Dissipation and concentration of vorticity from Stochastic Quantization

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CANONICAL QUANTIZATION

Quantum Hamiltonian ($\alpha :=$ coupling parameter)

$$\mathcal{H} = \sum_{i=1}^{N} \left\{ -\frac{\hbar^2}{2m} \nabla_i^2 + \Phi(\mathbf{r}_i) \right\} + \Phi_{int}(\mathbf{r}_1, ..., \mathbf{r}_N, \alpha)$$

($\mathcal{H}$ is bounded from below)

$$i\,\hbar \partial_t \hat{\Psi} = \left(-\frac{\hbar^2}{2m} \hat{\nabla}^2 + \Phi_{tot}^{\alpha,N} \right) \hat{\Psi}$$

where $\hat{\nabla} := (\nabla_1, \ldots, \nabla_N)$ and $\Phi_{tot}^{\alpha,N} := \sum_{i=1}^{N} \Phi(\mathbf{r}_i) + \Phi_{int}(\mathbf{r}_1, \ldots, \mathbf{r}_N, \alpha)$.

The basic object is the classical lagrangian

$$\mathcal{L}[\hat{q}^{cl}] = \sum_{i=1}^{N} \left\{ \frac{1}{2} m (\dot{\mathbf{q}}_i^{cl})^2(t) - \Phi(\mathbf{q}_i^{cl}(t)) \right\} - \Phi_{int}(\mathbf{q}_1^{cl}(t), ..., \mathbf{q}_N^{cl}(t), \alpha)$$

$\hat{q}^{cl}$:= classical N- body configuration.

Quantization comes from requiring that the configuration of the system evolves as a Markov diffusion (with diffusion coefficient equal to $\frac{\hbar}{m}$) and that it makes stationary the mean classical action

- Eulerian (or " stochastic control") approach, with pre-regularization of the action : Guerra and M., Phys. Rev. D ’83.
- Lagrangian (or "path-wise") approach, with self-regularization of the action : M. , Phys. Rev. D ’85.

diffusion

i) q is a solution of the stochastic differential equation

$$dq(t) = b(q(t), t)dt + \left(\frac{\hbar}{m}\right)^{\frac{1}{2}} dW(t)$$

ii) The drift b, is smooth

The Brownian Motion W models "quantum fluctuations"

ρ := time dependent probability density of the configuration
V := current velocity field

$$b = V + \frac{\hbar}{2m}\nabla \ln \rho$$

and the continuity equation holds

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho V)$$

Eq. of motion in the Lagrangian approach

$d = 3$) (ρ, V)

$$\begin{cases} \partial_t \rho = -\nabla \cdot (\rho V) \\ \partial_t V + (V \cdot \nabla) V - \frac{\hbar^2}{2m^2} \nabla \left(\frac{\nabla^2 \sqrt{\rho}}{\sqrt{\rho}} \right) + \\ \qquad + \frac{\hbar}{2m} (\nabla \ln \rho + \nabla) \wedge (\nabla \wedge V) = -\frac{1}{m} \nabla \Phi \end{cases}$$

$d = 3N$ ($\hat{\rho}, \hat{V}$)

$$\begin{cases} \partial_t \hat{\rho} = -\hat{\nabla} \cdot \left(\hat{\rho} \hat{V} \right) \\ \left[\partial_t \hat{V} + \left(\hat{V} \cdot \hat{\nabla} \right) \hat{V} - \frac{\hbar^2}{2m^2} \hat{\nabla} \left(\frac{\hat{\nabla}^2 \sqrt{\hat{\rho}}}{\sqrt{\hat{\rho}}} \right) \right]_k + \\ \qquad + \frac{\hbar}{2m} \sum_{p=1}^{3N} (\partial_p \ln \hat{\rho} + \partial_p) \left(\partial_k \hat{V}_p - \partial_p \hat{V}_k \right) = -\frac{1}{m} \partial_k \Phi_{tot}^{\alpha, N} \qquad k = 1, \ldots, 3N \end{cases}$$

Equations of motion in Schrödinger form

$d = 3$) (for simplicity of notations)

There exists S s.t. $\Psi := \rho^{\frac{1}{2}} \mathrm{e}^{\frac{\imath}{\hbar} S}, \qquad \mathcal{A} := mV - \nabla S$

$$\begin{cases} \imath\hbar\partial_t \Psi = \frac{1}{2m}(\imath\nabla + \mathcal{A})^2 \Psi + \Phi\Psi \\ \\ \partial_t \mathcal{A} = b_* \wedge (\nabla \wedge \mathcal{A}) - \frac{\hbar}{2m} \nabla \wedge (\nabla \wedge \mathcal{A}) \end{cases}$$

$$b_* := \frac{1}{m}[\nabla S - \mathcal{A} - \frac{\hbar}{2} \nabla \ln |\Psi|^2]$$ (Loffredo, M., JMP ’89)

Relaxation to dynamical equilibrium

Energy Theorem (Loffredo and M., JMP '89; extended in 2007)($d = 3$ for simplicity of notations)

$$E[\rho, V] := \int_{\mathbb{R}^{3N}} \left(\frac{1}{2} m V^2 + \frac{1}{2} m U^2 + \Phi \right) \rho \, dr$$

with $U := \frac{\hbar}{2m} \nabla \ln \rho$ (osmotic velocity).

