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Vortex dynamics by geometric and structural complexity analysis

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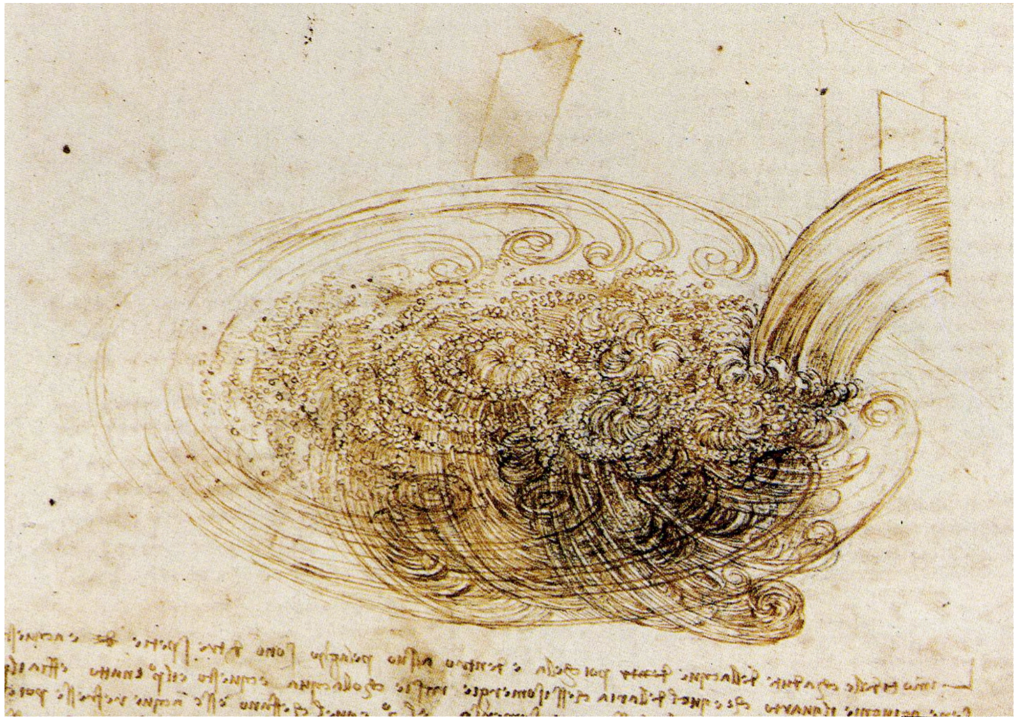
Aims

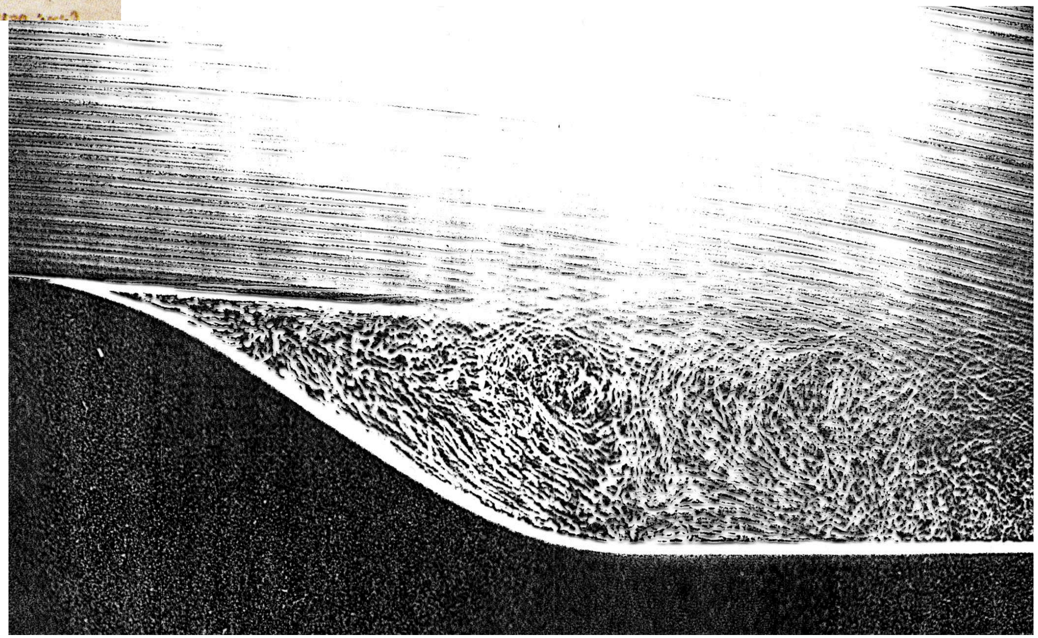
- ***Theoretical goals:***
  - *describe and classify complex morphologies;*
  - *study possible relationships between energy and complexity;*
  - *understand and predict energy localization and transfer;*
- ***Applications:***
  - *implement new visiometric tools and diagnostics;*
  - *develop real-time energy analysis of dynamical processes.*

Coherent structures

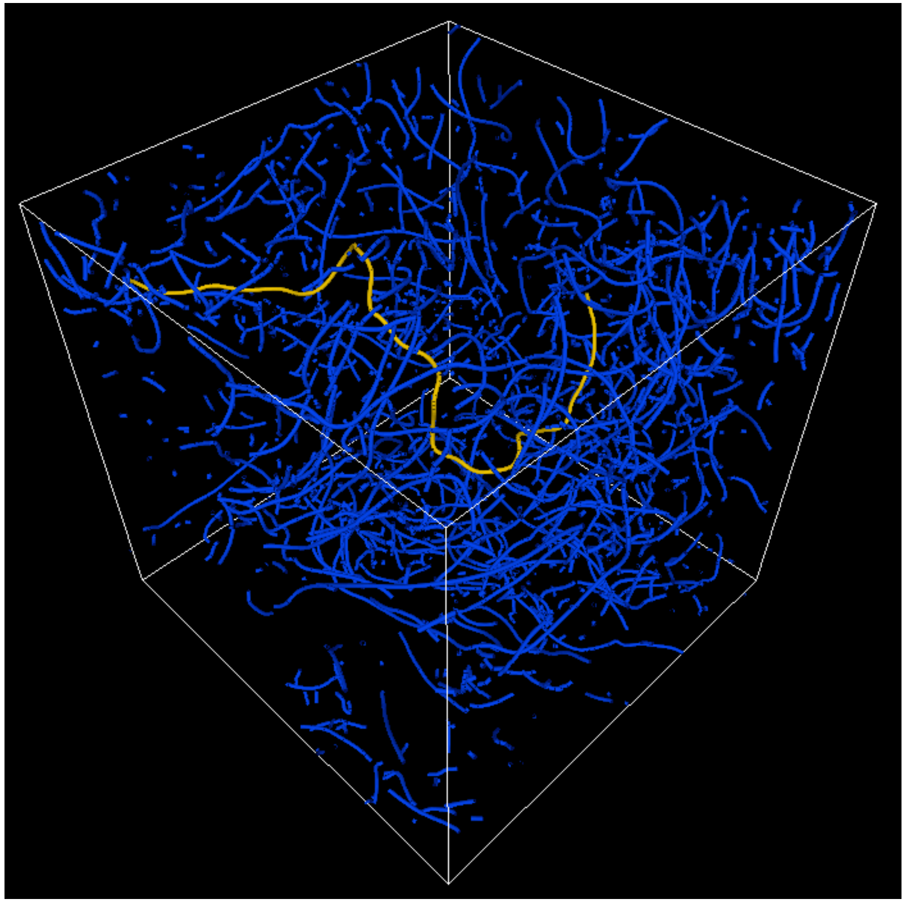


Leonardo da Vinci
(Water studies, 1506)

