(x)} \vee \underline{L(x, a)}$

Removing underlined rightmost literals obatin:

$\square$

3 Exercise 3

Given the constraint network $\mathcal{N}$:

- Variables: {X,Y,Z,W}, Domain $D_x = D_y = D_z = \{0, 1\}$ and $D_w = \{0, 1, 2\}$
- Constraints: $X > Y$, $X < W$, $X < Z$, $Y = Z$

describe an execution of AC-3. Is the resulting network arc consistent ? Is the resulting network consistent ? Motivate your answers.

Solution Exercise 3

Initialise the Queue: $Q = \{(X, W), (W, X), (X, Y), (Y, X), (Y, Z), (Z, Y), (X, Z), (Z, X)\}$
Process each couple of nodes in the queue using the revise procedure and put back in the queue couple of nodes according to domain modifications.

1. REV(X,W): no changes
2. REV(W,X): delete 0 from D_W, $D_W = \{1, 2\}$, W has no other constraints so no need to append anything to the Queue
3. REV(X,Y): delete 0 from D_X, $D_X = \{1\}$, append (W,X) to the Queue
4. REV(Y,X): delete 1 from D_Y, $D_Y = \{0\}$, should append (Z,Y) to the Queue but already present
5. REV(Y,Z): no changes
6. REV(Z,Y): delete 1 from D_Z, $D_Z = \{0\}$, should append (X,Z) to the Queue but already present
7. REV(X,Z): delete 1 from D_X, $D_X = \emptyset$
8. We can STOP, because being $D_X = \emptyset$ we can already claim the network is not arc consistent.

The resulting network is not arc consistent because D_X is empty. The resulting network is not consistent because it is not arc consistent and arc consistency is a necessary condition for consistency (although not sufficient).

4 Exercise 4 (25) [15]

Given the soft graph coloring problem below, apply the Branch and Bound algorithm to find an optimal solution. Use ordering $d = \{x_1, x_2, x_3, x_4\}$.

- Variables: $\{x_1, x_2, x_3, x_4\}$, Domain $D_i = \{0, 1\}$
- Constraints: $R_{1,2}, R_{1,3}, R_{2,3}, R_{2,4}$
- $F_{ij} : \{< (1, 1), -1 >, < (0, 1), 0 >, < (1, 0), 0 >, < (0, 0), -2 >\}$

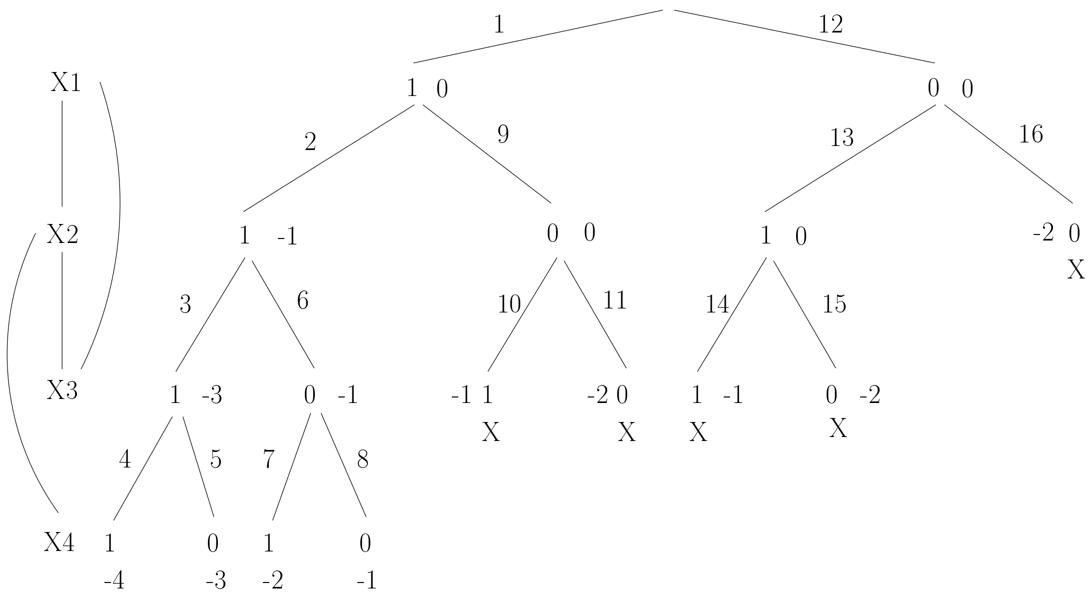

Figure 1: Execution of the branch and bound algorithm

Solution Exercise 4

We use the first choice function as the bounding evaluation function f. The execution of the branch and bound algorithm is explained in Figure 1 (we arbitrarily expand always 1 first). The optimal solution found by the algorithm is $x_1 = 1, x_2 = 1, x_3 = 0, x_4 = 0$ with a value of -1 The upper bound is initialised to $-\infty$ and updated to -4 at time step 4, then to -3 at time step 5, to -2 at time step 7 and finally to -1 at time step 8. Whenever the value of the bounding evaluation function is lower or equal to -1 the algorithm stops and back tracks.

5 Exercise 5

Given a combinatorial auction and the following set of bids:

- $B_1 = \{1, 2, 3\}, 5,\ B_2 = \{1, 4, 6\}, 4,\ B_3 = \{4, 5\}, 1,\ B_4 = \{2, 5\}, 3$

Answer to the following questions:

1. Provide a cost network formalisation of the winner determination problem, specifying the variables, their domain and the constraints (hard and soft).

