The Effect of the Trapping Potential on the Behaviour of a Fast Rotating Condensate

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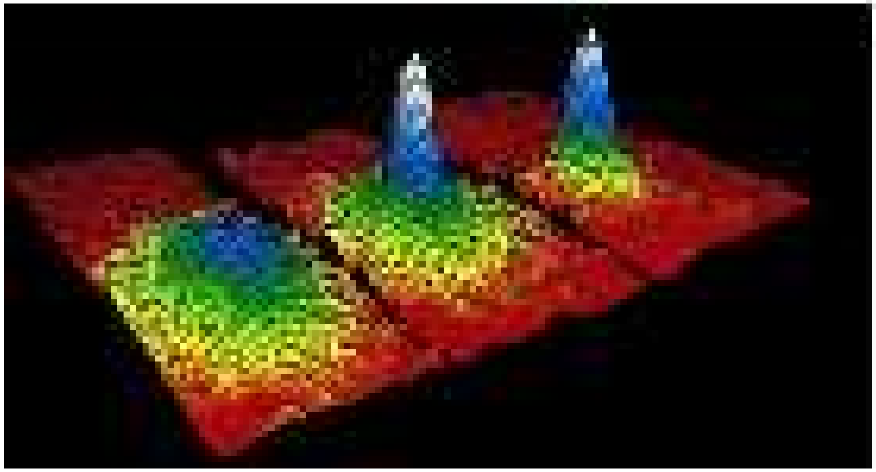

Work in collaboration with Amandine Aftalion: Project ANR VoLQuan

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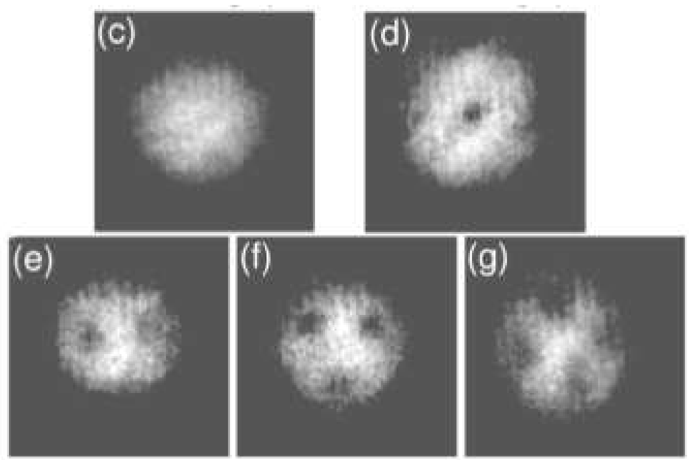
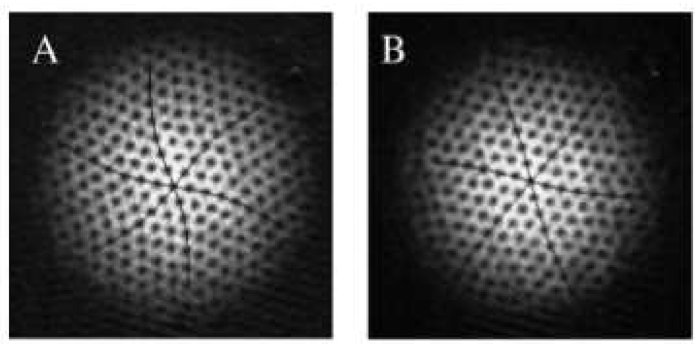
Introduction

Experimental examples of rotating condensates: Harmonic Traps



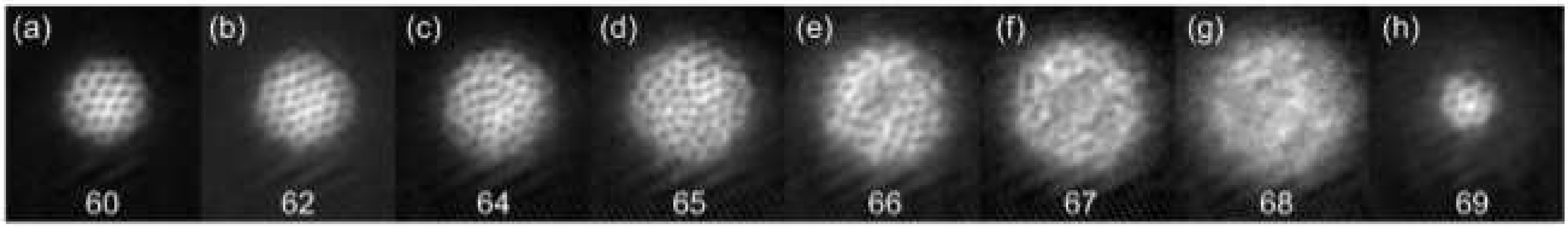
[Madison et al. PRL, 2000]



