On vortex knots and unknots

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Fluid flow evolution

We consider a *homogeneous, incompressible fluid* in $\mathbb{R}^3$:

velocity field $\quad \mathbf{u} = \mathbf{u}(\mathbf{X}, t) : \begin{cases} \nabla \times \mathbf{u} = \boldsymbol{\omega} \\ \nabla \cdot \mathbf{u} = 0 \\ \mathbf{u} = 0 \text{ as } \mathbf{X} \to \infty \end{cases} \quad \text{in } \mathbb{R}^3$

- Fluid motion is governed by:

Euler equations

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\,\mathbf{u} = -\frac{1}{\rho}\nabla p \tag{1}$$

- Under Euler equations **topology is conserved.**
  - **Kinetic energy** $\quad T = \frac{1}{2}\rho \int_V |\mathbf{u}|^2 \mathrm{d}V = cst;$
  - **Kinetic helicity** $\quad H = \int_V \mathbf{u} \cdot \boldsymbol{\omega}\, \mathrm{d}V = cst;$
  - **Vortex circulation** $\quad \Gamma = cst;$
  - **topological quantities:** knot type, crossing number, linking numbers.

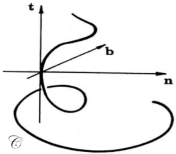
Biot-Savart Law

- Let $\mathscr{C}_t$ be a smooth **space curve** given by:

$$\mathscr{C}_t : \begin{cases} \mathbf{X}(s,t) := \mathbf{X}_t(s) \\ s \in [0,L] \to \mathbb{R}^3 \end{cases}$$

- The intrinsic reference on $\mathscr{C}_t$ is given by the **Frenet frame** $\left\{\hat{\mathrm{t}}, \hat{\mathrm{n}}, \hat{\mathrm{b}}\right\}$:

$$\begin{cases} \hat{\mathrm{t}} = \mathbf{X}'(s,t) = \dfrac{\partial \mathbf{X}}{\partial s} \\ \hat{\mathrm{n}} = \dfrac{\hat{\mathrm{t}}'}{c} \\ \hat{\mathrm{b}} = \hat{\mathrm{t}}' \times \hat{\mathrm{n}} \end{cases}$$



- Let identify $\mathscr{C}_t$ with a thin vortex filament of circulation Γ. The vortex line moves with a **self induced velocity**:

Biot-Savart (BS) law

$$\mathbf{u}(\mathbf{X},t) = \frac{\Gamma}{4\pi} \oint_{\mathscr{C}_t} \frac{\hat{\mathrm{t}} \times (\mathbf{X}(s) - \mathbf{X}(s^*))}{|\mathbf{X}(s) - \mathbf{X}(s^*)|^3} \mathrm{d}s \qquad (2)$$

Vortex filament motion under the Localized Induction Approximation (LIA)

- The Biot-Savart law can be simplified according to the following:

Asymptotic theory :
$$\begin{cases} \boldsymbol{\omega} = \omega_0 \hat{\mathbf{t}} & \omega_0 = \text{constant} \\ \dfrac{R}{a} = \delta \gg 1 & \text{thin tube approximation} \\ \dfrac{|X(s_i) - X(s_j)|}{|s_i - s_j|} = o(1) & \text{no self-intersection} \end{cases}$$

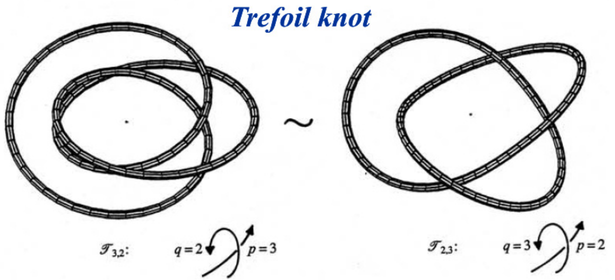
$$\Downarrow$$

LIA law
$$\mathbf{u}_{LIA} = \dot{\mathbf{X}}(s,t) \equiv \frac{\partial \mathbf{X}}{\partial t} = \frac{\Gamma}{4\pi} \ln \delta \; \mathbf{X}' \times \mathbf{X}'' = \frac{\Gamma}{4\pi} \ln \delta \; c\hat{\mathbf{b}} \, , \qquad (3)$$