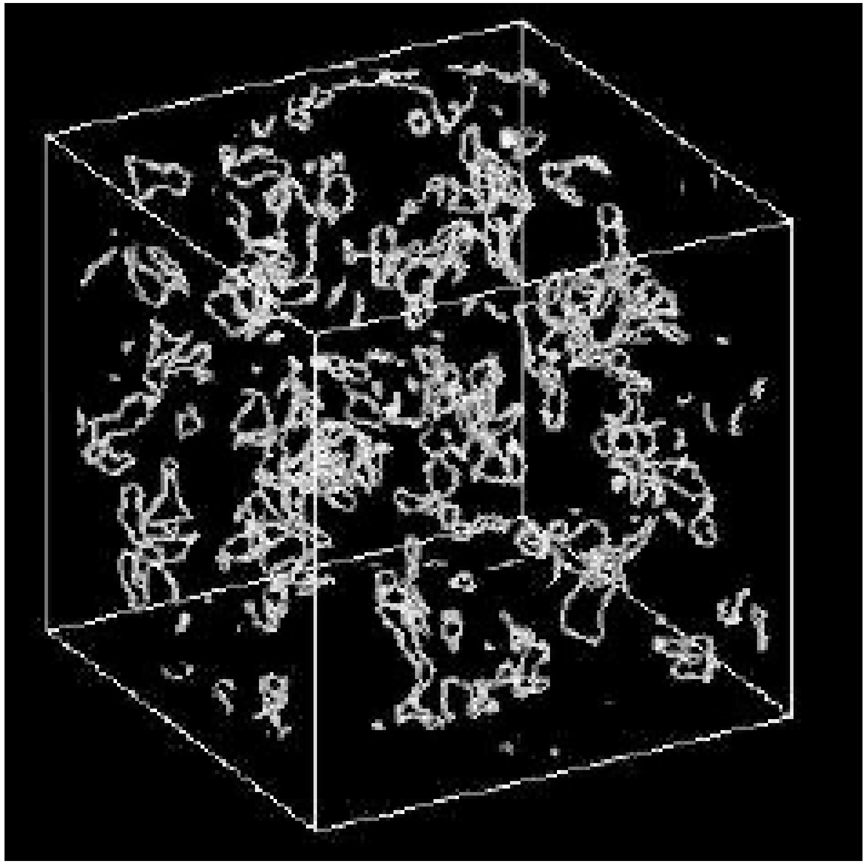
Werlé, ONERA, 1974
(Van Dyke, 1982)



Vorticity localization in classical and quantum fluids

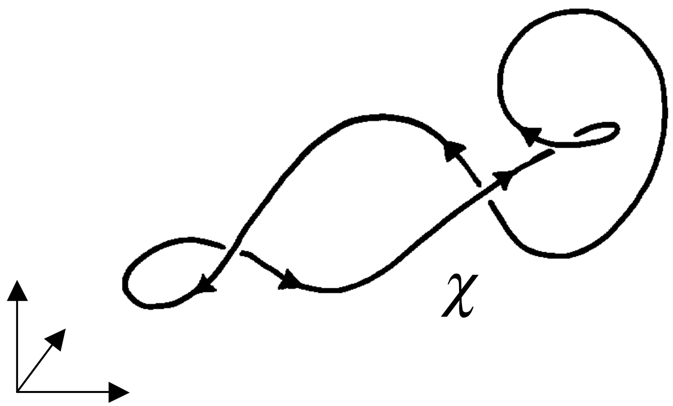


Kida et al.
(Toki-Kyoto, 2002)

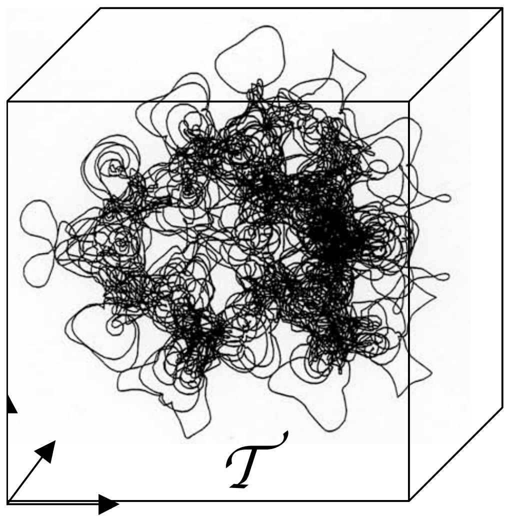


Miyazaki et al.
(Physica D, 2009)

Geometric approach to vortex filament motion



- Localized induction approximation (LIA) and intrinsic kinematics
- Solitons and integrable vortex dynamics
- Stationary solutions
- Topological properties and fluid invariants



Structural complexity analysis of vortex dynamics

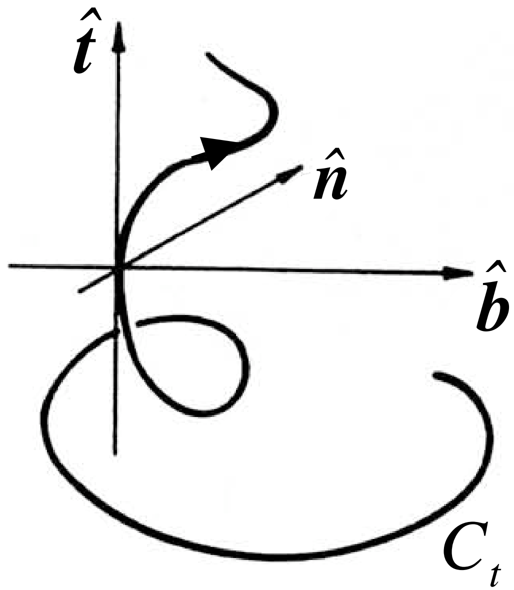
- Geometry and topology of fluid flows
- Measures of morphological complexity
- Energy/complexity relations
- Shapefinders and eigenvalue analysis
- Dynamical properties in terms of graph analysis

Localized induction approximation (LIA)

$$\left.\begin{array}{l}\textit{homogeneous}\\ \textit{incompressible}\\ \textit{inviscid}\end{array}\right\} \textit{fluid in } \mathbb{R}^3 : \qquad \boldsymbol{u} = \boldsymbol{u}(X,t) \left\{\begin{array}{l}\boldsymbol{\omega} = \nabla \times \boldsymbol{u}\\ \nabla \cdot \boldsymbol{u} = 0 \quad \textit{in } \mathbb{R}^3\\ \boldsymbol{u} = 0 \quad \textit{as } X \to \infty\end{array}\right.$$

- ***Space curve*** C_t ***, given by:***

$$C_t : \left\{\begin{array}{l} X(s,t) := X_t(s) \in \mathrm{C}^{\nu}\\ s \in [0,L] \to \mathbb{R}^3\end{array}\right.$$



- ***Intrinsic reference on*** C_t ***, given by:***
  (Frenet frame)

$$\hat{\boldsymbol{t}} := X'(s,t) = \frac{\partial X}{\partial s} \quad , \qquad \{\hat{\boldsymbol{t}}, \hat{\boldsymbol{n}}, \hat{\boldsymbol{b}}\}$$

- ***Vortex line on*** C_t ***:*** $\boldsymbol{\omega} = \omega_0 \hat{\boldsymbol{t}}$, ω_0 ***= constant***

$$\left.\begin{array}{l}\textit{asymptotic theory}\\ \quad (\ R/a \gg 1\ )\\ \textit{no self-intersections}\end{array}\right\} \textbf{\textit{LIA:}} \qquad \dot{X}(s,t) \equiv \frac{\partial X}{\partial t} \approx X' \times X'' = \mathrm{c}\hat{\boldsymbol{b}} \equiv \boldsymbol{u}_{\mathrm{LIA}}$$

(Da Rios, 1906; Hama & Arms, 1961)

Intrinsic equations under LIA and NLSE

- ***Intrinsic description:*** $\boldsymbol{u} = (u_t, u_n, u_b)$, $c(s,t)$ ***curvature,*** $\tau(s,t)$ ***torsion;***

  under LIA: $\boldsymbol{u}_{\mathrm{LIA}} = \mathrm{c}\hat{\boldsymbol{b}}$, $u_t = u_n = 0$, $u_b = \mathrm{c}$.

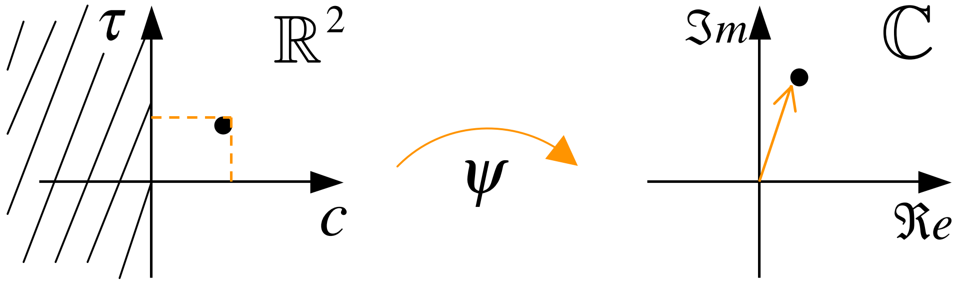
$$C_t := \varphi_t(C) \quad \begin{cases} \dot{c} = -(c\tau)' - c'\tau \\ \dot{\tau} = \left(\dfrac{c'' - c\tau^2}{c}\right)' + c'c \end{cases}$$

Da Rios, 1906
Betchov, 1965

- ***NLSE via Madelung transform:***

$$\psi(s,t) = c(s,t)\, \mathrm{e}^{\mathrm{i}\int_0^s \tau(\xi,t)\mathrm{d}\xi}$$

NLSE: $i\dot{\psi} + \psi'' + \frac{1}{2}|\psi|^2\psi = 0$



(Hasimoto, 1972)