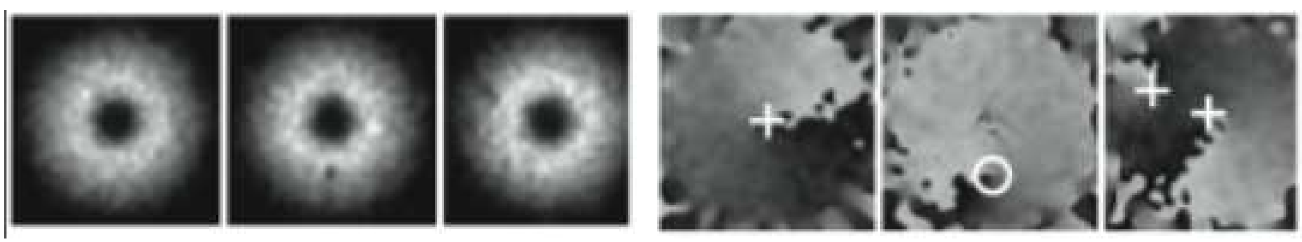
[Coddington et al. PRL, 2003]

Introduction

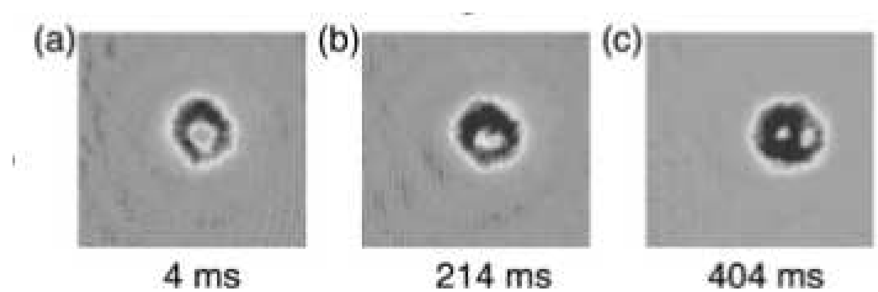
More experimental examples of rotating condensates: Harmonic Plus Gaussian traps



[Bretin et al. PRL, 2000]



[Weiler et al. Nature, 2008]



[Ryu et al. PRL, 2007]

Outline of the Talk

1. Fast Rotation in Harmonic Traps

- Minimisation of the Gross-Pitaevskii energy
- Lowest Landau level (LLL) analysis

2. Fast Rotation in Toroidal Traps (Annular condensates)

- Harmonic + Quartic and Harmonic + Gaussian traps
- LLL analysis vs. Thomas-Fermi analysis

Section 1: Fast Rotation in Harmonic Traps

The non-dimensional Gross-Pitaevskii energy is

$$E[\psi] = \int \left(\psi^* [H_\Omega \psi] + \frac{g}{2} |\psi|^4 \right) \quad d^2 r$$

for Hamiltonian

$$H_\Omega = -\frac{1}{2}\nabla^2 + \frac{r^2}{2} - \Omega L_z = -\frac{1}{2}(\nabla - iA)^2 + (1 - \Omega^2)\frac{r^2}{2}$$

K.E. P.E. rotation energy Coroilis restoring centrifugal

where $L_z = i(y\partial_x - x\partial_y)$ and $\boldsymbol{A} = \boldsymbol{\Omega} \times \boldsymbol{r}$ with $\boldsymbol{\Omega} = (0, 0, \Omega)$ and $\boldsymbol{r} = (x, y, 0)$.

The Landau Levels

A common eigenbasis of L_z and H_Ω is the Hermite functions

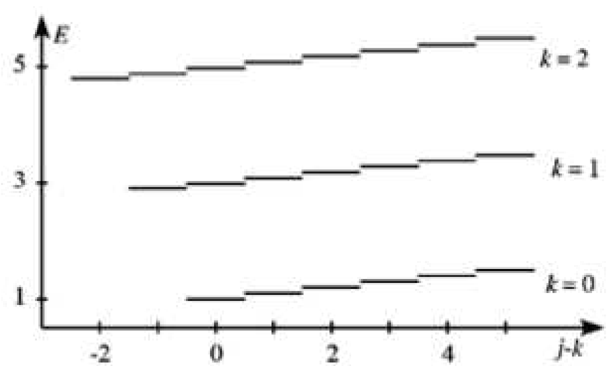
$$\phi_{j,k} = e^{r^2/2}(\partial_x + i\partial_y)^j(\partial_x - i\partial_y)^k(e^{-r^2}).$$

The eigenvalues for L_z are $j - k$, while for H_Ω, they are

$$E_{j,k} = 1 + (1 - \Omega)j + (1 + \Omega)k.$$

Suppose $\Omega = 1$. These are the Landau levels.

If $\Omega \sim 1$ then two adjacent levels are separated by ~ 2. However the distance between two adjacent states is $1 - \Omega \ll 1$.



The Lowest Landau Level

We are interested in the lowest energy state: the lowest Landau level (LLL). This occurs when $k = 0$.

Any function ψ of the LLL is a linear combination of the $\phi_{j,0}$'s and we can write

$$\psi(\boldsymbol{r}) = e^{-r^2/2} P(u) = e^{-r^2/2} \prod_{j=1}^{n} (u - u_j)$$

for P an analytic function of $u = x + iy$ and u_j the n complex zeros of P.

Each u_j is the position of a vortex!

