Quantum Turing Machines

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Outline

1 Introduction

2 The Quantum Turing Machine: Formal Description

3 The Computational Power of QTM

4 Equivalence Results on Quantum Computational Models

5 Bibliography

General

- The Quantum Turing Machine was defined by David Deutsch (1985) as precise model of a quantum physical computer.
- There are two ways of thinking about QTM:
  - the quantum physical analogue of a Probabilistic Turing Machine;
  - computation as transformation in a space of complex superposition of configuration.

The QTM is the most general model for a computing device based on quantum physic.

Basic Definitions

Computable Numbers

Definition

1. A real number $x \in \mathbb{R}$ is *computable* iff there is a Deterministic Turing Machine which on input 1^n computes a binary representation of an integer $m \in \mathbb{Z}$ such that $|m/2^n - x| \leq 1/2^n$. Let $\tilde{\mathbb{R}}$ be the set of computable real numbers.
2. A real number $x \in \mathbb{R}$ is *polynomial-time computable* iff there is a Deterministic Polytime Turing Machine which on input 1^n computes a binary representation of an integer $m \in \mathbb{Z}$ such that $|m/2^n - x| \leq 1/2^n$. Let $\mathbf{P}\mathbb{R}$ be the set of polynomial time real numbers.

Basic Definitions

Definition

1. A complex number $z = x + iy$ is *computable* iff $x, y \in \tilde{\mathbb{R}}$. Let $\tilde{\mathbb{C}}$ be the set of computable complex numbers.
2. complex number $z = x + iy$ is *polynomial-time computable* iff $x, y \in \mathbf{P}\mathbb{R}$. Let $\mathbf{P}\mathbb{C}$ be the set of polynomial time computable complex numbers.
3. a normalized vector ϕ in any Hilbert space $\ell^2(\mathcal{S})$ is *computable* (*polynomial computable*) if the range of ϕ (a function from $\mathcal{S}$ to complex numbers) is $\tilde{\mathbb{C}}$ ($\mathbf{P}\mathbb{C}$).

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QTM: General

Definition

A QTM is a triplet (Σ, Q, δ), where:

- Σ is an alphabet (with an identified symbol $\#$);
- Q is a set of states (with q_0 initial state, q_f final state, $q_0 \neq q_f$);
- δ is the transition function

$$\delta : Q \times \Sigma \rightarrow \tilde{\mathbb{C}}^{Q \times \Sigma \times \{L,R\}}$$

- The QTM has a two-way infinite tape of cells indexed by $\mathbb{Z}$ and a read-write tape head that moves along the tapes.

QTM: The Time Evolution Operator

Let M be a QTM and let S be the inner product space of finite complex linear combination of M with the Euclidean norm. We call each element $\phi \in S$ a *superposition* of M.

Definition

A QTM M defines a linear operator $U_M{:}S \rightarrow U_M$ called the time evolution operator in the following way: if if M starts in configuration C with current state p and scanned symbol σ, then after one step M will be in a superposition $\psi = \sum_i \alpha_i c_i$, where each non-zero α_i correspond a transition $\delta(p, \sigma, \tau, q, d)$, and c_i is the new configuration obtained by applying the transition to c.

QTM: The Time Evolution Operator

Note:

- the set **C** of configurations of M is an orthonormal basis for S;
- each superposition $\psi \in S$ can be represented as a vector of complex numbers indexed by configurations;
- U_M can be represented as a square matrix with columns and rows indexed by configurations where the matrix element from a column c and a row c' gives the amplitudes with which configuration c leads to configuration c' in a single step of M;

Definition

We say that a QTM M is *well-formed* if the time evolution operator U_M is unitary

QTM: computation as unitary transformation

- QTM is obviously reversible.
- An efficient QTM implements any given unitary transformation, approximating it by a product of simple unitary transformations.
- The "'super-power' of quantum computation: reversibility, quantum parallelism, and interference of computational paths.
- It is possible to define an Universal QTM.