Torus knots

Theorem (Massey, 1967)

A closed, non-self intersecting curve embedded in a torus Π, that cuts a meridian at $p > 1$ points and a longitude at $q > 1$ points (p and q relatively prime integers), is a **non-trivial knot** $\mathscr{T}_{p,q}$*, with winding number $w = q/p$.*

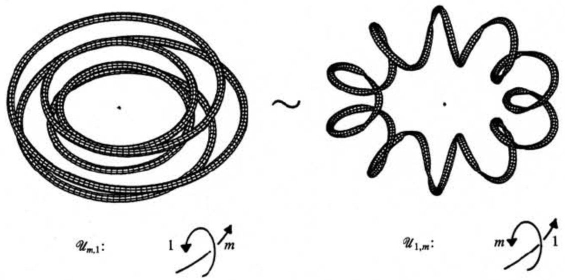


- $p > 1$ *longitudinal* wraps and $q > 1$ *meridional* wraps;
- For given p, q (p, q coprimes) $\mathscr{T}_{p,q} \sim \mathscr{T}_{q,p}$ **topologically equivalent**.

Torus unknots

- Case $p = 1$ or $q = 1$:

Unknots $\mathscr{U}_{1,m} \sim \mathscr{U}_{m,1} \sim \mathscr{U}_0$ $(circle)$



Vortex torus knot solution under LIA

Theorem (Kida, 1981)

Let $\mathscr{K}_v$ denote the embedding of a knotted vortex filament in an ideal fluid. If $\mathscr{K}_v$ evolves under LIA, then there exist a class of **steady solutions** *in the shape of* **torus knots** $\mathscr{K}_v \equiv \mathscr{T}_{p,q}$ *in terms of incomplete elliptic integrals:*

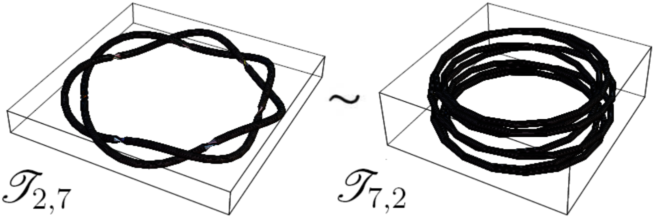
$$\textbf{Kida 1981}: \begin{cases} r^2 = F_r(J) \\ \alpha = F_\alpha(E) \\ z = F_z(\Pi) \end{cases}$$

- Solution $\mathscr{T}_{p,q}$ in explicit **analytic closed form**, based on **linear perturbation** from the circular solution $\mathscr{U}_0$ of NLS:

$$\textbf{Ricca 1993}: \begin{cases} r = r_0 + \epsilon r_1 \ , \\ \alpha = \dfrac{s}{r_0} + \epsilon \alpha_1 \ , \\ z = \dfrac{\hat{t}}{r_0} + \epsilon z_1 \ , \end{cases} \Rightarrow \begin{cases} r = r_0 + \epsilon \sin(w\phi) \ , \\ \alpha = \dfrac{s}{r_0} + \epsilon \dfrac{1}{wr_0} \cos(w\phi) \ , \\ z = \dfrac{\hat{t}}{r_0} + \epsilon \left(1 + \dfrac{1}{w^2}\right)^{1/2} \cos(w\phi) \ , \end{cases}$$

Theorem (Ricca, 1993)

Let $\mathscr{T}_{p,q}$ denote the embedding of a small-amplitude vortex torus knot $\mathscr{K}_v$ evolving under LIA. $\mathscr{T}_{p,q}$ is steady and stable under linear perturbation iff $q > p$ $(w > 1)$.



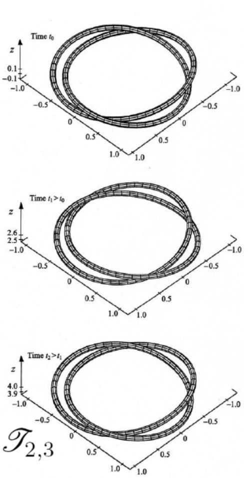
Definition

A vortex structure is said to be Lyapunov (structurally) stable if it evolves under signature-preserving flows that conserve its topology, geometric signature and vortex coherency.