- ***Da Rios-Betchov eqs. from NLSE:***

  by taking log-derivative of $\psi(s,t)$

$$\Rightarrow \quad C_t := \varphi_t(C) \quad \begin{cases} \dot{c} = f_c(s,t) \\ \dot{\tau} = f_\tau(s,t) \end{cases}$$

Higher-order LIAs and integrable geometric dynamics

- ***Higher-order effects:***

$$\boldsymbol{u} = (\alpha + \beta s)c\hat{\boldsymbol{b}} + (\gamma + \delta s)\hat{\boldsymbol{t}} + \mu\left(\tfrac{1}{2}c^2\hat{\boldsymbol{t}} + c'\hat{\boldsymbol{n}} + c\tau\hat{\boldsymbol{b}}\right)$$

β, δ ***: inhomogeneities***
(Lakshmanan & Ganesan, 1985)

μ ***: axial flow and vorticity***
(Fukumoto & Miyazaki, 1991)

γ ***: non-linear stretching***
(Onuki, 1985)

- ***Higher-order LIAs:***

$$\boldsymbol{u}^{(0)} = \boldsymbol{u}_{\mathrm{LIA}}$$

$$\boldsymbol{u}^{(j+1)} = \sum_{\nu=1}^{j+1} \mathscr{F}\left(\frac{\partial^\nu \boldsymbol{u}^{(0)}}{\partial s^\nu}\right) = \mathscr{R}\left(\boldsymbol{u}^{(0)}\right) \longleftrightarrow \dot{\psi} = \mathscr{P}_\psi\left(u_n e^{i\Theta}\right) + \mathscr{Q}_\psi\left(u_b e^{i\Theta}\right)$$

(Langer & Perline, 1991)

(Nakayama *et al.*, 1992)

- ***Integrable geometric dynamics:***

$$\boldsymbol{u} = \left(u_t, u_n, u_b\right) \Rightarrow C_t \coloneqq \varphi_t(C) \quad \begin{cases} \dot{s} = f_s(t) \\ \dot{c} = f_c(s,t) \\ \dot{\tau} = f_\tau(s,t) \end{cases}$$

Germano, 1983
Ricca, 1991

Conserved quantities from integrable dynamics

$A = \oint \boldsymbol{X} \times \boldsymbol{X}' \, \mathrm{d}s =$ ***const.*** (**Arms & Hama, 1965**) ; $L = \oint \mathrm{d}s =$ ***const.*** (**Marsden & Weistein, 1983**)

- ***Geometric invariants:***

$$\mathscr{P}(L) = \sum_{j=0}^{\infty} \mathscr{I}_j \, L^j$$

(Langer & Perline, 1991; Ricca, 1991)

$$\begin{cases} \mathscr{I}_0 = \oint \tau \, \mathrm{d}s \\ \mathscr{I}_1 = \oint c^2 \, \mathrm{d}s \\ \mathscr{I}_2 = \oint c^2 \tau \, \mathrm{d}s \\ \mathscr{I}_3 = \oint \left(\dfrac{c^4}{4} - c'^2 - c^2 \tau^2 \right) \mathrm{d}s \\ \dots \quad \dots \end{cases}$$

- ***Physical conserved quantities:***

(**Fukumoto, 1987 Ricca, 1992**)

kinetic energy: $\mathscr{K} = \dfrac{1}{2} \int \|\boldsymbol{u}\|^2 \, \mathrm{d}^3 \boldsymbol{X}$

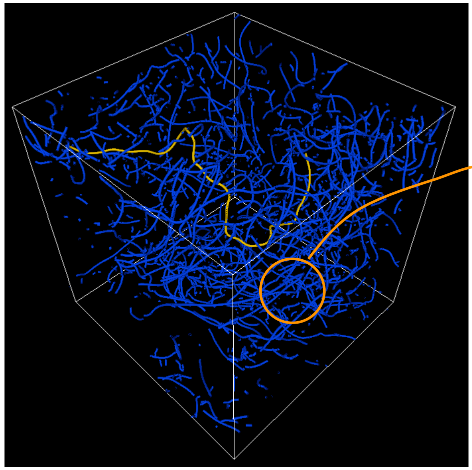
pseudo-helicity: $\mathscr{H} = \int \boldsymbol{u} \cdot (\nabla \times \boldsymbol{u}) \, \mathrm{d}^3 \boldsymbol{X}$

enstrophy: $\mathscr{E} = \int \boldsymbol{\omega} \, \mathrm{d}^3 \boldsymbol{X}$

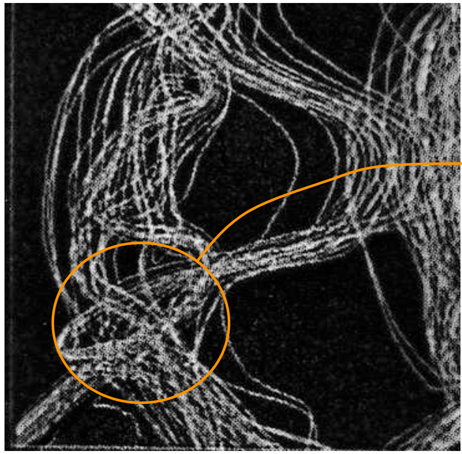
linear momentum: $\boldsymbol{P} = \dfrac{1}{2} \int (\boldsymbol{X} \times \boldsymbol{\omega}) \mathrm{d}^3 \boldsymbol{X}$

angular momentum: $\boldsymbol{M} = \dfrac{1}{3} \int \boldsymbol{X} \times (\boldsymbol{X} \times \boldsymbol{\omega}) \mathrm{d}^3 \boldsymbol{X}$

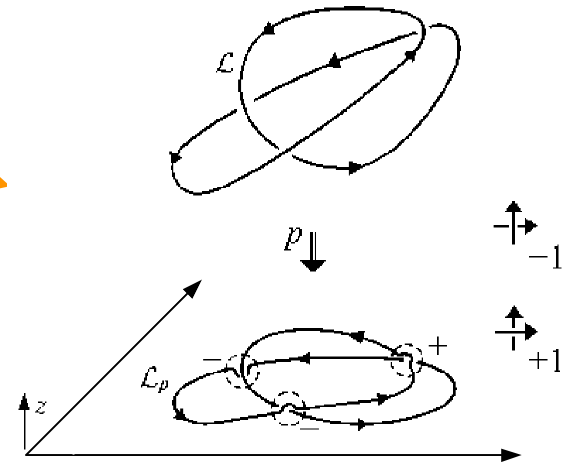
Structural complexity analysis of 3D vortex tangles



experiment
or simulation



domain extraction



projected diagram
analysis

- Measures of structural complexity:

Algebraic information:
- average crossing number

Geometric information:
- tropicity directions
- coiling, writhing
- alignment
- signed area

Topological information:
- minimal crossing number
- linking numbers
- topological changes

Dynamical systems analysis:
- topological entropy
- eigenvalue analysis

Energy and helicity of a vortex tangle

- ***Kinetic energy:*** $E(\mathcal{T}) \equiv \frac{1}{2}\int_{\mathcal{T}} \|\boldsymbol{u}\|^2 \, \mathrm{d}^3\boldsymbol{X} = \int_{\mathcal{T}} \boldsymbol{u}\cdot(\boldsymbol{X}\times\boldsymbol{\omega}) \, \mathrm{d}^3\boldsymbol{X}$;

  from Lamb (1932), we have: $E(\mathcal{T}) \approx \frac{\Gamma^2}{8\pi}\sum_{ij}\iint_{\chi_i\chi_j} \frac{\hat{\boldsymbol{t}}_i\cdot\hat{\boldsymbol{t}}_j}{|\boldsymbol{X}_i-\boldsymbol{X}_j|}\,\mathrm{d}s_i\mathrm{d}s_j$,

  with total length of vortex filaments given by $L(\mathcal{T}) = \sum_i \int_{\chi_i} \hat{\boldsymbol{t}}_i \, \mathrm{d}s_i$.