Observation of QTM

- When a QTM M in superposition $\psi = \sum_i \alpha_i c_i$ is *observed* or *measured*, a configuration c_i is observed with probability α_i. Moreover the superposition of m is updated to $\psi' = c_i$.
- It is also possible to perform a partial measurement; for example, suppose we want to observe the first cell (which contains 0 or 1). Suppose the super position is $\psi = \sum_i \alpha 0_i c 0_i + \psi = \sum_i \alpha 1_i c 1_i$. *If* 0 *isobserved*,Pr[0]=$\psi = \sum_i |\alpha 0_i|^2$ and the new superposition is given by $1/\sqrt{Pr[0]}\psi = \sum_i \alpha 0_i c 0_i$.
- In general, the output of a QTM is a sample from a probability distribution.

Quantum Turing Machine: Input/Output Conventions

Quantum Turing machines need some input/output conventions.

Definition

- We consider *final configuration* any configuration in a QTM M in the final state q_f.
- We say that a QTM M *halts with running time* T on input x if when M is run with input x, at time T the superposition contains only final configurations, and at any times $T_i < T$ the superposition contains no final configurations.

Bernstein and Vazirani in [1] give also careful definitions on the output of QTM.

Quantum Turing Machine: Input/Output Conventions

Definition (Stationariety and Normal Form)

- A QTM M is called *well behaved* if it halts on all input strings in a final superposition where each configuration has the tape head in the same cell.
- If this cell is always the start cell, we call M *stationary*.
- We say that M is in *normal form* if it is well formed and q_f always leads back to q_0.

Definition (Unidirectionality)

A QTM is called *unidirectional* if each state can be entered from only one direction.

Quantum Turing Machine: Input/Output Conventions

Definition (Multitrack TM)

A *multitrack* TM with k tracks is a TM whose alphabet Σ is of the form $\Sigma_1 \times \ldots \times \Sigma_k$ with a special blank symbol $\#$ in each Σ_i such that the blank in Σ is $(\#, \ldots, \#)$. The input is specified by specifying the string on each track. So the TM on input $x_1; \ldots; x_k \in \Pi_{i=1\ldots k}(\Sigma_i - \#)$ is started in the initial configuration with the non-blank portion of the i-th coordinate of the tape containing the string x_i starting in the start cell.

Termination of QTM

How we can verify that the QTM *M* effectively halts? Bernstein and Vazirani in [1] write:"*This can be accomplished by performing a measurement to check whether the machine is in the final state* q_f. *Making this partial measurement does not have any other effect on the computation*".
This can be "implemented" with an *observation cell* in which we can perform a partial measurement.

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Accepting languages with QTM

Definition

Let M a stationary, normal form, multitrack QTM whose last track has alphabet $\{\#, 0, 1\}$. If we run M with string x in the first track and the empty string elsewhere, wait until M halts and then observe the last track of the start cell: we will see a 1 with probability p. We will say that M accepts x with probability p and rejects x with probability $1 - p$.

Definition

We say that a QTM M *accepts a language* $\mathcal{L}$ *with probability* p, if M accepts with probability at least p every string $x \in \mathcal{L}$, and rejects with probability at least p every string $x \notin \mathcal{L}$.

Quantum Complexity Classes

Definition

The class *EQP* is the set of the languages $\mathcal{L}$ accepted by polynomial QTM *M* with probability 1.

EQP is the error-free (or exact) quantum polynomial-time complexity classes.

Definition

The class *BQP* is the set of the languages $\mathcal{L}$ accepted by polynomial QTM *M* with probability 2/3.

Quantum Complexity Classes

Definition

The class *ZQP* is the set of the languages $\mathcal{L}$ accepted by polynomial QTM *M* such that, for every string *x*:

- if $x \in \mathcal{L}$, then *M* accepts *x* with probability $p > 2/3$ and rejects with probability $p = 0$;
- if $x \notin \mathcal{L}$, then *M* rejects *x* with probability $p > 2/3$ and accepts with probability $p = 0$.

The class *ZQP* is the zero-error extension of the class *BQP*. In fact the QTM never gives a wrong answer, but in each case with probability 1/3 gives a "don't-know" answer (clearly, in this case we need to have three answers).