2. Is the resulting cost network acyclic ? Motivate your answer

3. Solve the cost network with a solution technique of your choice.

Solution Exercise 5

question 1

A possible cost network formalisation of the winner determination problem for the above comabinatorial auction is the following:

- Variables: $\{v_1, v_2, v_3, v_4\}$ where $D_i = \{0, 1\}$
- Hard Constraints: $R_{1,2}$, $R_{1,4}$, $R_{2,3}$, $R_{3,4}$, where $R_{i,j} = \{(0,0),(1,0),(0,1)\}$
- Soft Constraints: $r_1(v_1) = \{< (0), 0 >, < (1), 5 >\}$, $r_2(v_2) = \{< (0), 0 >, < (1), 4 >\}$, $r_3(v_3) = \{< (0), 0 >, < (1), 1 >\}$, $r_4(v_4) = \{< (0), 0 >, < (1), 3 >\}$

Where $v_i = 1$ indicates that bid B_i was selected and $v_i = 0$ indicates that bid B_i was not selected. Hard constraints indicate that bids that share an item can not be both selected, and soft constraints indicates that when bid B_i is selected the auctioneer gains the value of that bid. The primal graph of the cost network is reported in figure 2

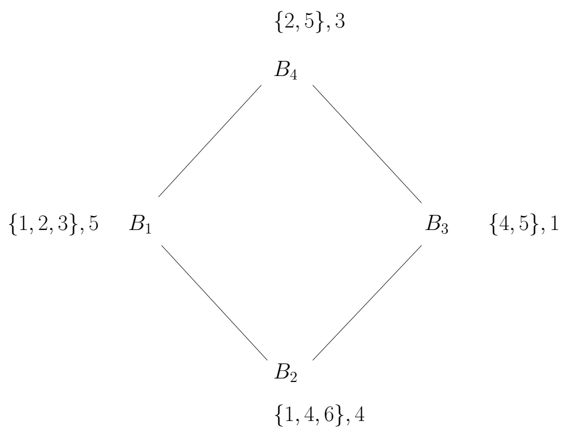

Figure 2: Primal graph for the winner determination problem

question 2

The network is not acyclic. One way to show this is that the primal graph of the cost network is not chordal. This is because there is a cycle of length four (v_1, v_2, v_3, v_4) without chords (i.e., there is no edge between (v_1, v_3) and (v_2, v_4)).

question 3

We choose bucket elimination and we choose the variable ordering $d = v_1, v_2, v_3, v_4$. The first phase of the Bucket Elimination algorithm is then to allocate functions to buckets, going from last variable to first in the ordering d. We then process each bucket by maximising out the variable of the bucket from the sum of functions that the bucket contains and allocate new created function in lower buckets:

- Bucket 4, $v_4 : R_{1,4}, R_{3,4};\ r_4(v_4) \rightarrow H^4(v_1, v_3) = \{< (0,0), 3, v_4 = 1 >, < (0,1), 0, v_4 = 0 >, < (1,0), 0, v_4 = 0 >, < (1,1), 0, v_4 = 0 >\}$
- Bucket 3, $v_3 : R_{2,3},\ r_3(v_3) + H^4(v_1, v_3) \rightarrow H^3(v_1, v_2) = \{< (0,0), 3, v_3 = 0 >, < (0,1), 3, v_3 = 0 >, < (1,0), 1, v_3 = 1 >, < (1,1), 0, v_3 = 0 >\}$
- Bucket 2, $v_2 : R_{1,2},\ r_2(v_2) + H^3(v_1, v_2) \rightarrow H^2(v_1) = \{< (0), 7, v_2 = 1 >, < (1), 1, v_2 = 0 >\}$
- Bucket 1, $v_1 : r_1(v_1) + H^2(v_1) \rightarrow M = \max_{v_1} H(v_1)$ where $H(v_1) = \{< (0), 7 >, < (1), 6 >\}$

We then need to propagate the values which maximise the sum of functions in each bucket up:

- Bucket 1, $v_1 : v_1^* = \arg\max_{v_1}[r_1(v_1) + H^2(v_1)]$ gives $v_1^* = 0$
- Bucket 2, $v_2 : v_2^* = \arg\max_{v_2}[r_2(v_2) + H^3(v_1^* = 0, v_2)]$ subject to $R'_{1,2} = \{(v_1^* = 0, 0), (v_1^* = 0, 1)\}$ gives $v^*2 = 1$
- Bucket 3, $v_3 : v_3^* = \arg\max_{v_3}[r_3(v_3) + H^4(v_1^* = 0, v_3)]$ subject to $R'_{2,3} = \{(v_2^* = 1, 0)\}$ gives $v^*3 = 0$
- Bucket 4, $v_4 : v_4^* = \arg\max_{v_4}[r_4(v_4)]$ subject to $R'_{1,4} = (v_1^* = 0, 0), (v_1^* = 0, 1), R'_{3,4} = (v_3^* = 0, 0), (v_3^* = 0, 1)$ gives $v^*4 = 1$

The optimal solution is then $\{v_1 = v_1^* = 0, v_2 = v_2^* = 1, v_3 = v_3^* = 0, v_4 = v_4^* = 1\}$ which is equivalent to selecting bids B_2 and B_4 and gives a revenue of 7