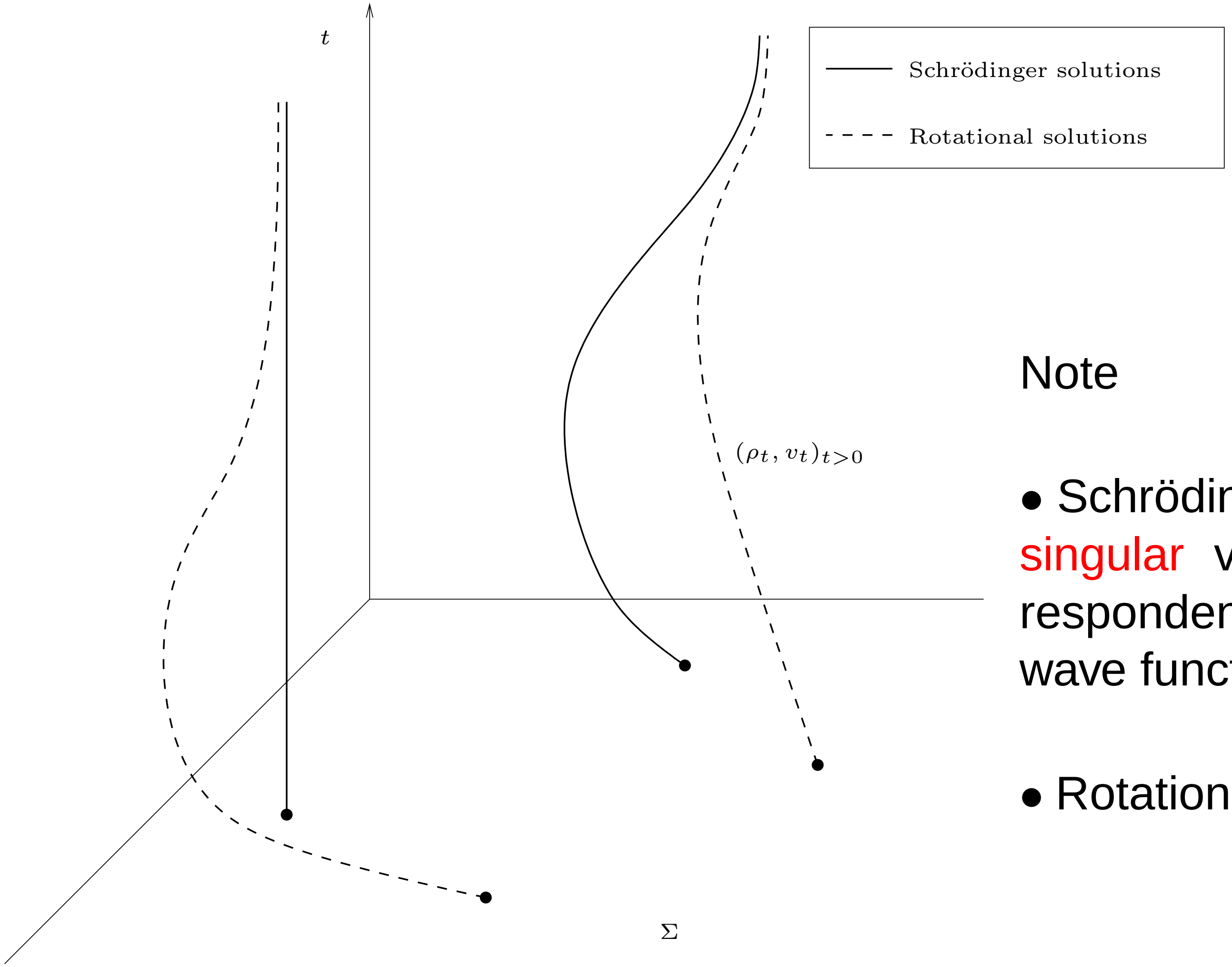
$$\frac{d}{dt} E[\rho, V] = -\frac{\hbar}{2} \int_{\mathbb{R}^{3N}} (\nabla \wedge V)^2 \rho dr$$

NOTE: Irrotational solutions conserve the energy and

$$E = < \Psi, \mathcal{H} \Psi >$$

For generic initial data Schrödinger solutions act as an attracting set, which corresponds to DYNAMICAL EQUILIBRIUM.

SCHRÖDINGER VERSUS ROTATIONAL SOLUTIONS



Note

• Schrödinger solutions : singular velocity field in correspondence of nodes of the wave function.

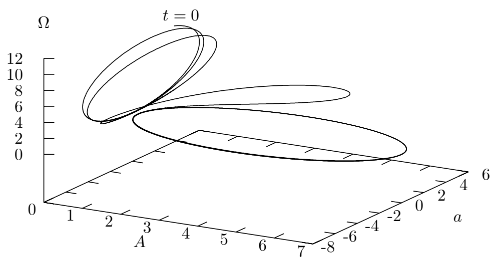
• Rotational solutions : smooth

Non trivial behavior of vorticity 1

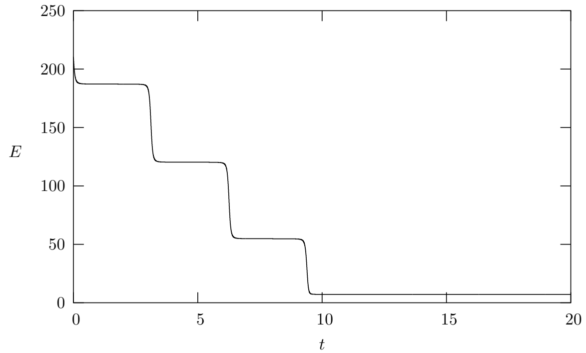
1) For $\rho > 0$: the vorticity $\nabla \wedge \mathcal{A}$ does not go to zero monotonically
(firstly conjectured by Guerra in 1992)
“gaussian solutions to the bidimensional harmonic oscillator”
(M. and Ugolini, AHP ’94) (global existence and center manifold)

$$\Phi := \Phi(r) = \frac{1}{2} k^2 r^2$$

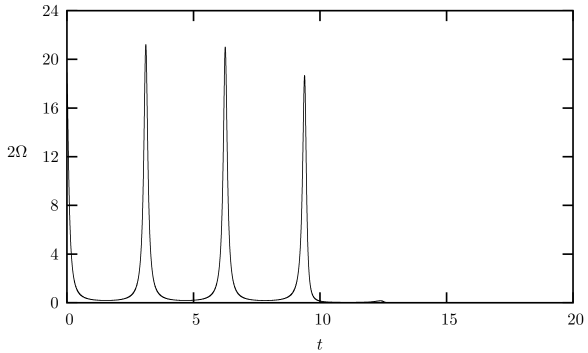
$$\rho(r,t) = \frac{A}{\pi} exp(-Ar^2), \quad V(r,t) = ar\hat{r} - \alpha r\hat{\theta}$$