Link to Ginzberg-Landau Problems

The GP energy can be rewritten by introducing a parameter ϵ,

$$\epsilon = \left(\frac{1}{4g}\right)^{2/5}$$

If we rescale $R = \frac{1}{\sqrt{\epsilon}} r$ and $u(r) = R^{3/2}\psi(Rr)$ then we get

$$\begin{aligned} E[u] &= \int \frac{1}{2}|\nabla u|^2 - \Omega L_z + \frac{1}{2\epsilon^2} r^2 |u|^2 + \frac{1}{4\epsilon^2}|u|^4 \\ &= \int \frac{1}{2}|\nabla u|^2 - \Omega L_z + \frac{1}{4\epsilon^2}\left(|u|^2 - \rho_t\right)^2 \end{aligned}$$

where $\rho_t = \rho_0 - r^2$.

Thomas-Fermi analysis considers the limit $\epsilon \to 0$. However the LLL considers the asymptotic limit $\Omega \to 1$.

Minimisation of the GP Energy (1)

Remember the GP energy is

$$E[\psi] = \int \left(\psi^* [H_\Omega \psi] + \frac{g}{2} |\psi|^4 \right) \quad d^2 r$$

for which the wave function ψ minimising $E[\psi]$ is

$$H_\Omega \psi(\boldsymbol{r}) + g|\psi(\boldsymbol{r})|^2 \psi(\boldsymbol{r}) = \mu \psi(\boldsymbol{r})$$

where the chemical potential μ can be determined from the normalisation condition.

We can then write the energy in the LLL as

$$E_{LLL} = \Omega + \int \left((1 - \Omega) r^2 |\psi|^2 + \frac{g}{2} |\psi|^4 \right) \quad d^2 r$$

Minimisation of the GP Energy (2)

The minimisation of

$$E_{LLL} = \Omega + \int \left((1-\Omega) r^2 |\psi|^2 + \frac{g}{2} |\psi|^4 \right) \quad d^2 r$$

is equivalent to minimising

$$E[\psi] = \frac{E_{LLL}[\psi] - \Omega}{1-\Omega} = \int r^2 |\psi|^2 + \frac{\lambda}{2} |\psi|^4 \quad d^2 r$$

where $\lambda = \frac{g}{1-\Omega}$.

Minimisation of E_{LLL} depends only on one parameter λ: a combination of g and Ω.

We get that

$$\min[E] = \frac{2\sqrt{2}}{3\sqrt{\pi}} \sqrt{\lambda}$$

Minimisation of the GP Energy (3)

The density and radius of the disk are then

$$
\begin{aligned}
|\psi|^2 &= \frac{2}{\pi R_0^2}\left(1 - \frac{r^2}{R_0^2}\right) \\
R_0 &= \left(\frac{2\lambda}{\pi}\right)^{1/4}
\end{aligned}
$$

This is an inverted parabola! However it is not in the space of minimisation: to alleviate this we need to add vortices.

Note

- The LLL analysis is valid for $g(1-\Omega) \ll 1$ and is obtained by balancing K.E. and P.E. terms. We have that the vortex size is of the same order as the intervortex distance (i.e. interaction between vortices becomes important). Vortices need not be small.

- In contrast a TF analysis neglects the K.E. and finds a balance between the P.E. and interaction energy. We require large g so that vortices are small.

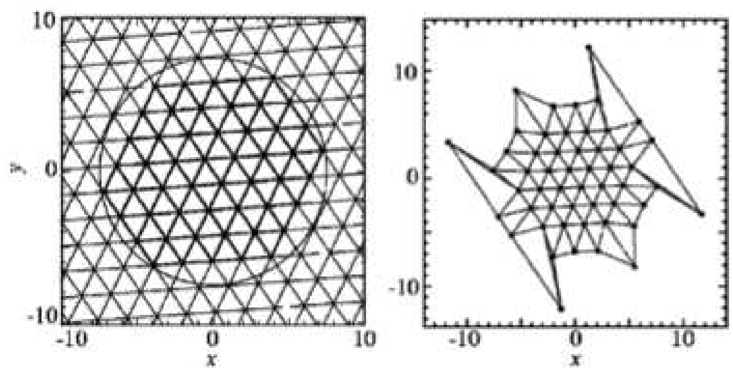
Gaussian or Inverted Parabola?

Is the lattice regular or distorted?

A regular lattice provides a Gaussian decay. [Ho, PRL 2001].

However the optimum energy is obtained for the inverted parabola. In this case the lattice is distorted towards the edges and extends to infinity. The mean behaviour is that of an inverted parabola. To get an inverted parabola in the LLL, we require vortices.

regular ⟶



⟵ distorted (lower energy)

So a distorted lattice can change the decay to an inverted parabola and improve the energy. [Wanatabe et al. PRL 2004; Cooper et al. PRA 2004; Aftalion et al. PRA 2005].

The lattice extends to infinity.

Section 2: Fast Rotation in Toroidal Traps

Introducing an extra term to the harmonic trap removes the possibility of a singularity occurring when $\Omega \to 1$. There are two types of trap that can create an annular condensate;

- Harmonic Plus Quartic: $V(r) = \pm\frac{1}{2}r^2 + kr^4$

- Harmonic Plus Gaussian: $V(r) = \frac{1}{2}r^2 + A\exp(-l^2r^2)$

for constants k, l and A.