- *If* $\boldsymbol{\omega} \equiv \nabla\times\boldsymbol{u} = \varpi_0\hat{\boldsymbol{t}}$ *, then the kinetic helicity is given by (Moffatt, 1969; Moffatt & Ricca, 1992):*

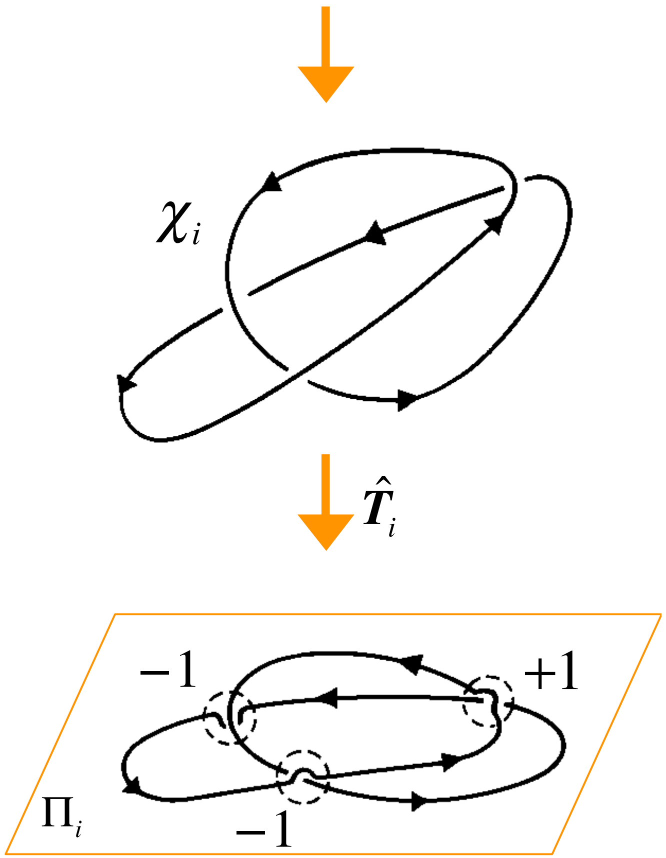
$$\mathcal{H}(\mathcal{T}) \equiv \int_{\mathcal{T}} \boldsymbol{u}\cdot\boldsymbol{\omega} \, \mathrm{d}^3\boldsymbol{X} = \sum_i Lk_i\Gamma_i^2 + 2\sum_{i\neq j} Lk_{ij}\Gamma_i\Gamma_j$$

where
- $Lk_i = Lk(\chi_i,\chi_i) = Wr_i + Tw_i$ — ***Calugareanu-White invariant***
- $Lk_{ij} = Lk(\chi_i,\chi_j)$ — ***Gauss linking number***
- Γ_i — ***i-th vortex circulation***

Tangle analysis by indented projections

Let $\Pi_i = \Pi(\hat{\boldsymbol{T}}_i)$ ***be the "indented"*** $\hat{\boldsymbol{T}}_i$***–projection of the oriented tangle component*** χ_i ***; assign the value*** $\boldsymbol{\varepsilon}_{\mathrm{r}} = \pm 1$ ***to each apparent crossing in*** Π_i ***.***

$$\mathcal{T} = \bigcup_i \chi_i$$



- ***writing:***

$$Wr_i = Wr(\chi_i) = \left\langle \sum_{\mathrm{r}\in\chi_i} \varepsilon_{\mathrm{r}} \right\rangle \quad , \quad Wr = Wr(\mathcal{T}) = \left\langle \sum_{\mathrm{r}\in\mathcal{T}} \varepsilon_{\mathrm{r}} \right\rangle \; ;$$

- ***linking:***

$$Lk_{ij} = Lk(\chi_i, \chi_j) = \frac{1}{2} \sum_{\substack{r\in\chi_i\cap\chi_j \\ i\neq j}} \varepsilon_r , \qquad Lk_{tot} = \sum_{\substack{r\in\mathcal{T} \\ i\neq j}} \left| Lk_{ij} \right| \; ;$$

- ***average crossing number:***

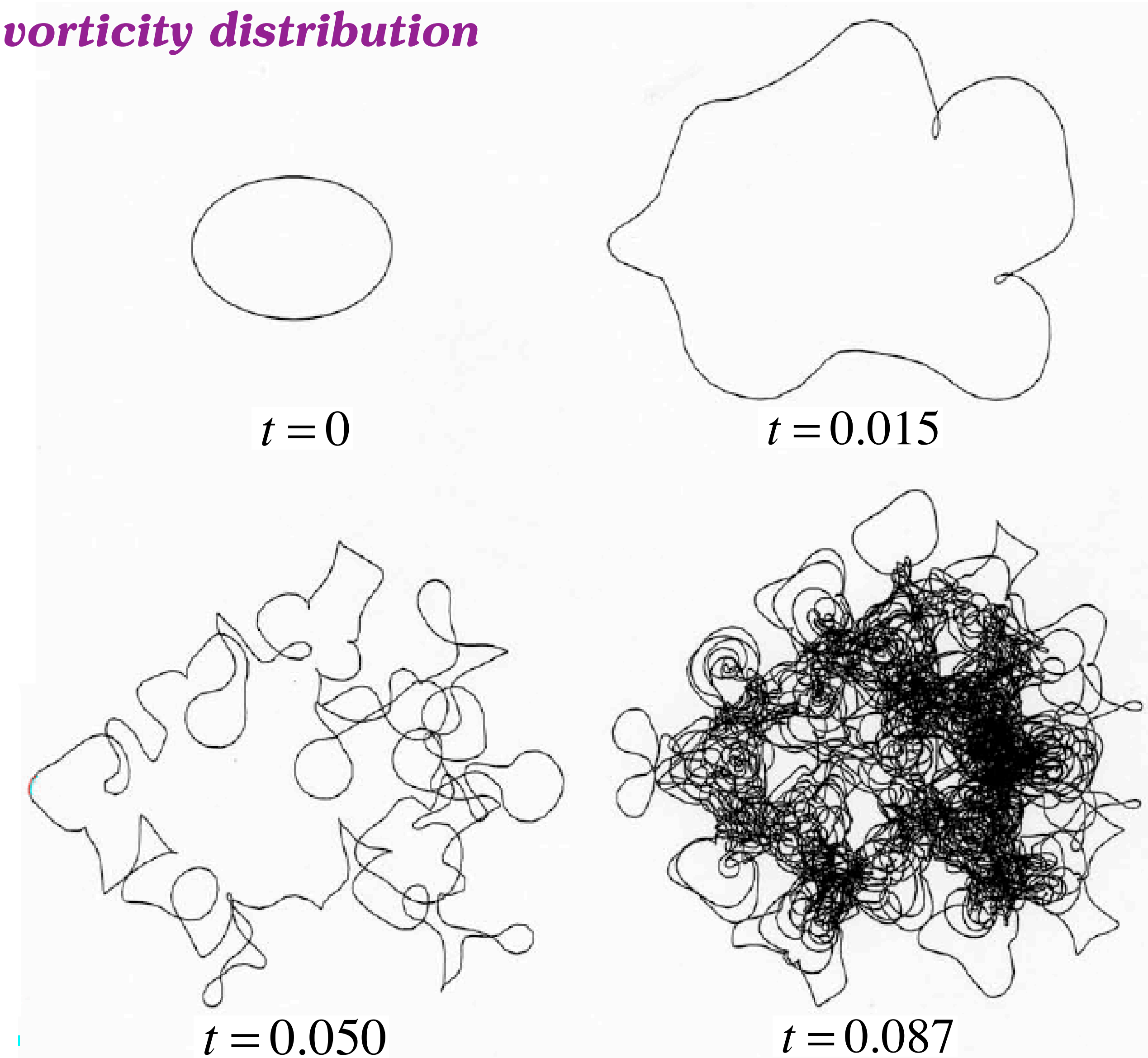
$$\bar{C}_{ij} = \bar{C}(\chi_i, \chi_j) = \left\langle \sum_{r\in\chi_i\cap\chi_j} \left|\varepsilon_r\right| \right\rangle , \qquad \bar{C} = \sum_{r\in\mathcal{T}} \bar{C}_{ij} \; ;$$