Quantum Complexity Classes

Some results:

- The inclusions $EQP \subseteq ZQP \subseteq BQP$ obviously hold.
- The relationship with classical complexity classes is the following:

$$P \subseteq BPP \subseteq BQP \subseteq PSPACE$$

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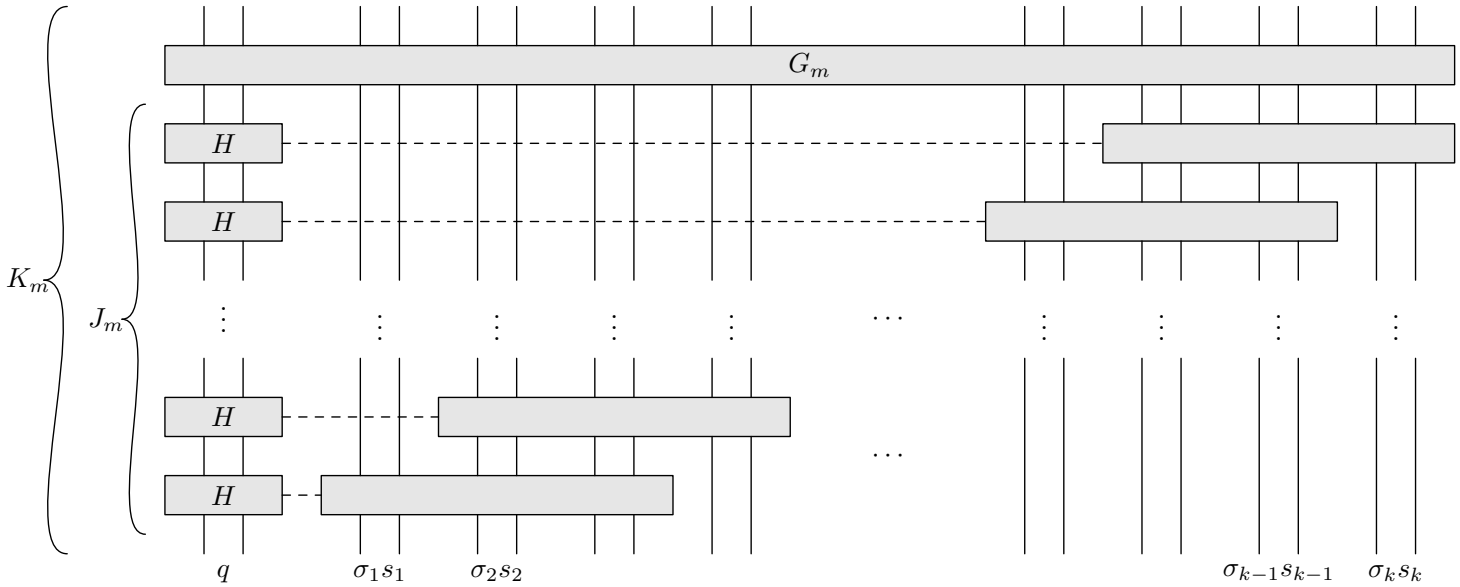
QTM and Quantum Circuit Families

- On total computations, the most relevant quantum computational models, i.e. QTM and Quantum Circuit Families are equivalent.
- In [4] Yao propose an interesting encoding of the QTM in terms of Quantum Circuit Families.

QTM and Quantum Circuit Families



- The quantum circuit computing one step of the simulation.

The Perfect Equivalence

The interesting case of polynomial time quantum computations has been largely investigate by Nishimura and Ozawa [3]. It is possible to define a "perfect equivalence" result between polynomial time QTM and a particular class of Quantum Circuit Families.

Definition (Polynomial-Time QTM)

A *polynomial time* QTM M is a QTM which on every input x halts in time T with T polynomial in the length of x.

The Perfect Equivalence

The perfect equivalence needs some hypothesis:

- Amplitudes of the QTM M have to be in $\mathbf{P}\mathbb{C}$.
- We restrict Quantum Circuit Families to the sub-class of the *Finitely Generated* one.

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