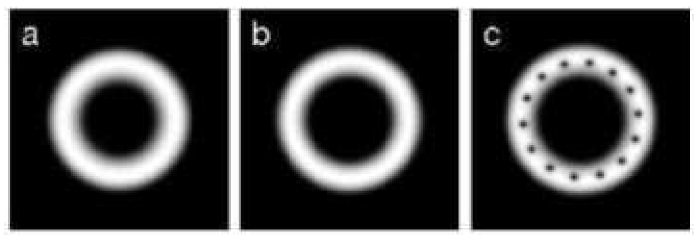
The expression for the energy is

$$\int \frac{1}{2}|\nabla\psi|^2 + V(r)|\psi|^2 + \frac{1}{2}g|\psi|^4 - \Omega L_z$$

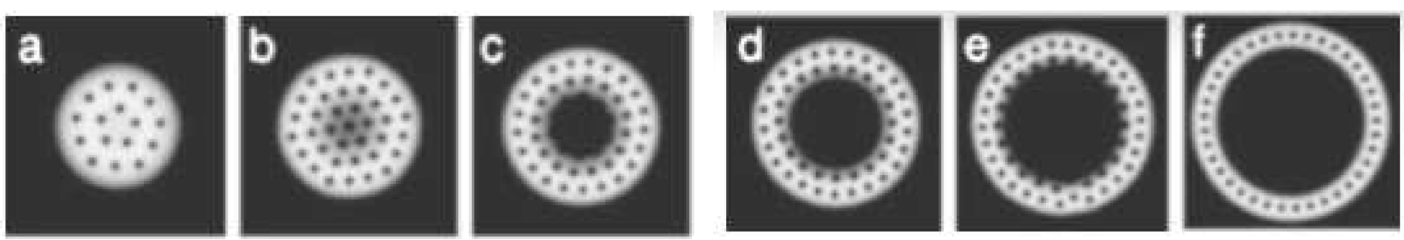
Harmonic Plus Quartic Traps

$V(r) = \pm\frac{1}{2}r^2 + kr^4$

If we take the negative sign (a 'Mexican hat' potential) then the condensate is always annular (provided $\mu < 0$), irrespective of the angular rotational velocity. [Cozzini et al. PRA, 2006].



If we take the positive sign then the condensate ground state at $\Omega = 0$ is circular. As Ω increases then an annulus will develop. [Fetter et al. PRA, 2005].



Ω increasing $\longrightarrow$

Thomas-Fermi analysis (large g)

We can work with the normalisation condition:

$$\begin{aligned} 1 &= \int_0^{R_2} |\psi|^2 \quad r dr d\theta \\ \Rightarrow \Omega_c^2 &= 1 + 2\sqrt{k}\left(\frac{3\sqrt{k}g}{2\pi}\right)^{1/3}. \end{aligned}$$

Thus Ω_c is the value of the rotational velocity at which the central hole first appears. If $\Omega > \Omega_c$ we can again use the normalisation condition, now integrating $\int_{R_1}^{R_2}$ to give that

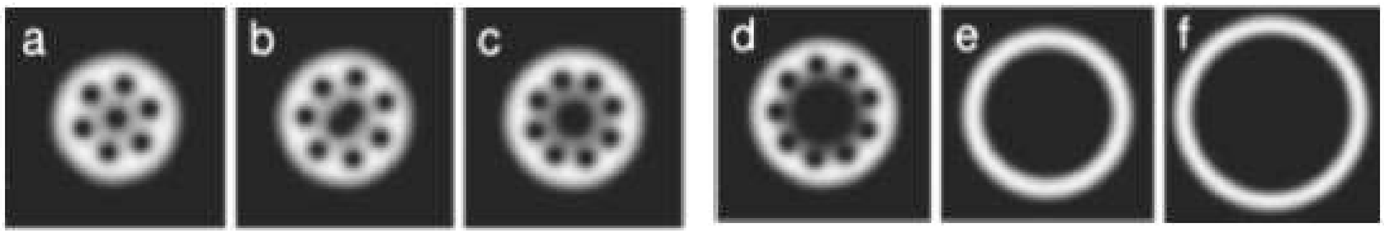
$$\begin{aligned} \text{Area} &= \pi(R_2^2 - R_1^2) = \frac{2\pi}{\sqrt{k}}\left(\frac{3\sqrt{k}g}{2\pi}\right)^{1/3} \qquad \leftarrow \text{constant!} \\ \text{Width} &= R_2 - R_1 = \left(\frac{(\Omega_c^2 - 1)^2}{k[\Omega^2 - 1 + \sqrt{(\Omega^2 - 1)^2 - (\Omega_c^2 - 1)^2}]}\right). \end{aligned}$$

[Fetter et al. PRA, 2005]