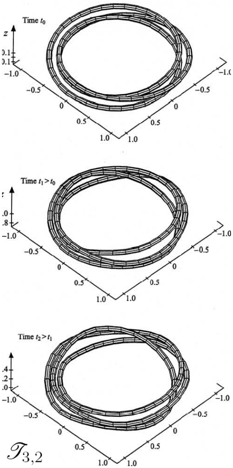
Numerical study of evolution of vortex knots

- The evolution of the vortex is obtained by **integrating** the equation of motion forward in time at each point from the initial vortex configuration;
- Typical **time step**: $\Delta t \in [0.001, 0.1]$ chosen in relation of value $N \in [100, 400] =$ number of sub-division of initial vortex configuration;
- The **calculation** is **non-dimensionalized**: $r_0 = 1$, $\Gamma = 1$, $\epsilon_0 = 0.1$;
- $\boldsymbol{T_{tot}}$: **time elapsed** at the occurrence of the first knot self-intersection;
- $\boldsymbol{D_{tot}}$: **distance travelled** in time T_{tot};
- **Convergence** tested both in **space** and **time** (independently form N and Δt);
- **Computation** performed on Intel(R) Core(2) Duo CPU T7300 2.00 GHZ, RAM 1.99 GB;

Torus knots $\mathscr{T}_{2,3}$ and $\mathscr{T}_{3,2}$ under Biot-Savart (BS) law