trajectory



energy vs. time



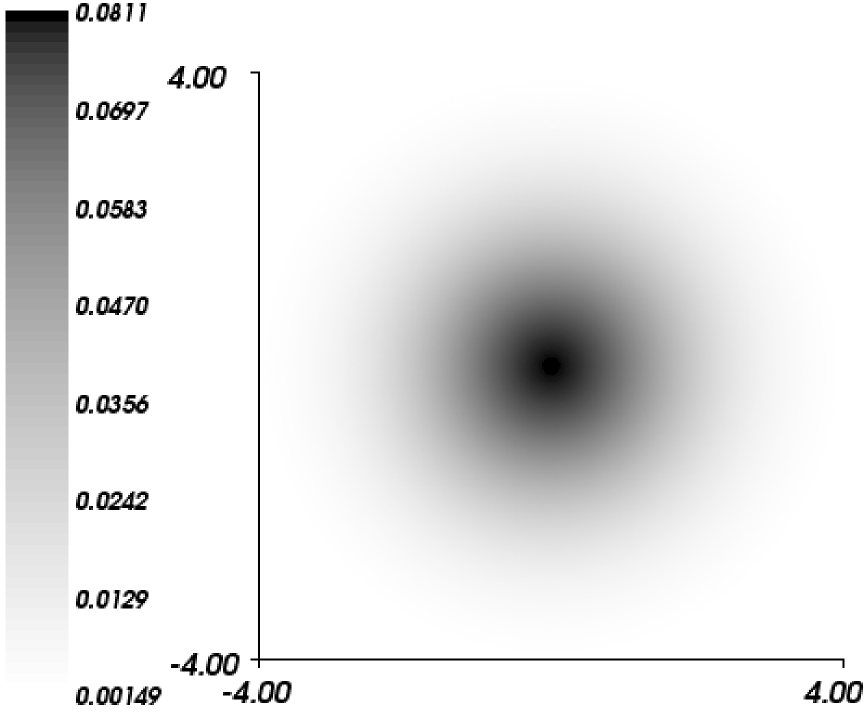
vorticity vs. time

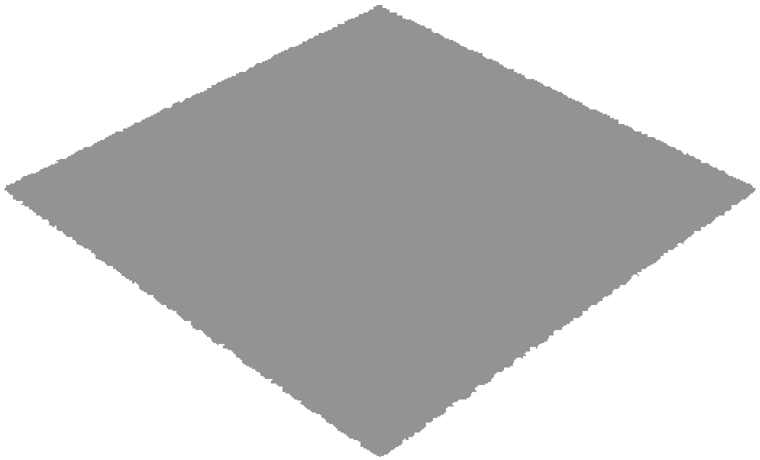
Non trivial behavior of vorticity 2

2) The vorticity can concentrate in the zeroes of the density

Non gaussian solutions to the bidimensional harmonic oscillator, numerical results:
[Caliari, Inverso and M. 2004, *New J. Phys.*, Vol 6, no. 69,
http://www.iop.org/EJ/journal/NJP]

Numerical experiment (t=0)

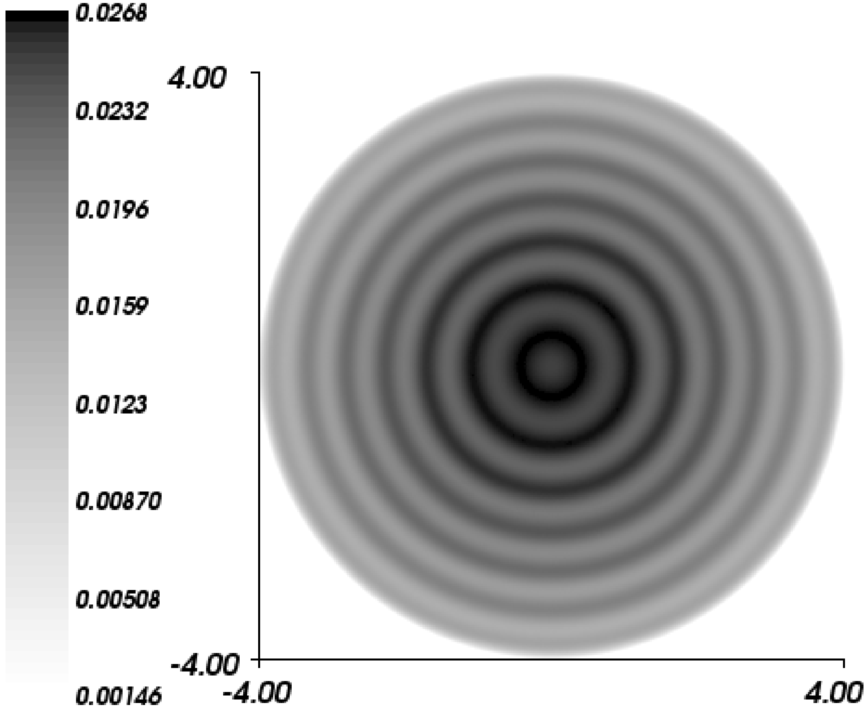




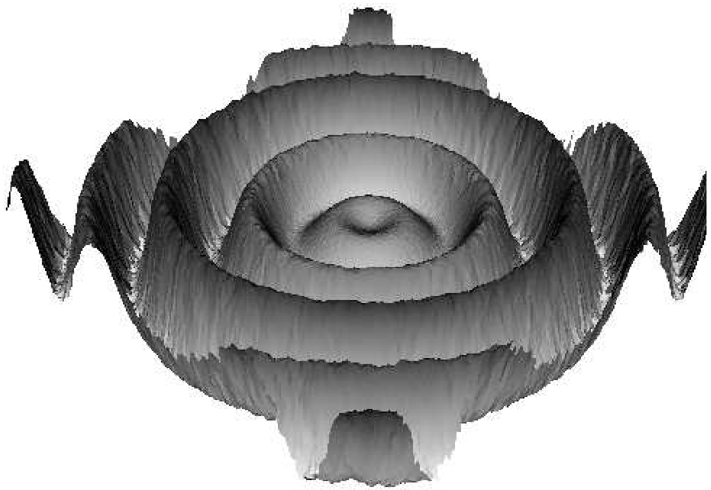
ρ at time $t = 0$, $E = 195$

$-\nabla \wedge v = 2\Omega_o$

Numerical experiment (t=0.08)