- ***estimated values:***

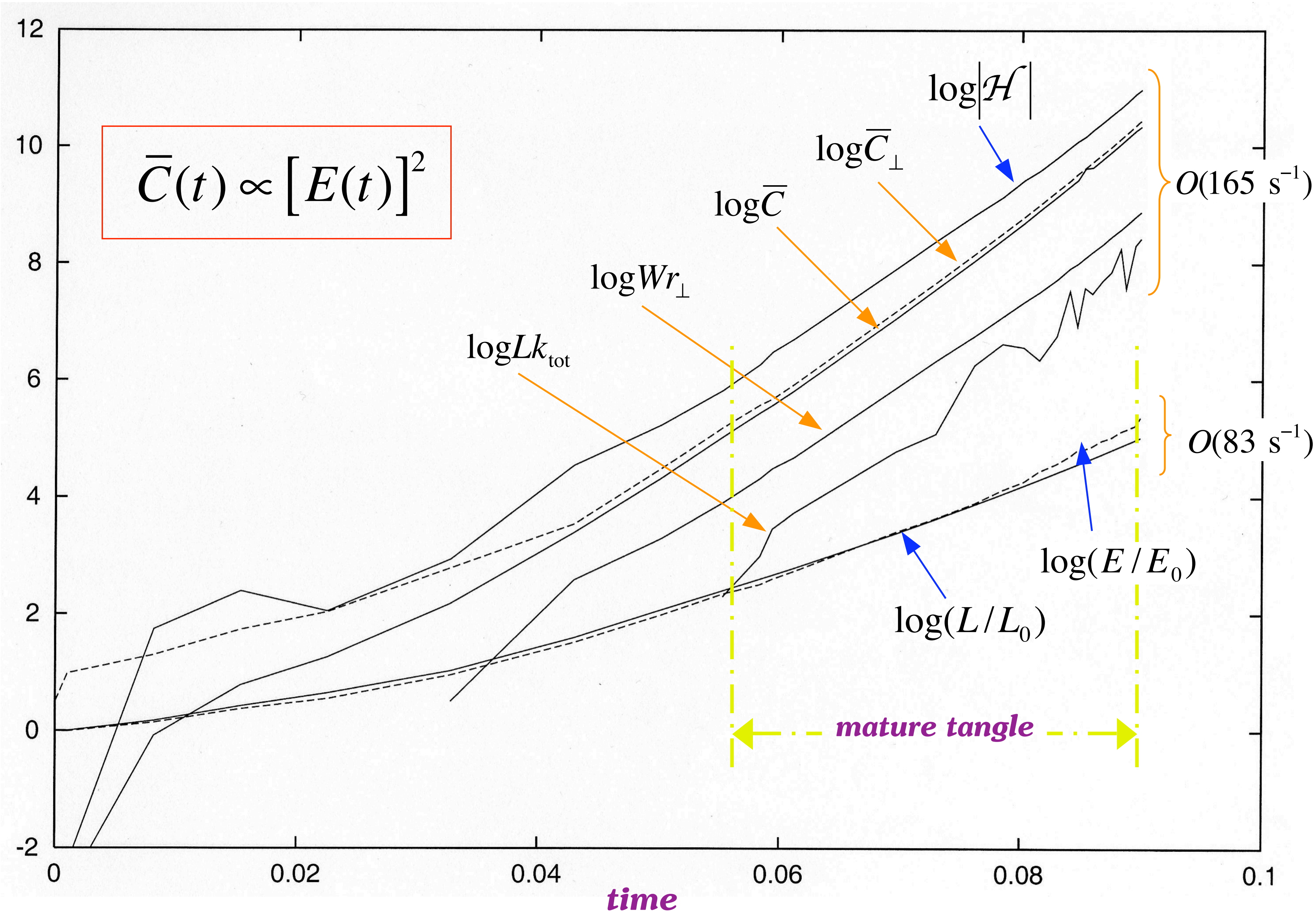
$$\overline{Wr}_{\perp} = \left(\sum_{\mathrm{r}\in\mathcal{T}} \boldsymbol{\varepsilon}_{\mathrm{r}} \right)_{\perp} , \qquad \bar{C}_{\perp} = \left(\sum_{\mathrm{r}\in\mathcal{T}} \left|\varepsilon_{\mathrm{r}}\right| \right)_{\perp} .$$

A vortex tangle test case *(Barenghi, Ricca & Samuels, 2001)*

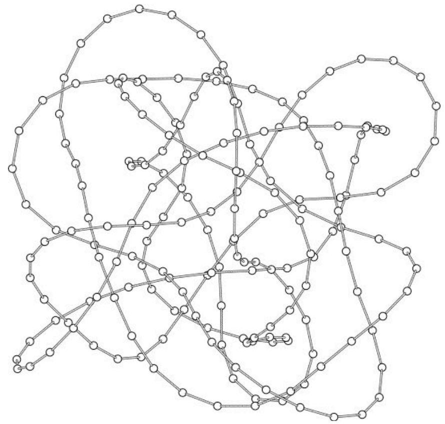
- *ABC-type flow field super-imposed on initial vorticity distribution*



Vortex tangle test case: energy-complexity relation



Energy-complexity relation



- $\bar{C}(t) \propto [E(t)]^2$ **: since**

$$\bar{C} = \frac{1}{4\pi} \sum_{ij} \iint_{\chi_i \chi_j} \frac{\left| (\boldsymbol{X}_i - \boldsymbol{X}_j) \cdot \hat{\boldsymbol{t}}_i \times \hat{\boldsymbol{t}}_j \right|}{\left| \boldsymbol{X}_i - \boldsymbol{X}_j \right|^3} \, \mathrm{d}s_i \mathrm{d}s_j \quad ,$$

we have

$$\delta \bar{C} \propto \left(\frac{\delta s}{\left| \boldsymbol{X}_i - \boldsymbol{X}_j \right|} \right)^2 \quad , \quad \textbf{and} \qquad \delta(E / E_0) \propto \frac{\delta s}{\left| \boldsymbol{X}_i - \boldsymbol{X}_j \right|} \quad ;$$

hence

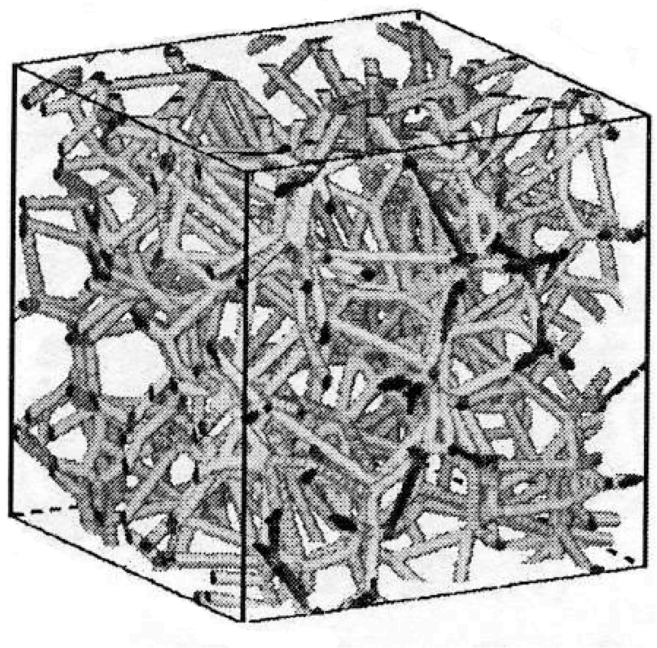
$$\bar{C} \propto (E / E_0)^2 \, .$$

Helicity-complexity bounds

If $Lk_{ii} = 0$ **,** $\Gamma_i = \Gamma$ $\quad \forall \chi_i \in \mathcal{T}$ **, then we have (Ricca, 2008):**

- $$|H(t)| \leq 2\Gamma^2 \bar{C}(t) \quad \textbf{and} \quad \left| \frac{\mathrm{d}|H(t)|}{\mathrm{d}t} \right| \leq 2\Gamma^2 \frac{\mathrm{d}\bar{C}(t)}{\mathrm{d}t} \quad .$$

Tropicity interpretation of eigenvalues



- **Eigenvalue analysis: let** $\lambda_1 \geq \lambda_2 \geq \lambda_3$.

  From (Vilanova et al., 2006), we have:

$$C_f = \frac{\lambda_1 - \lambda_2}{\sum_i \lambda_i} , \quad C_p = \frac{2(\lambda_2 - \lambda_3)}{\sum_i \lambda_i} , \quad C_s = \frac{3\lambda_3}{\sum_i \lambda_i} .$$

- **Shapefinders: let**

$$V = \int_{\mathcal{D}} \mathrm{d}^3 \boldsymbol{X} , \quad A = \int_{\partial\mathcal{D}} \mathrm{d}^2 \boldsymbol{X} , \quad H_G = \frac{1}{2}\int_{\partial\mathcal{D}} \left(\frac{1}{R_1} + \frac{1}{R_2} \right) \mathrm{d}^2 \boldsymbol{X}, \quad \chi_E = \frac{1}{2}\int_{\partial\mathcal{D}} \frac{1}{R_1 R_2} \mathrm{d}^2 \boldsymbol{X} .$$

Then

$$L = \frac{H_G}{4\pi\chi_E} , \quad W = \frac{A}{H_G} , \quad T = \frac{3V}{A} ;$$

(Sahni et al., 1998)

If $\mathcal{D}$ **is convex and** $L \geq W \geq T > 0$ **, we can define**

$$C_F = \frac{L - W}{L + W + T} , \quad C_P = \frac{2(W - T)}{L + W + T} , \quad C_S = \frac{3T}{L + W + T} ,$$

so that

$$\lambda_1 \propto L , \quad \lambda_2 \propto W , \quad \lambda_3 \propto T .$$

Interpretation of momenta in terms of projected areas

- ***Linear momentum:***

$$\boldsymbol{P} \equiv \frac{1}{2}\int_{\mathcal{T}} \boldsymbol{X}\times\boldsymbol{\omega}\,\mathrm{d}^3\boldsymbol{X} = \frac{1}{2}\sum_i \Gamma_i \oint_{\chi_i} \boldsymbol{X}\times\boldsymbol{X}'\,\mathrm{d}s\ ,$$