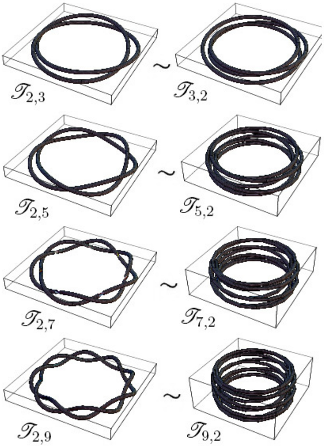


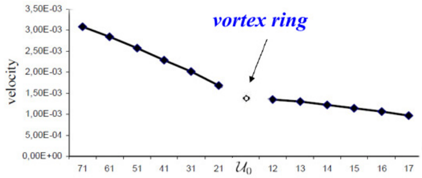
$\mathscr{T}_{2,3}$

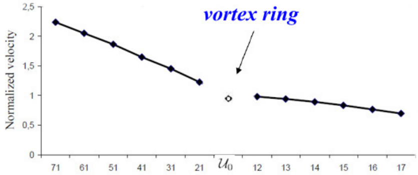


$\mathscr{T}_{3,2}$

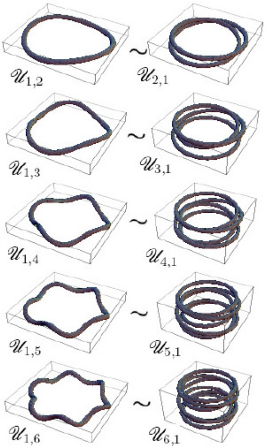
BS evolution of vortex knots velocity

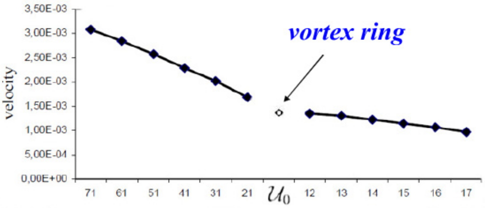


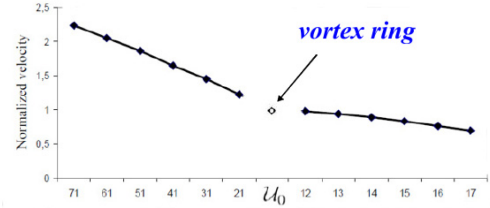




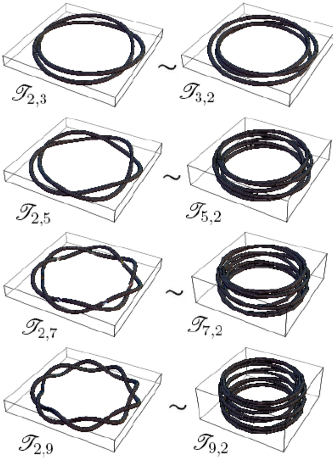
BS evolution of vortex unknots velocity

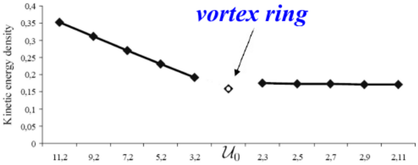


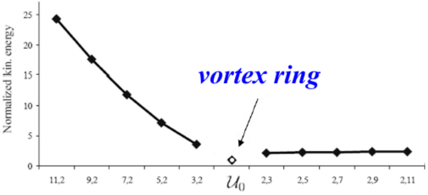




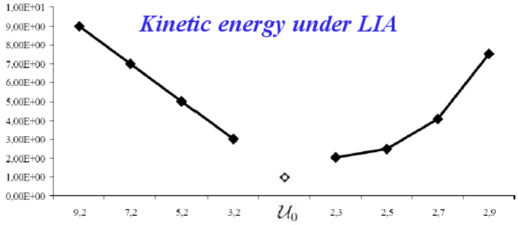
BS evolution of vortex knots kinetic energy

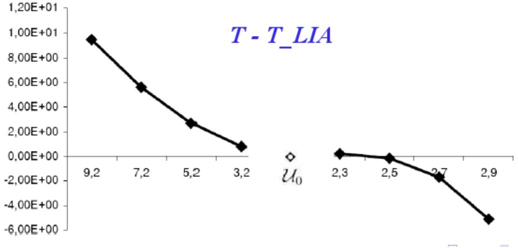




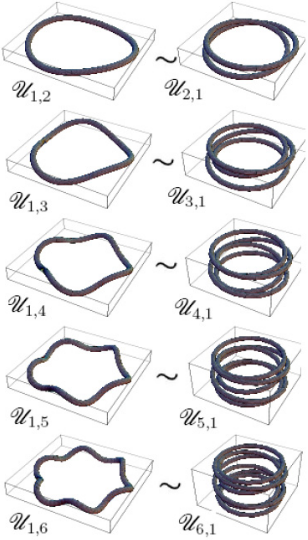


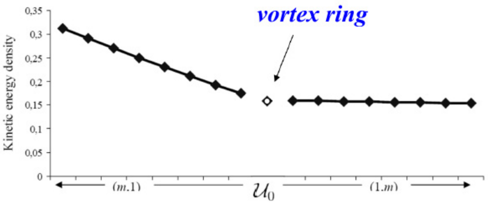
LIA evolution of vortex knots kinetic energy

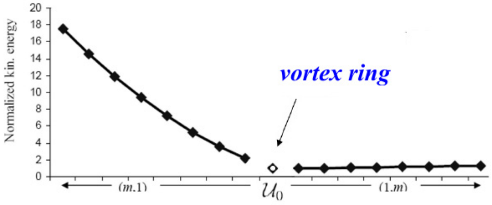




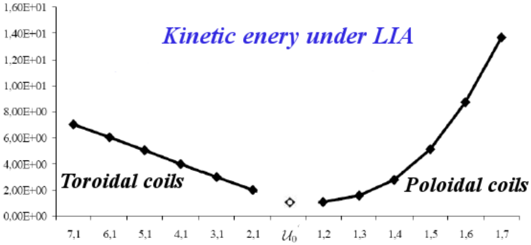
BS evolution of vortex unknots kinetic energy