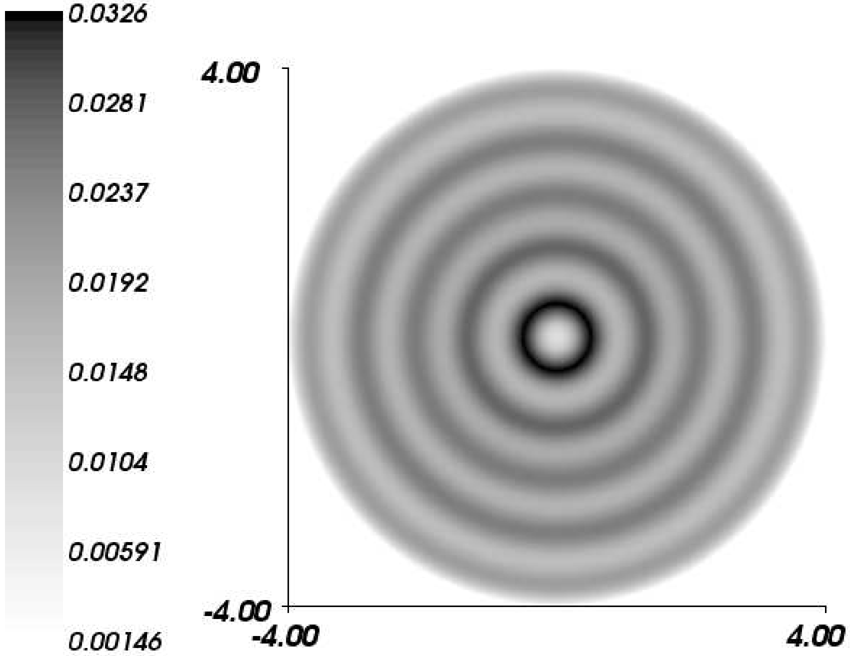


ρ at time $t = 0.08$, $E = 43$

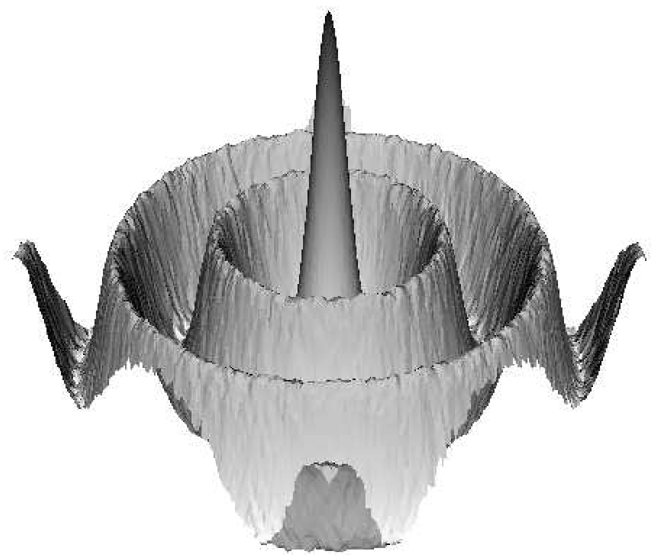


$-\nabla \wedge v$

Numerical experiment (t=0.14)

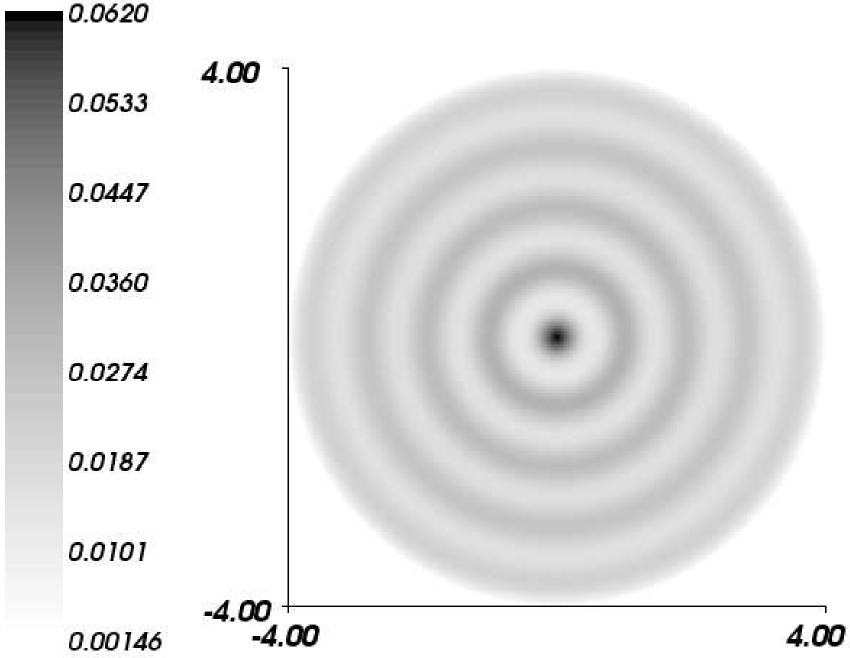


ρ at time $t = 0.14$, $E = 17$

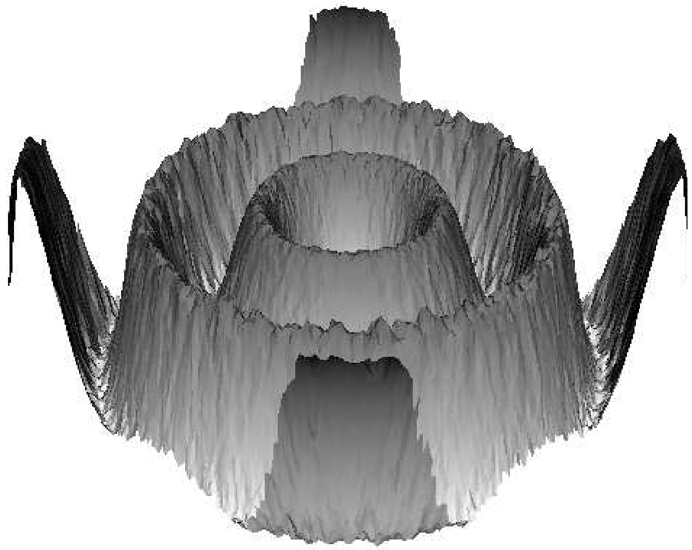


$-\nabla \wedge v$

Numerical experiment (t=0.16)