- ***Angular momentum:***

$$\boldsymbol{M} \equiv \frac{1}{3}\int_{\mathcal{T}} \boldsymbol{X}\times(\boldsymbol{X}\times\boldsymbol{\omega})\,\mathrm{d}^3\boldsymbol{X} = \frac{1}{3}\sum_i \Gamma_i \oint_{\chi_i} \boldsymbol{X}\times(\boldsymbol{X}\times\boldsymbol{X}')\,\mathrm{d}s\ .$$

- ***Under Euler (and LIA) equations:*** $\dfrac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}t}=0\ ,\qquad \dfrac{\mathrm{d}\boldsymbol{M}}{\mathrm{d}t}=0\ .$

Plane graph by standard projection

$$\Lambda \equiv p(\mathcal{T}):\ \mathbb{R}^3 \to \mathbb{R}^2 \qquad \left\{\begin{array}{l} \Lambda_{xy} \equiv p_z(\mathcal{T}) \\ \mathcal{A}_{xy} \equiv \mathcal{A}(\Lambda_{xy}) \end{array}\right\}_{xy}\ ,\qquad \left\{\ \cdots\ \right\}_{yz}\ ,\qquad \left\{\ \cdots\ \right\}_{zx}\ .$$

Then (Arms & Hama, 1965; Ricca, 1992):

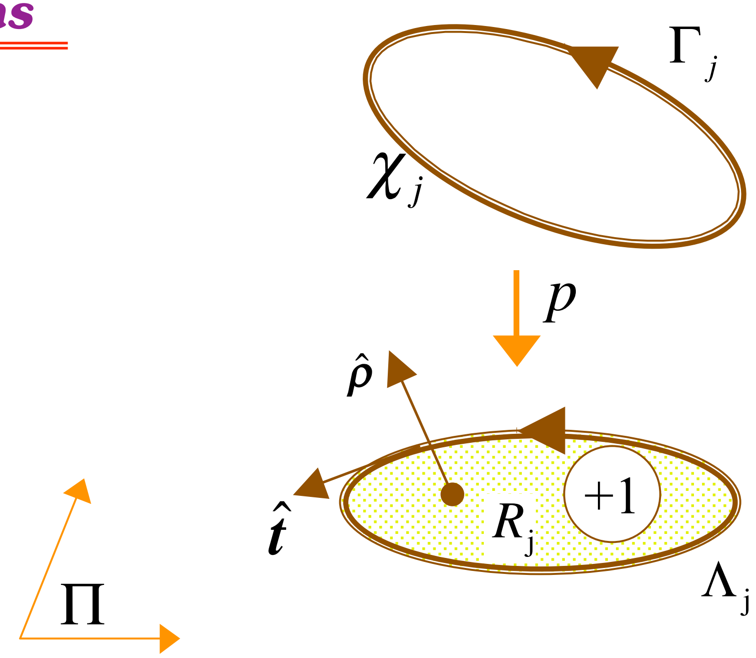
- $P_{xy} = \overline{\Gamma}\mathcal{A}_{xy}\ ,\ P_{yz} = \overline{\Gamma}\mathcal{A}_{yz}\ ,\ P_{zx} = \overline{\Gamma}\mathcal{A}_{zx}\ ;$
- $M_{xy} = \overline{\Gamma}d_z\mathcal{A}_{xy}\ ,\ M_{yz} = \overline{\Gamma}d_x\mathcal{A}_{yz}\ ,\ M_{zx} = \overline{\Gamma}d_y\mathcal{A}_{zx}\ .$

Signed area and weight of projected graphs

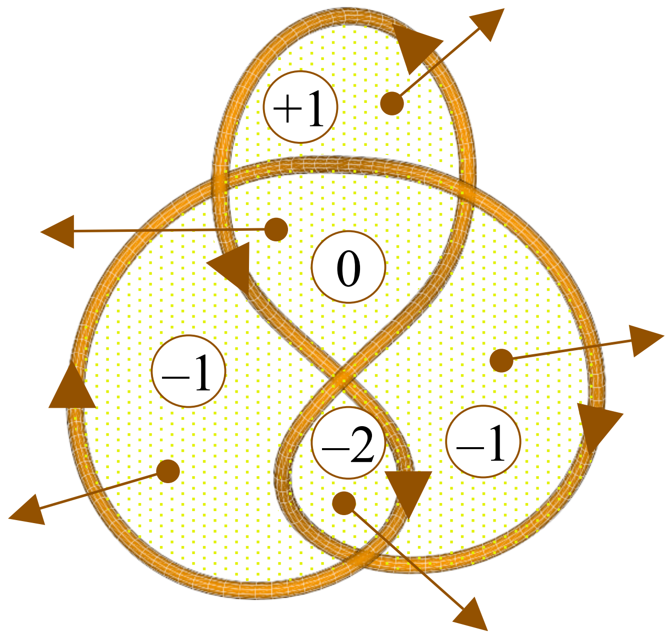
Let $\Lambda_j = p(\chi_j)$ *and* $R_j \equiv \operatorname{int}\left(\Lambda_j \bigcap \Pi\right)$ *:*

- ***Cauchy index*** $\mathcal{I}_j = \mathcal{I}\left(\Lambda_j\right)$ *: at each intersection* $\hat{\rho} \bigcap \vec{\Lambda}_j$ *assign* $\varepsilon_{\mathrm{i}} = \pm 1$ *and take*

$$\mathcal{I}_j = \mathcal{I}\left(\Lambda_j\right) = \sum_{\hat{\rho} \bigcap \vec{\Lambda}_j} \varepsilon_i \ .$$



Example



- ***Signed area:*** $\mathcal{A}_j\left(\Lambda\right) = \sum_j \mathcal{I}_j \mathcal{A}_j(R_j)$

- ***Weighted circulation:*** $\gamma_j = \dfrac{\sum_k \Gamma_k L_{k,j}}{L_j}$

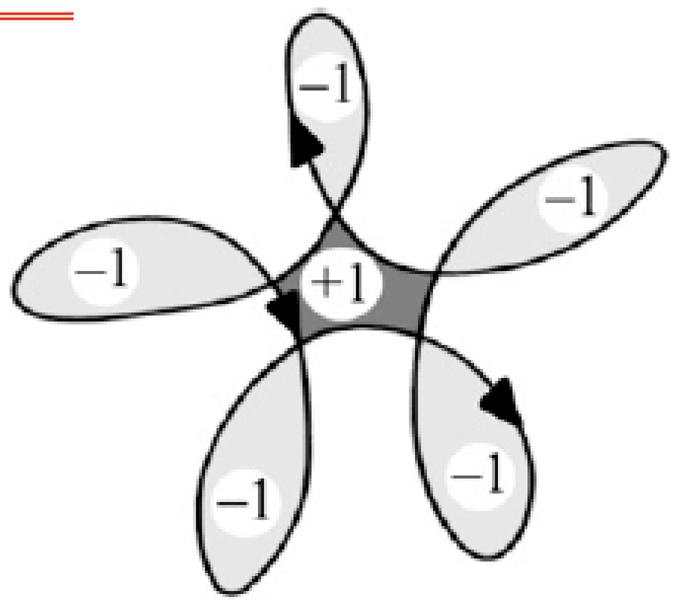
Momenta by signed area interpretation: geometric method (Ricca, 2008)

Let $\mathcal{T} = \bigcup \chi_i$ *be a vortex tangle. Then, by considering the associated principal projected graphs, we have*

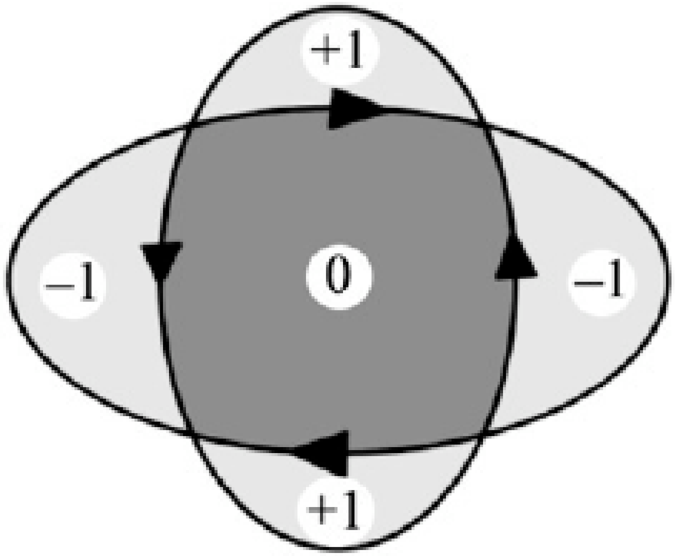
$$\boldsymbol{P} = \left(P_{xy}, P_{yz}, P_{zx}\right): \quad P_{xy} = \sum_j \gamma_j \mathcal{I}_j \mathcal{A}_{xy}\left(R_j\right), \quad P_{yz} = \ldots, \quad P_{zx} = \ldots \quad ;$$

$$\boldsymbol{M} = \left(M_{xy}, M_{yz}, M_{zx}\right): \quad M_{xy} = d_z \sum_j \gamma_j \mathcal{I}_j \mathcal{A}_{xy}\left(R_j\right), \quad M_{yz} = \ldots \quad , \quad M_{zx} = \ldots \quad .$$