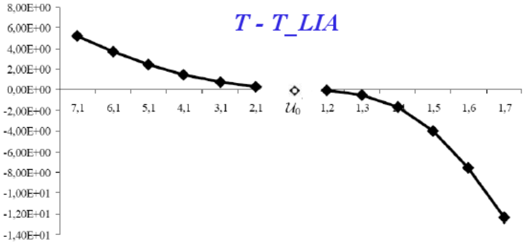




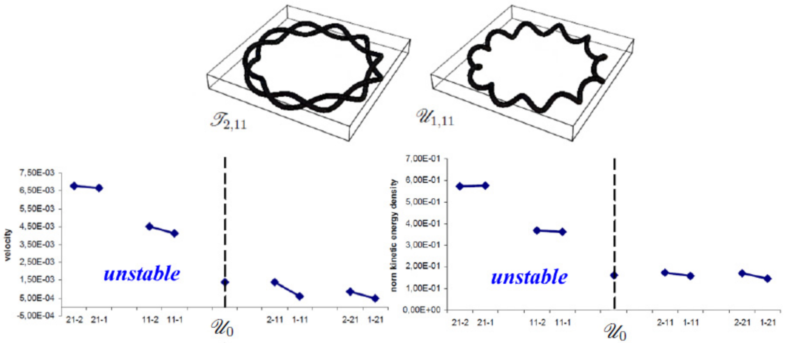


LIA evolution of vortex unknots kinetic energy

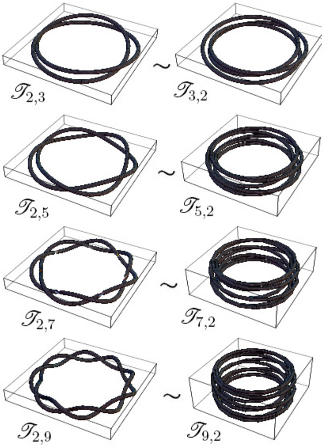


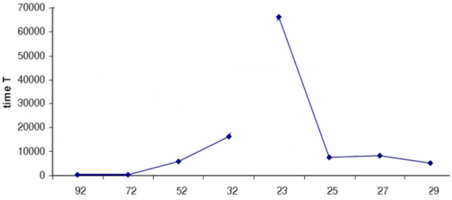


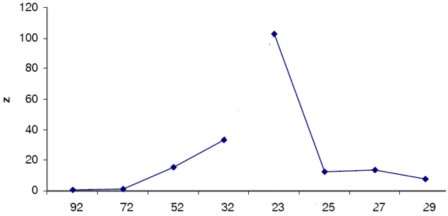
Role of topology in global dynamics



Stability of vortex knots



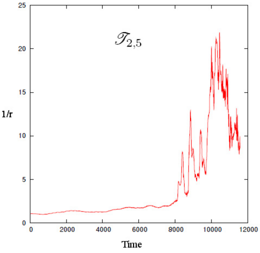


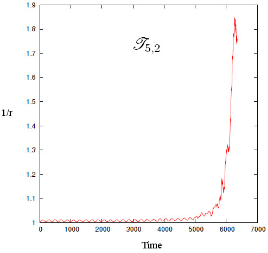


Stability of vortex knots

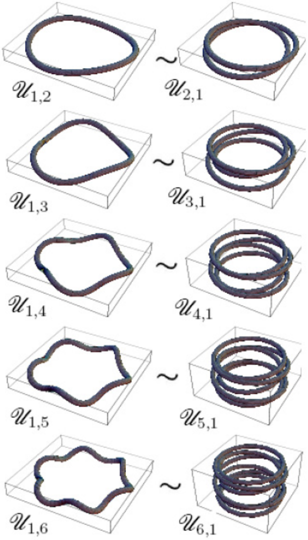
- the space travelled before a reconnection event decreases with increasing complexity;
- $\mathscr{T}_{p,2}$ is **faster** than its isotope $\mathscr{T}_{2,p}$;
- $\mathscr{T}_{p,2}$ is **not** necessarely **less stable** than its isotope $\mathscr{T}_{2,p}$;

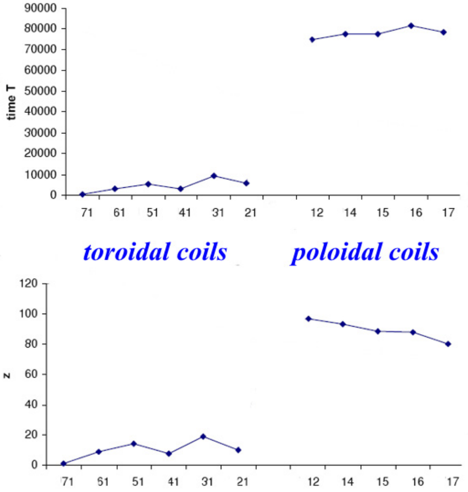
p	q	N	L	z	t
2	5	385	12,86	0,1414588451E+02	0,8761708008E+04
5	2	219	31,33	0,1548856926E+02	0,5851551758E+04





Stability of vortex unknots





Future work

Future work

- Role of **chirality** under LIA and BS evolution;
- **Structural stability** under LIA and BS law.
- **vortex bundles**.

References

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