Numerical Examples

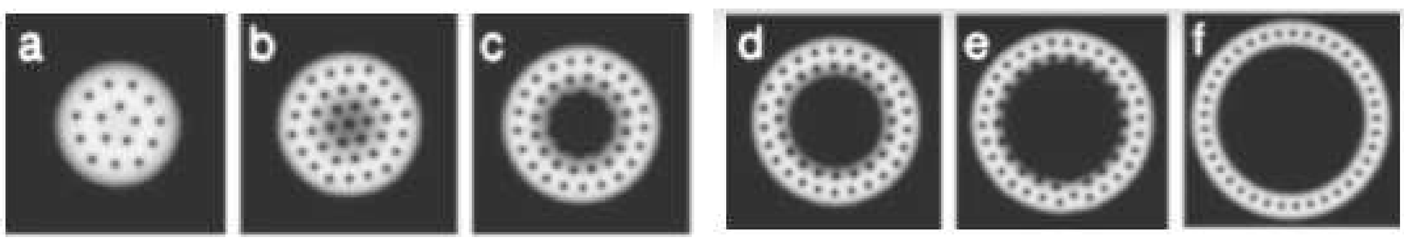
Examples from Fetter et al. PRA, 2005.

1. Low interaction strength $g = 80$ and $k = 0.5$.



Ω increasing $\longrightarrow$

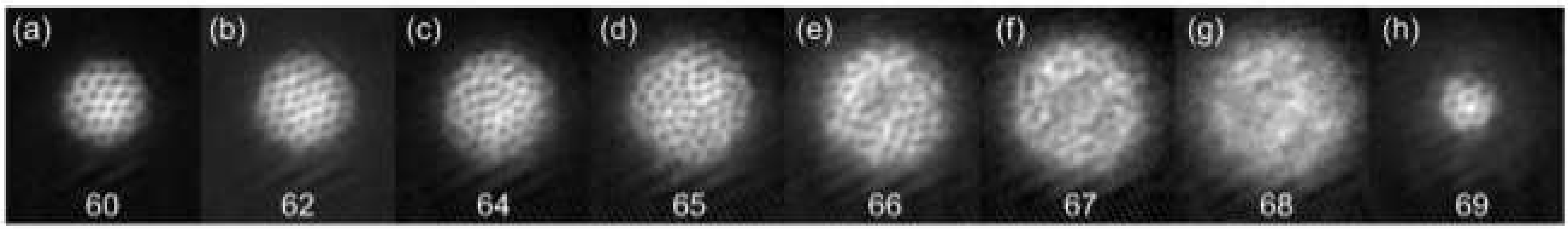
2. High interaction strength $g = 1000$ and $k = 0.5$.



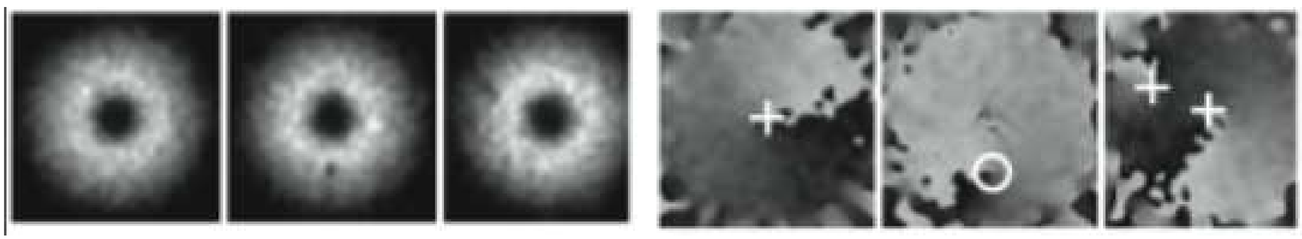
Ω increasing $\longrightarrow$

Harmonic Plus Gaussian Traps

Experimental Examples of Vortices in Toroidal Condensates:



[Bretin et al. PRL, 2004]



[Weiler et al. Nature, 2008]

All these examples use a harmonic plus Gaussian potential trap

$$V(r) = \frac{1}{2}r^2 + A\exp(-l^2r^2).$$

Inner Boundary Existence

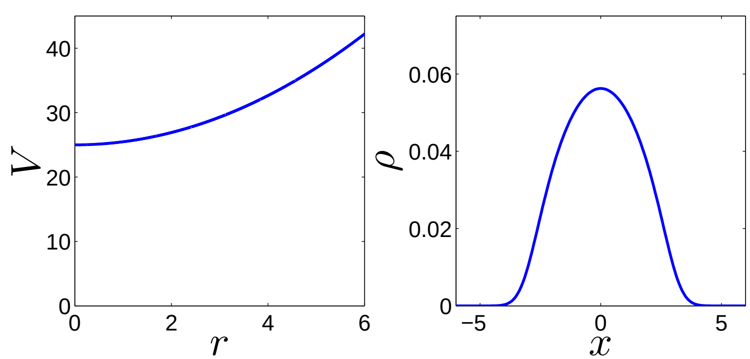
The existence of the inner boundary comes from the effective potential:

$$V(r) = A\exp(-l^2 r^2) + (1-\Omega^2)r^2/2$$

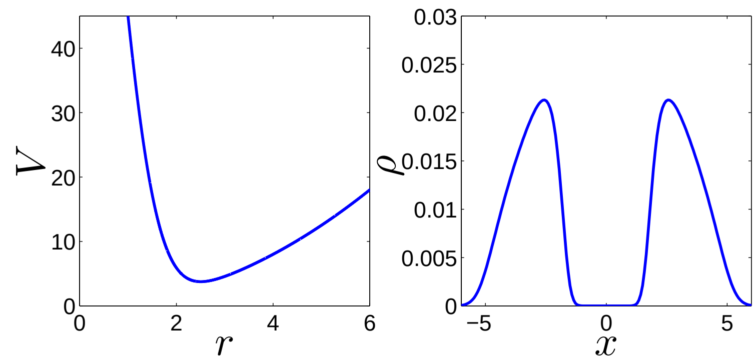
Look for the minimum:

$$\frac{\partial V}{\partial r} = 0 \quad \Rightarrow \quad r^2 = \frac{1}{l^2}\log\left(\frac{2Al^2}{1-\Omega^2}\right)$$

$$\Rightarrow \quad \frac{2Al^2}{1-\Omega^2} > 1+\delta \qquad \text{for an inner boundary to exist}$$



$g = 100,\ A = 25,\ l = 0.03$



$g = 500,\ A = 100,\ l = 0.9$

Disk Condensate

Suppose that A, l and Ω are such that the condensate is a disk. Thus $\frac{2Al^2}{1-\Omega^2} < 1+\delta$. Since Ω not large, then we can use the TF approximation:

$$g|\psi|^2 = \mu + \frac{1}{2}[(1-\Omega^2) - A\exp(-l^2r^2)]$$

The normalisation condition: $\int_0^{R_2} |\psi|^2 r dr d\theta = 1$. Substitute the TF density expression to see

$$\frac{g}{\pi} = \frac{1}{4}(1-\Omega^2)R_2^4 + \frac{A}{2}\left(R_2^2 e^{-l^2R_2^2} + \frac{1}{l^2}(e^{-l^2R_2^2} - 1)\right)$$

There are two solutions depending on the size of $l^2R_2^2$.

Suppose $l^2R_2^2$ small: then $R_2 \sim \left[\frac{4g}{\pi(1-\Omega^2-2Al^2)}\right]^{1/4}$

Suppose $l^2R_2^2$ large: then $R_2 \sim \left[\frac{4}{(1-\Omega^2)}\left(\frac{g}{\pi} + \frac{A}{l^2}\right)\right]^{1/4}$

Annular Condensate (1)

As $\Omega \to 1$, or we choose different parameters A, l then the condensate is annular. Thus $\frac{2Al^2}{1-\Omega^2} > 1 + \delta$.

LLL analysis: Start again with the TF density expression

$$g|\psi|^2 = \mu + \frac{1}{2}[(1-\Omega^2) - A\exp(-l^2r^2)]$$

and use the normalisation condition integrating $\int_{R_1}^{R_2}$. This gives

$$\frac{g}{2\pi} = \frac{\mu}{2}(R_2^2 - R_1^2) + \frac{(\Omega^2-1)}{8}(R_2^4 - R_1^4) + \frac{A}{2l^2}\left(e^{-l^2R_2^2} - e^{-l^2R_1^2}\right)$$

Define the 'area' of the annulus $X = R_2^2 - R_1^2$ such that

$$g = \frac{\pi X(\Omega^2-1)}{2}\left[\frac{Xe^{-l^2X}}{e^{-l^2X}-1} + \frac{1}{l^2} - \frac{X}{2}\right].$$

It turns out that the important parameter is $\frac{gl^4}{\pi(1-\Omega^2)}$.

Annular Condensate (2)

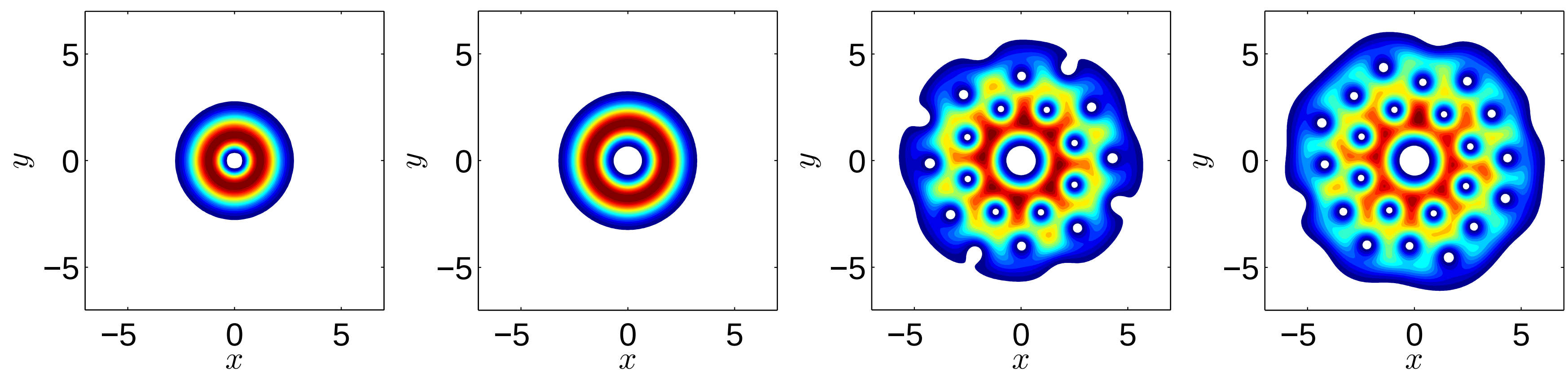
If $\frac{gl^4}{\pi(1-\Omega^2)}$ is large (equivalent to l^2X large) then