Examples



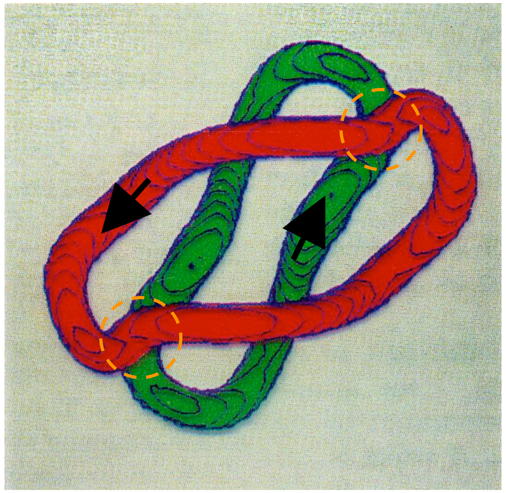
(a)



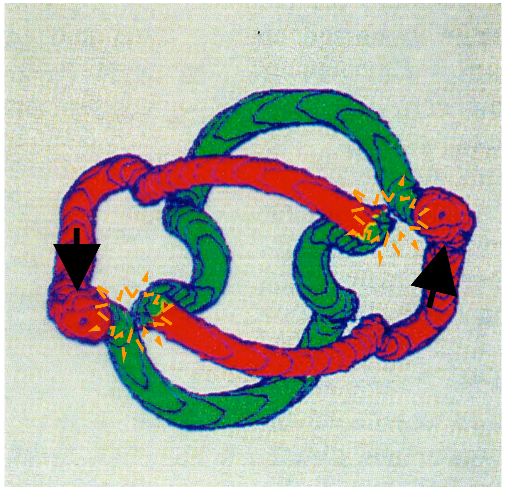
(b)

Vortex analysis by geomertic method

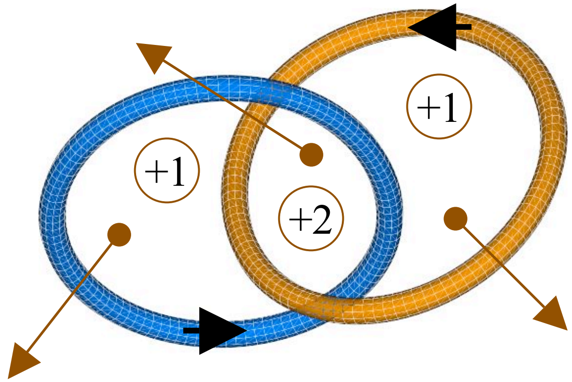
- *Following Aref & Zawadzki (Nature, 1991):*



before

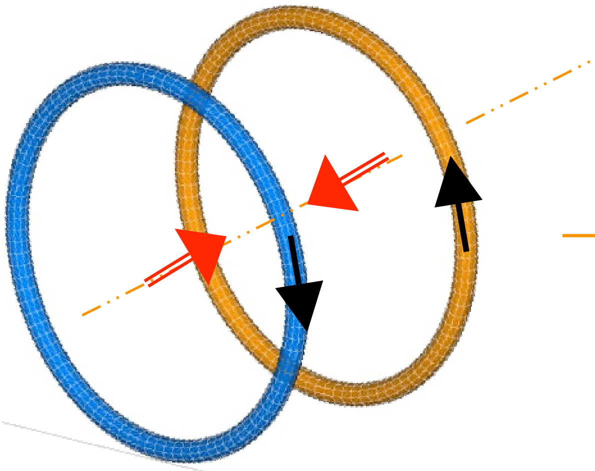


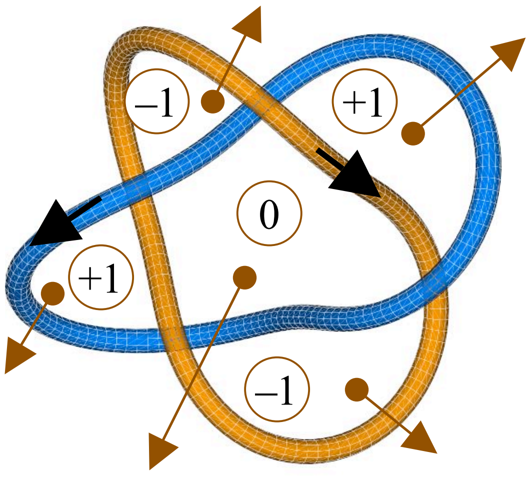
after

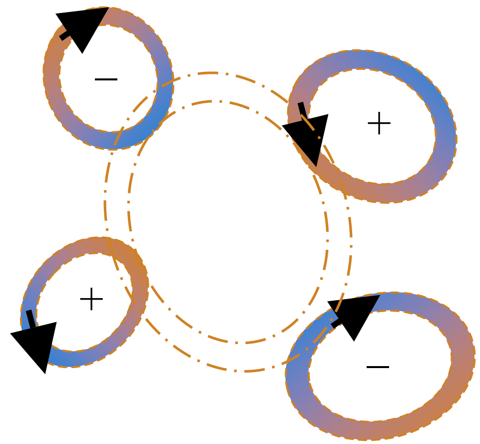


weighted areas

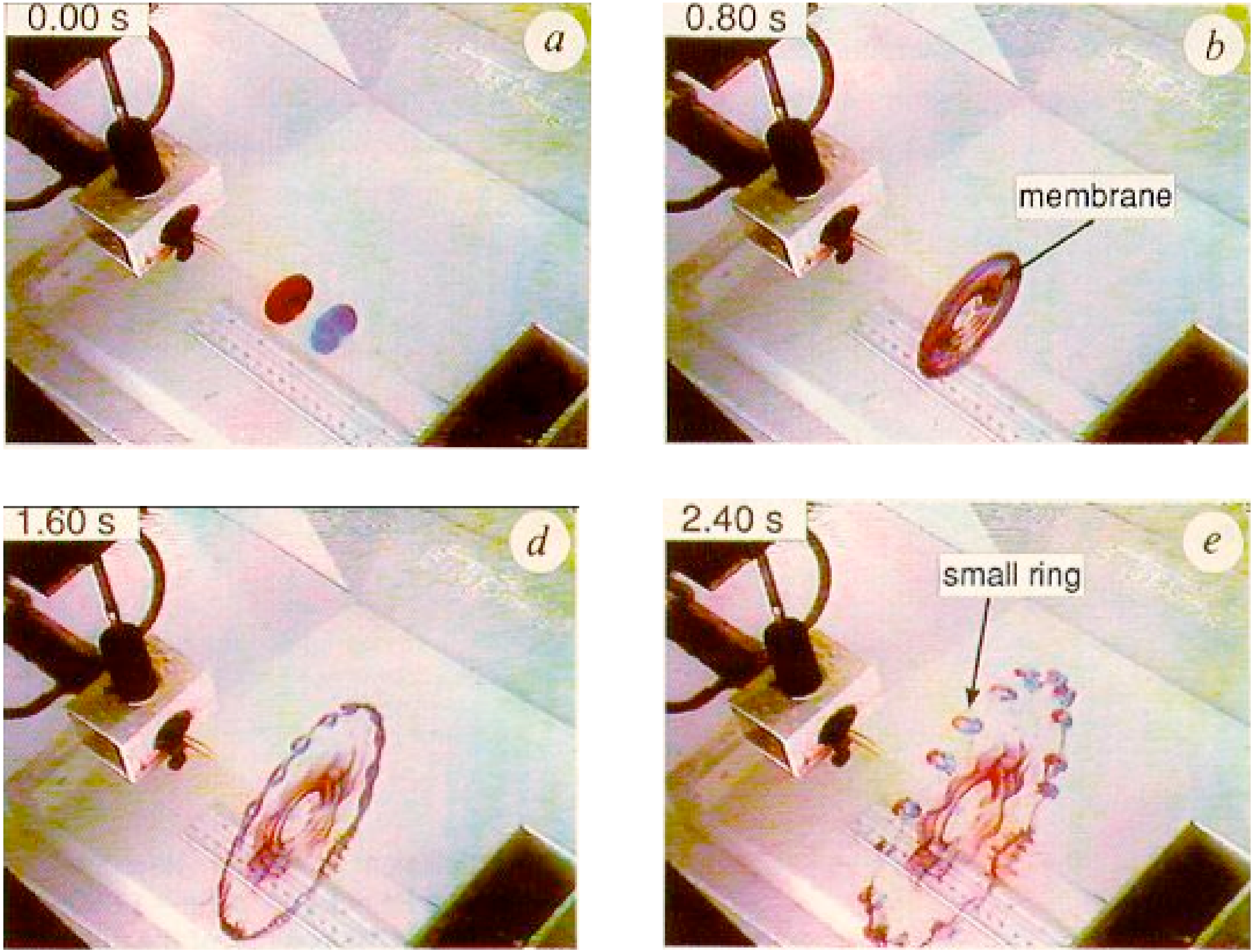
- *Zero-area momentum* ($\boldsymbol{P} = 0$) *interaction:*