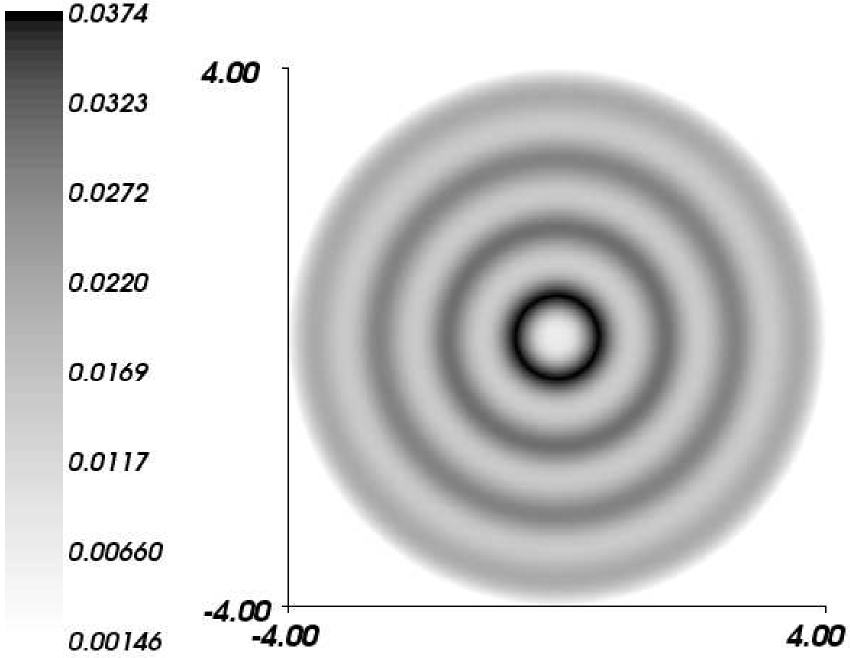


ρ at time $t = 0.16$, $E = 15$

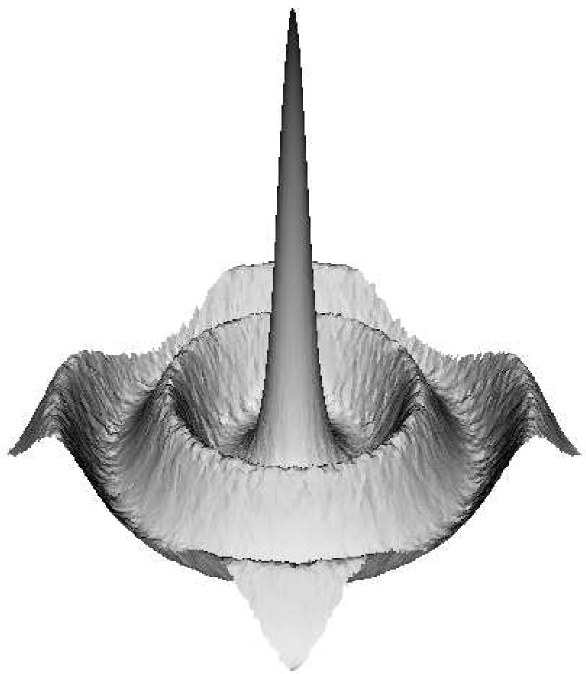


$-\nabla \wedge v$

Numerical experiment (t=0.19)

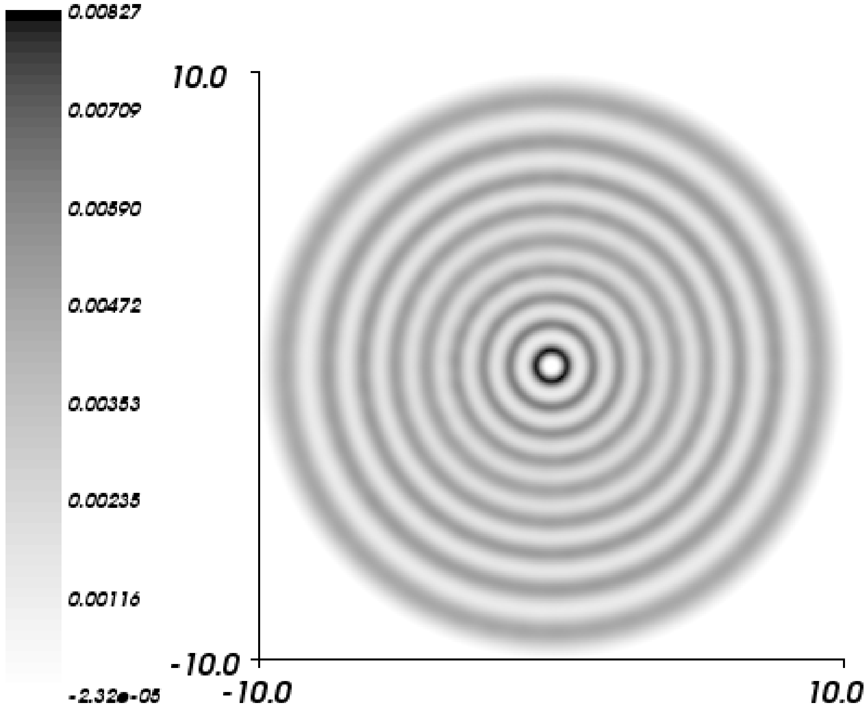


ρ at time $t = 0.19$, $E = 11$

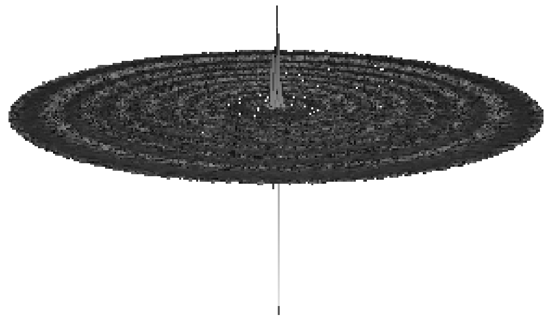


$-\nabla \wedge v$

Numerical experiment (t=0.23)