$$\begin{aligned} R_1 &\sim \frac{1}{l}\left[\ln\left(A\sqrt{\frac{\pi}{g(1-\Omega^2)}}\right)\right]^{1/2} \\ R_2 &\sim \left(\frac{4g}{\pi(1-\Omega^2)}\right)^{1/4} \\ d &\sim \left(\frac{2g}{\pi(1-\Omega^2)}\right)^{1/4}. \end{aligned}$$

Thus R_1, R_2 and d all become large (note that d and R_2 have the same form as $\Omega \to 1$, while R_1 has a slower rate of growth).

Remember that for validity of the LLL, we need $g(1-\Omega) \ll 1$. In essence g small and $\Omega \to 1$. So if $\frac{gl^4}{\pi(1-\Omega^2)}$ is small (equivalent to l^2X small) then this is violated.

Some Pictures (1)

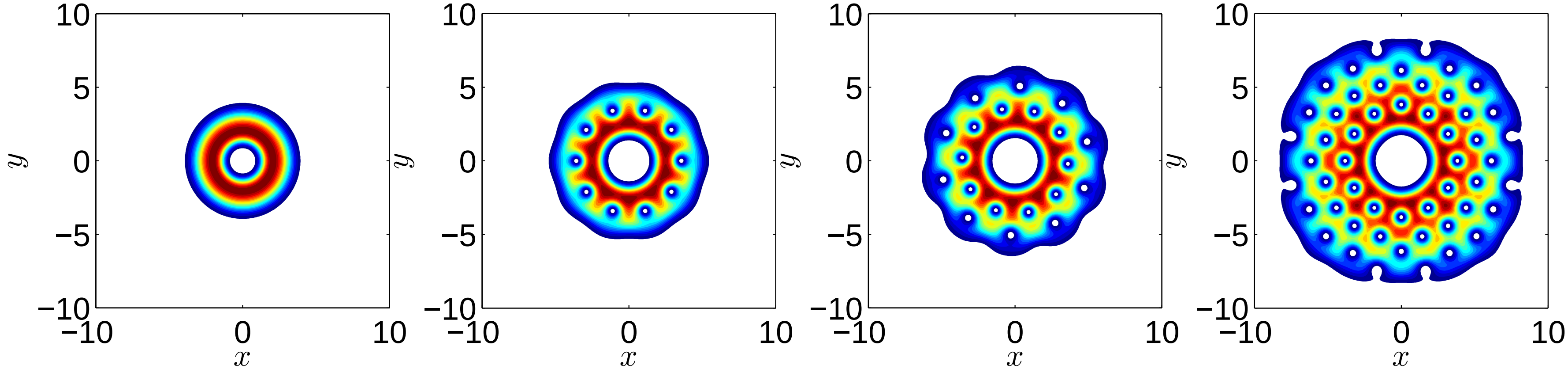


Ω increasing $\longrightarrow$

$g = 14$, $A = 1000$, $l = 5$ [g small].

Expansion of condensate cloud with rings of vortices around central hole.

Some Pictures (2)