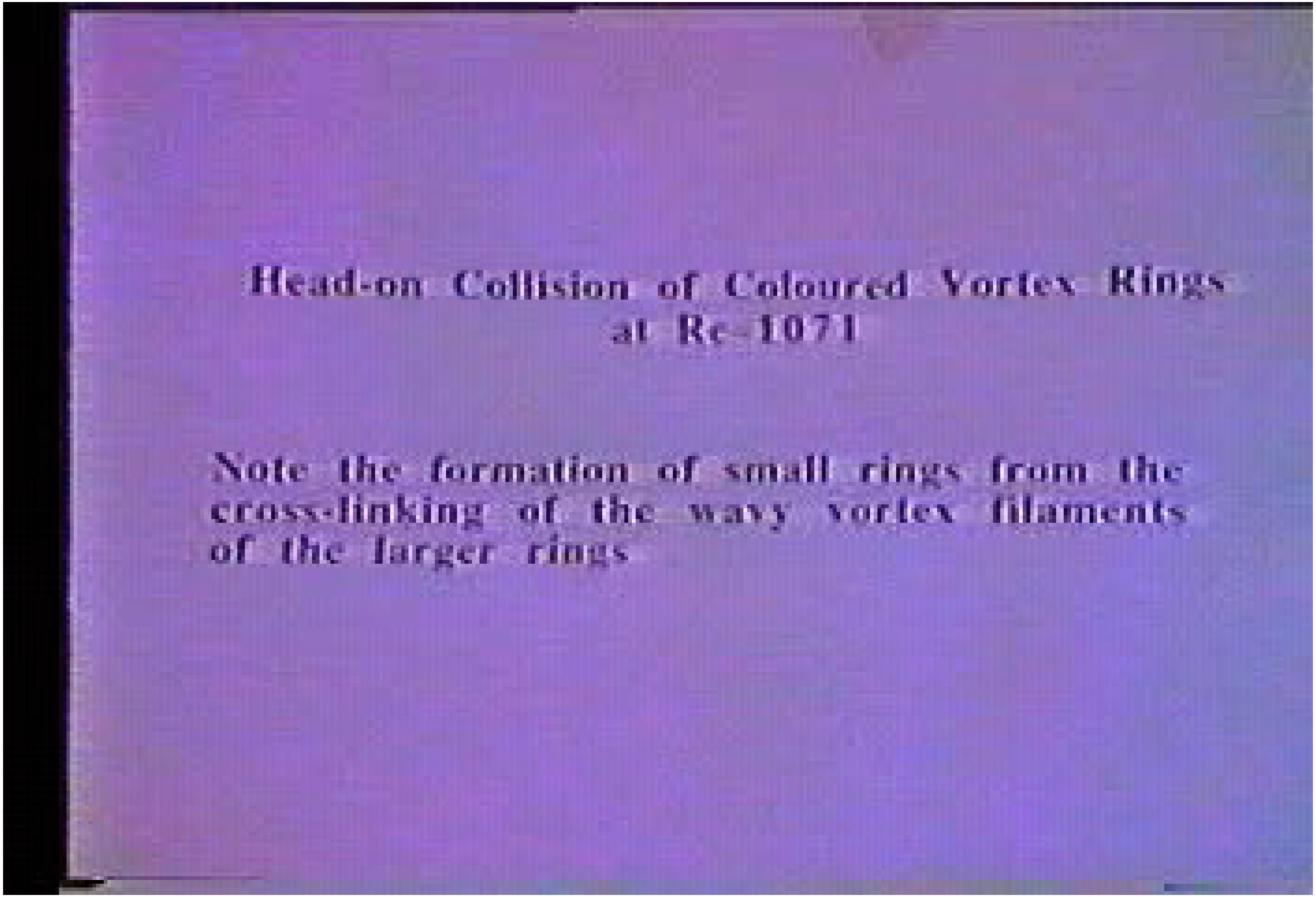




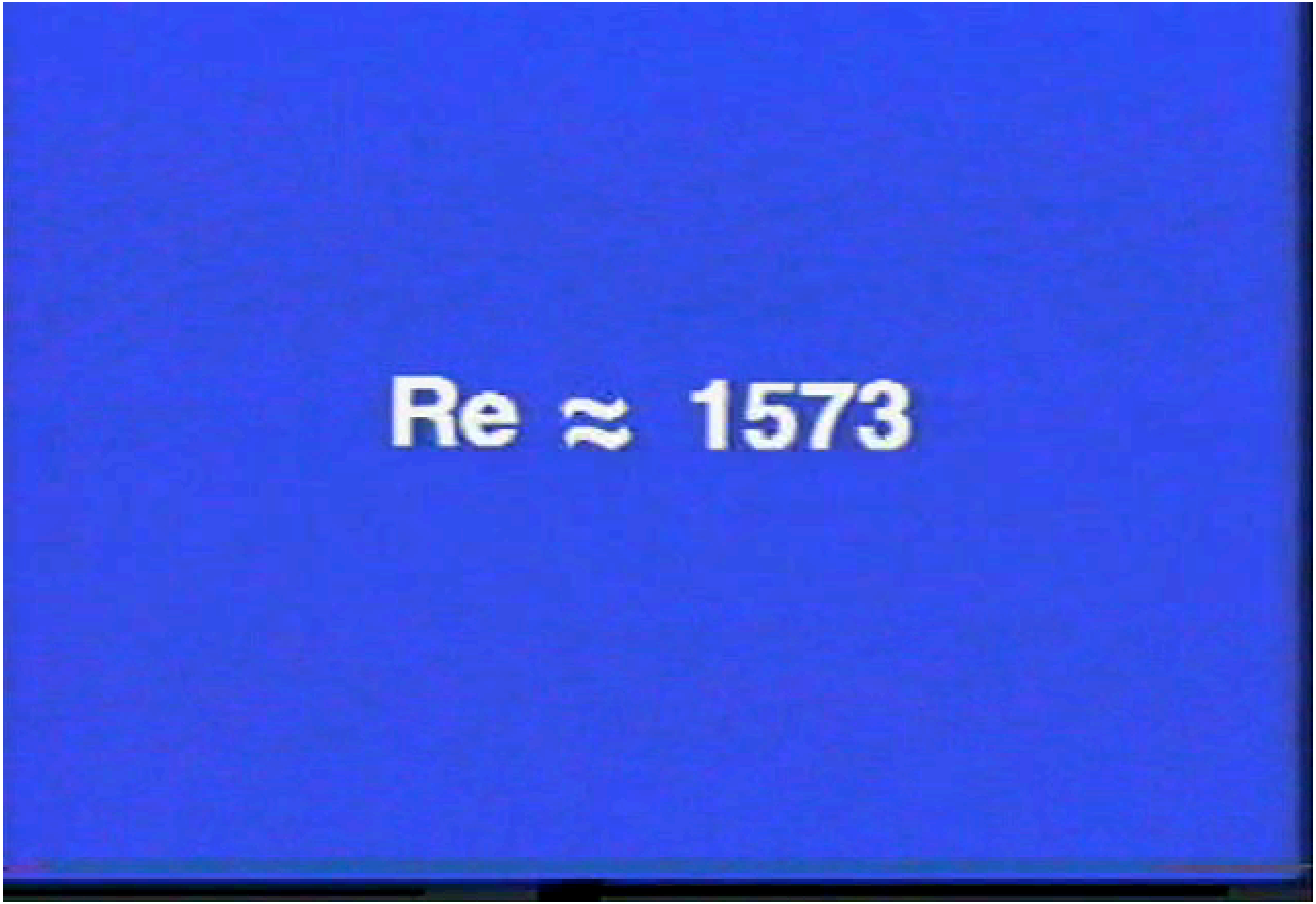
Head-on collision and breakdown of vortex rings (Lim, Nature, 1992)



Head-on collision of vortex rings (Lim, Nature, 1992): *Re=1071*



Head-on collision of vortex rings **(Lim, Nature, 1992)**: *Re=1573*



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