ρ at time $t = 0.23$, $E = 9$



$-\nabla \wedge v$

Cubic Schrödinger equation with vorticity

(Caliari,Loffredo , M. and Zuccher : New.J. Phys. (10) 2008) (for the connection with the N-body problem see some results in Loffredo and M. *Journal of Physics A: Mathematical and Theoretical* **40** 2007)

- $\Psi := \rho^{\frac{1}{2}} \mathrm{e}^{\frac{\imath}{\hbar} S}, \qquad \mathcal{A} := mV - \nabla S$

$$\begin{cases} \imath\,\hbar \partial_t \Psi = \frac{1}{2m}(\imath \nabla + \mathcal{A})^2 \Psi + (g|\Psi|^2 + \Phi)\Psi \\ \partial_t \mathcal{A} = b_* \wedge (\nabla \wedge \mathcal{A}) - \frac{\hbar}{2m} \nabla \wedge (\nabla \wedge \mathcal{A}) \end{cases}$$

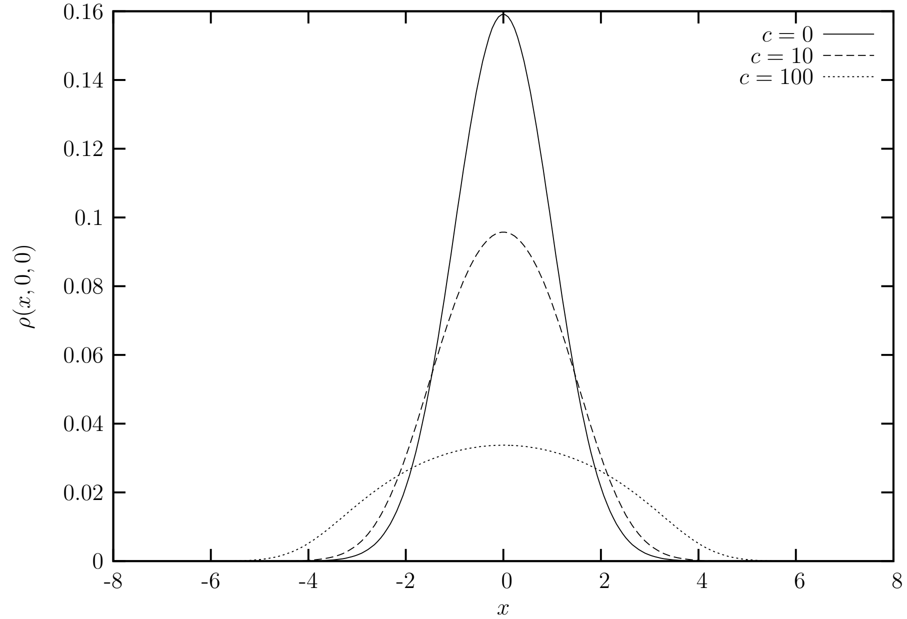
$$b_* := \frac{1}{m}[\nabla S - \mathcal{A} - \frac{\hbar}{2} \nabla \ln |\Psi|^2]$$

Extension of the energy theorem (dissipation independent of g !)

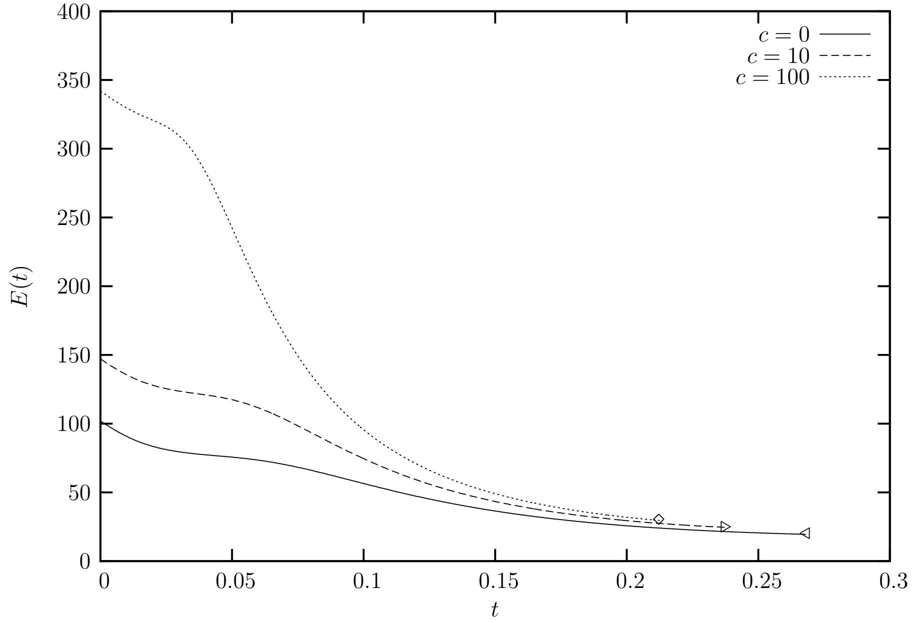
$$\mathcal{E}_{\mathcal{A}}[\rho, S, t] := \left[\frac{(\nabla S - \mathcal{A})^2}{2} + \frac{\hbar^2}{2m} (\nabla \log \rho)^2 + \Phi + \frac{g}{2}\rho \right] \rho$$

$$\frac{d}{dt} \int \mathcal{E}_{\mathcal{A}}[\rho, S, t] dr = -\frac{\hbar}{2} \int (\nabla \wedge \mathcal{A})^2 \rho dr$$

exp

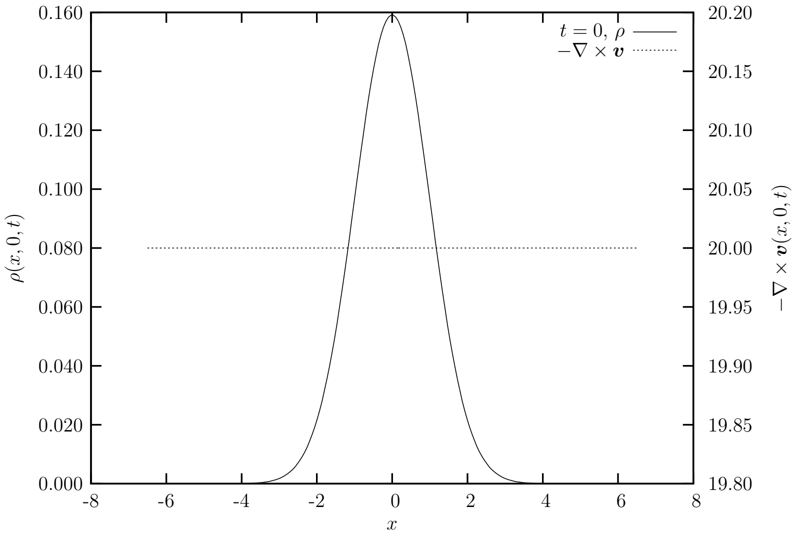


Profiles of the two-dimensional ground state densities (t=0) for different values of the coupling constant