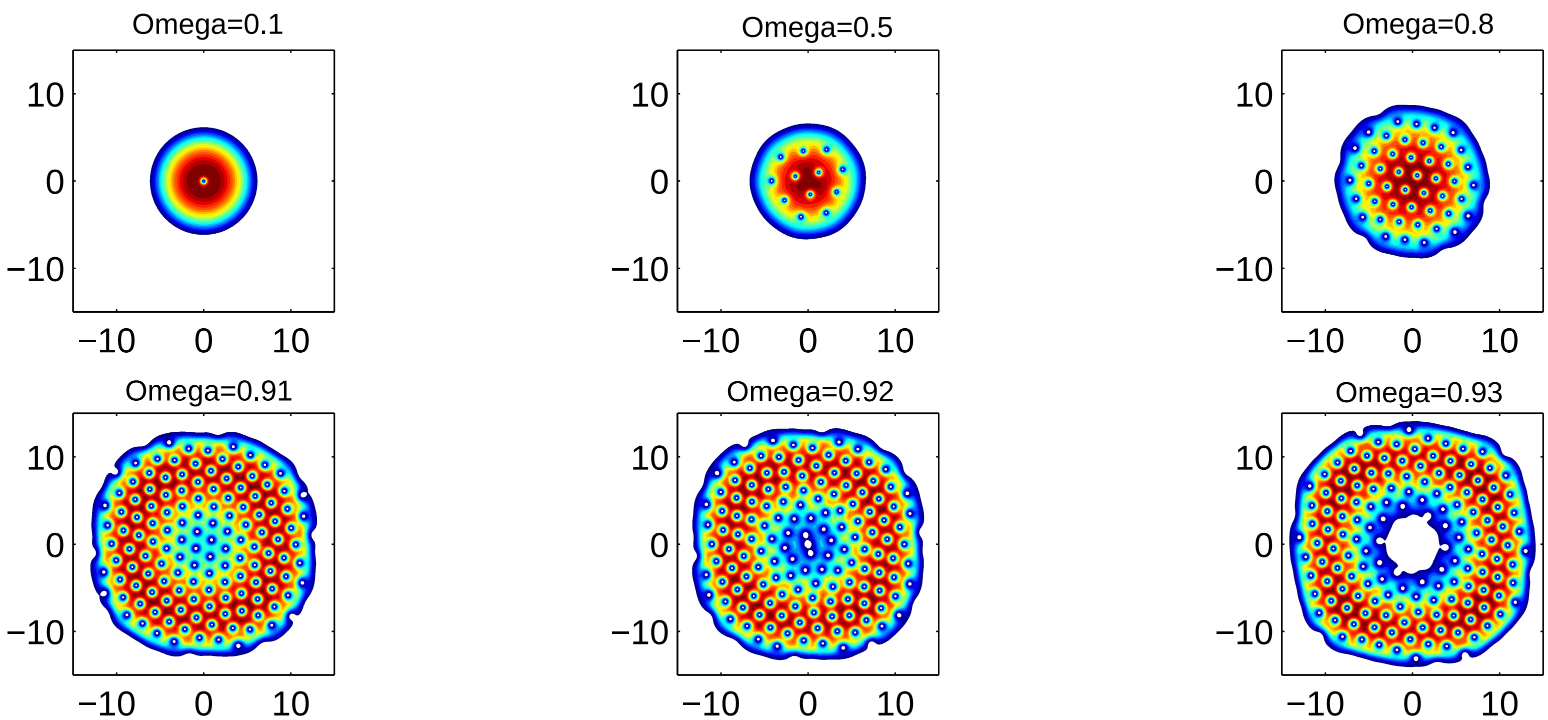
Ω increasing $\longrightarrow$

$g = 100$, $A = 25$, $l = 1$ [g large].

Qualitatively the same physics for large g.

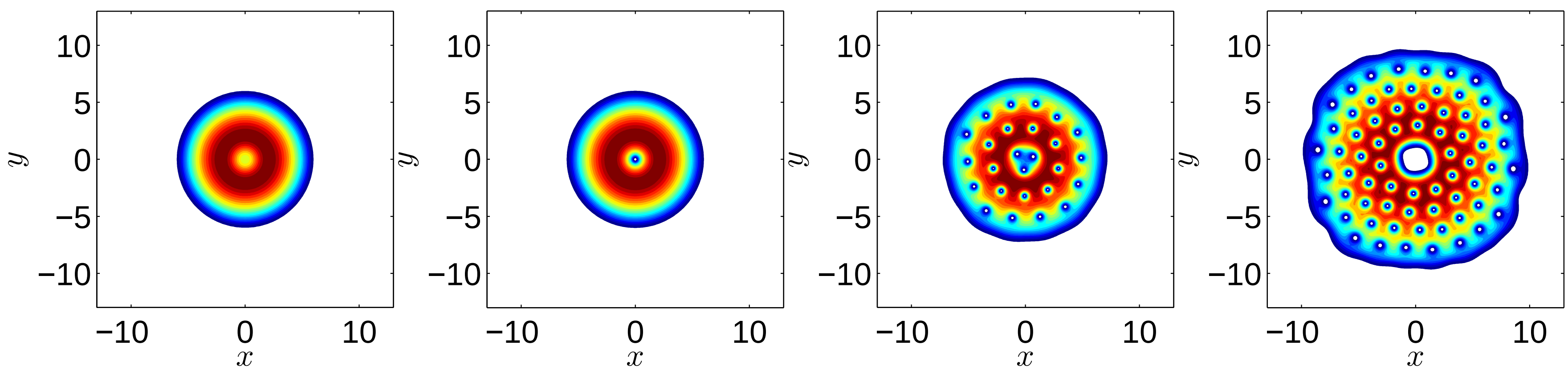
Some Pictures (3)

$g = 955.95$, $A = 24.83$, $l = 0.07$ [parameters of the Bretin et al. PRL (2004) experiment]



Development of a central hole as Ω is increased.

Some Pictures (4)



Ω increasing $\longrightarrow$

$g = 1000$, $A = 10$, $l = 0.75$ [g large].

The parameters create a local minimum of the density at the centre of the condensate. Numerically we find the single vortex state (at the centre) has a lower energy than a ring of vortices. As Ω increases we see the same qualitative behaviour as before.

Conclusions

- Rotating condensates expand and vortex lattice structures are created within.

- In a Harmonic trap, under the analysis of the LLL, one sees an inverted triangular lattice that is distorted at the edges. As the rotational angular velocity is increased, the number of vortices and the extent of the condensate increase.

- For annular condensates, as Ω gets large, their nature depends on the type of trap used. When the trap is harmonic plus quartic then the condensate expands infinitely and the annulus becomes infinitely thin. In this case a giant vortex is created in the central hole of the condensate.

- When the trap considered is harmonic plus Gaussian, then not only do the inner and outer boundaries of the condensate increase as Ω increases, but also the width of the annulus increases.