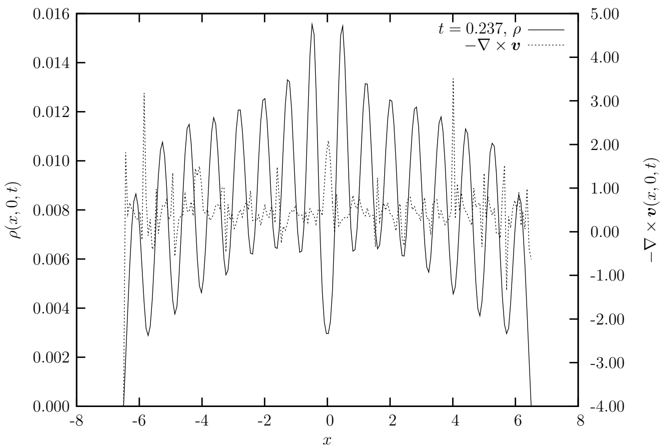


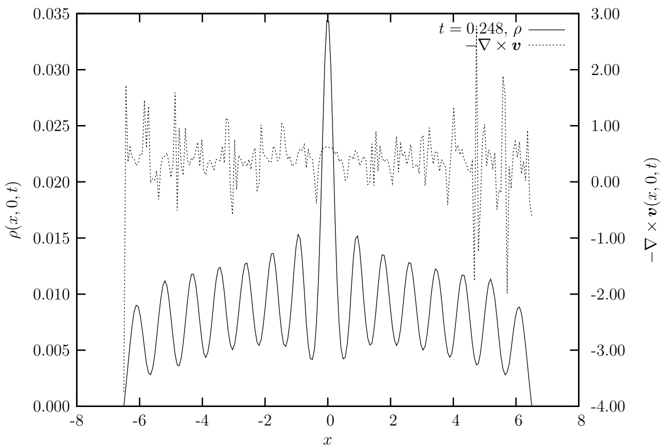
Energy as a function of time

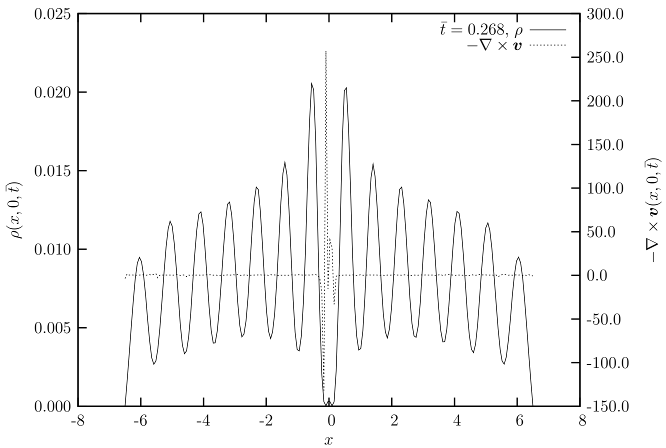
exp1



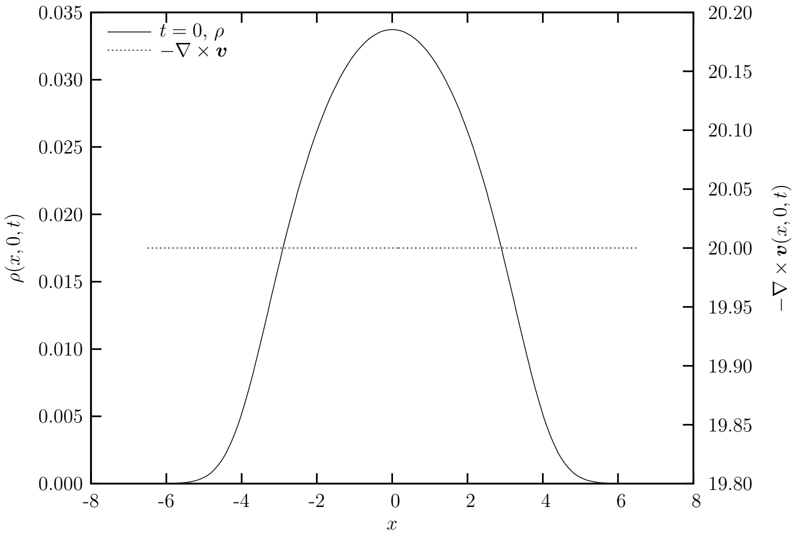
Density versus vorticity ($c = 0$)



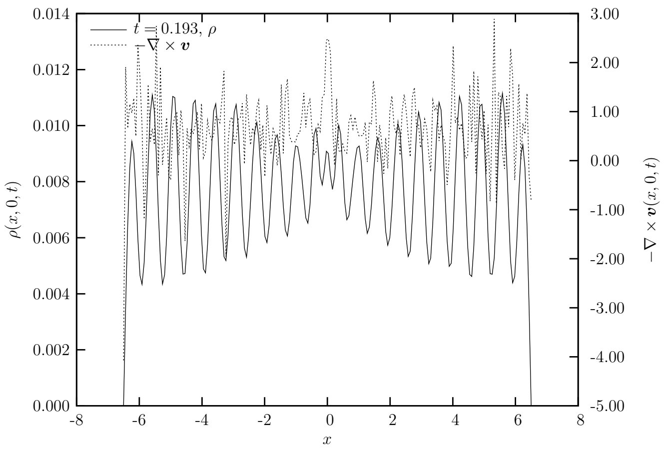


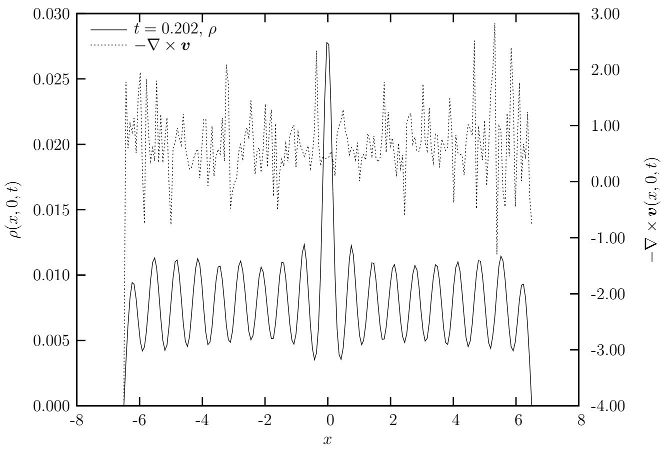


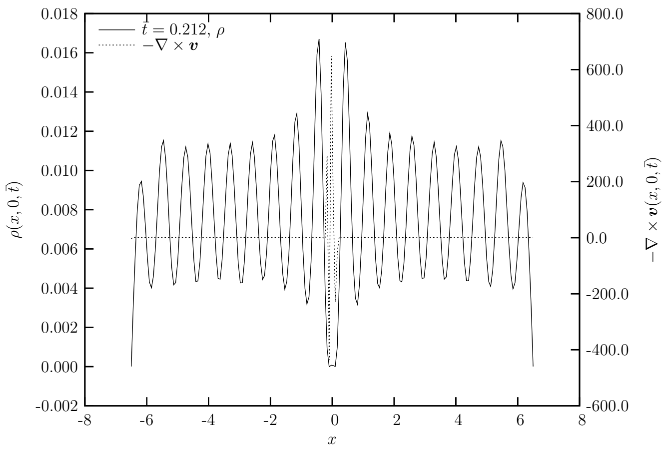
exp2



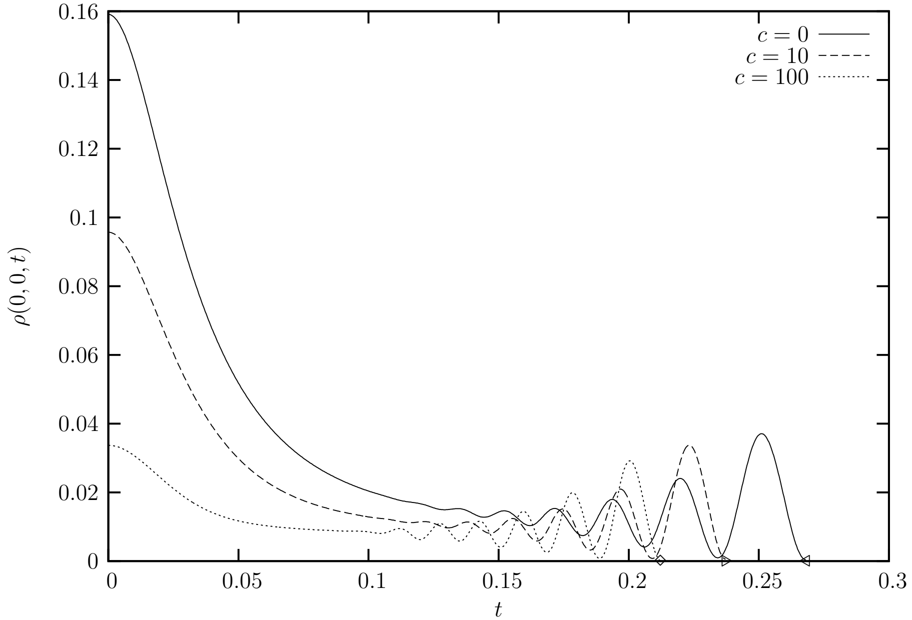
Density versus vorticity ($c = 100$)



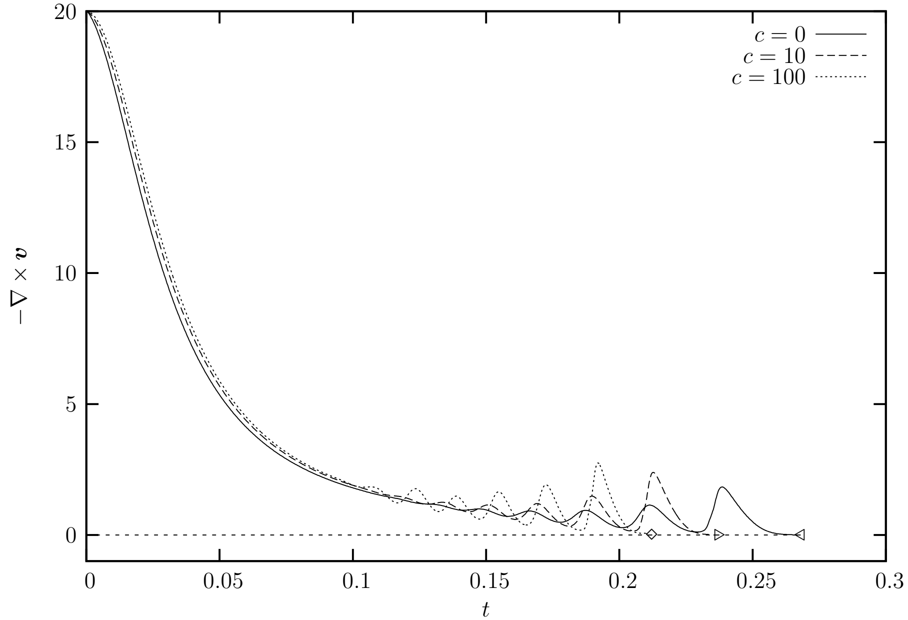




exp3



Density at the origin as a function of time



Vorticity $(-\nabla \wedge v)$ at the origin as a function of time

comment

Application to conceptually simple experiments where the condensation occurs after the stirring of a normal cloud ?

(P.C. Haljan, I. Coddington, P. Engels and E.A. Cornell*, Phys. Rev. Lett. 87, 210403 (2001))