Vortices in Rotating Bose-Einstein Condensates
A Review of (Recent) Mathematical Results

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Outline

1. General Setting and Background: the Gross-Pitaevskii (GP) Theory for Rotating Bose-Einstein (BE) Condensates [A].
2. Vortices and Rotational Symmetry Breaking.
3. The Thomas-Fermi (TF) Limit of the GP Theory:
   - Harmonic trapping potentials [IM].
   - (Strongly) Anharmonic trapping potentials [CY].

Main References

- [A] A. Aftalion, *Vortices in Bose-Einstein Condensates*, 2006.
- [CY] M.C., J. Yngvason, *J. Phys. A* **41** (2008), 445002.
- [IM] R. Ignat, V. Millot, *J. Funct. Anal.* **233** (2006), 260–306.
- Physics: A.L. Fetter, *Rev. Mod. Phys.* **81** (2009), 647–691.
- Numerics: W. Bao, in *Dynamics in Models of Coarsening, Coagulation, Condensation and Quantization*, 2007.

The GP Theory of a Rotating BE Condensate in a Trap

- 2d BE condensate rotating along the $\hat{z}-$axis with angular velocity Ω.
- External trap given by a potential $V(\vec{r})$ ($V(\vec{r}) \to \infty$ as $r \to \infty$).
- The *stationary* ground state properties of a rotating BE condensate can be described through minimizers of the GP energy functional (in the non-inertial rotating frame).
- $L = -i(x\partial_y - y\partial_x)$ is the $z-$component of the angular momentum.
- ε^{-2} is the coupling parameter ($\propto$ scattering length).
- The TF limit is $\varepsilon \to 0$ (large coupling and/or fast rotation).

The GP Energy Functional

$$\mathcal{E}^{\mathrm{GP}}[\Psi] = \int_{\mathbb{R}^2} \mathrm{d}\vec{r} \left\{ |\nabla\Psi|^2 - \Psi^*\Omega L\Psi + V(\vec{r})|\Psi|^2 + \frac{|\Psi|^4}{\varepsilon^2} \right\}.$$

Minimization of the GP Functional

$$\mathcal{E}^{\mathrm{GP}}[\Psi] = \int_{\mathbb{R}^2} \mathrm{d}\vec{r} \left\{ \left| \left(\nabla - i\vec{A}_\Omega \right) \Psi \right|^2 + \left[V(\vec{r}) - \frac{\Omega^2 r^2}{4} \right] |\Psi|^2 + \frac{|\Psi|^4}{\varepsilon^2} \right\}$$

- The vector potential is $\vec{A}_\Omega = \Omega \hat{e}_z \wedge \vec{r}/2$.
- GP ground state energy $E^{\mathrm{GP}} = \inf_{\|\Psi\|_2=1} \mathcal{E}^{\mathrm{GP}}[\Psi]$.
- Ψ^{GP} stands for *any* corresponding minimizer.

Boundedness from Below of $\mathcal{E}^{\mathrm{GP}}$

Assume that $V \in L^2_{\mathrm{loc}}(\mathbb{R}^3)$ and *either* $V(r) \simeq r^2$, $\Omega < 2$, *or* $V(r) \simeq r^s$, $s > 2$, as $r \to \infty \Longrightarrow \mathcal{E}^{\mathrm{GP}}$ is bounded from below. Usually one considers

- the harmonic potential $V(r) = r^2$ (upper bound on Ω!),
- (strongly) anharmonic (homogeneous) potentials $V(r) \simeq r^s$, $s > 2$ (no upper bound on Ω!): the simplest example is a confinement to the unitary disc $\mathcal{B}_1$ with Neumann boundary conditions ($s = \infty$).

Existence of a Minimizer Ψ^{GP}

Under the same hypothesis on V, $\exists$ (at least) one minimizer Ψ^{GP}.

- Ψ^{GP} solves the GP time-independent equation

$$-\Delta\Psi^{\mathrm{GP}} - \Omega L\Psi^{\mathrm{GP}} + V\Psi^{\mathrm{GP}} + 2\varepsilon^{-2}\left|\Psi^{\mathrm{GP}}\right|^2\Psi^{\mathrm{GP}} = \mu^{\mathrm{GP}}\Psi^{\mathrm{GP}}.$$

  The *chemical potential* μ^{GP} is fixed by the L^2 normalization:

$$\mu^{\mathrm{GP}} = E^{\mathrm{GP}} + \varepsilon^{-2}\|\Psi^{\mathrm{GP}}\|_4^4.$$

- If V is *smooth*, the same is Ψ^{GP}.

Uniqueness of Ψ^{GP}

The minimizer is *not* necessarily unique.

- Non-uniqueness is due to the presence of the *angular momentum* term. If $\Omega = 0$, $\mathcal{E}^{\mathrm{GP}}$ is strictly convex in $\rho = |\Psi|^2$.
- Non-uniqueness is strictly related to the occurence of isolated vortices $\Longleftrightarrow$ rotational symmetry breaking.

Ginzburg-Landau vs. Gross-Pitaevskii

- [GL] Minimize $\mathcal{E}^{\mathrm{GL}}$ w.r.t. u *and* $\vec{A}$ with $h = \mathrm{curl}(\vec{A})$, $h_{\mathrm{ex}} = \Omega$ (uniform magnetic field),

$$\mathcal{E}^{\mathrm{GL}}[u, \vec{A}] = \int_{\mathcal{D}} \mathrm{d}\vec{r} \left\{ \left| \left(\nabla - i\vec{A} \right) u \right|^2 + |h - h_{\mathrm{ex}}|^2 + \varepsilon^{-2}(1 - |u|^2)^2 \right\}.$$

- [GP] Minimize $\mathcal{E}^{\mathrm{GP}}$ w.r.t. L^2 normalized $\Psi(\vec{r}) = \sqrt{\rho(r)}u(\vec{r})$,

$$\mathcal{E}^{\mathrm{GP}}[\Psi] = \mathcal{E}^{\mathrm{TF}}[\rho] + \int_{\mathcal{S}} \mathrm{d}\vec{r} \left\{ \left| \left(\nabla - i\vec{A}_{\Omega} \right) u \right|^2 + \varepsilon^{-2}\rho \left(1 - |u|^2 \right) \right\} \rho.$$

- Main Differences:
  - In GL there is an *additional* minimization w.r.t. $\vec{A}$, whereas in GP $\vec{A}$ is *given* $\Longrightarrow$ major differences in the minimizers but the ground state energies can be *close* in certain regimes.
  - *No* L^2 normalization in GL $\Longrightarrow$ *no* chemical potential in GL $\Longrightarrow$ *no* density ρ in GL. *Not* only a technicality since there is some *physics* behind it!

Vortices and Rotational Symmetry Breaking

$$\mathcal{E}^{\mathrm{GP}}[\Psi] = \int_{\mathbb{R}^2} \mathrm{d}\vec{r} \left\{ \left| \left(\nabla - i\vec{A}_\Omega \right) \Psi \right|^2 + \left[V(r) - \frac{\Omega^2 r^2}{4} \right] |\Psi|^2 + \frac{|\Psi|^4}{\varepsilon^2} \right\}$$

Rotational Symmetry Breaking

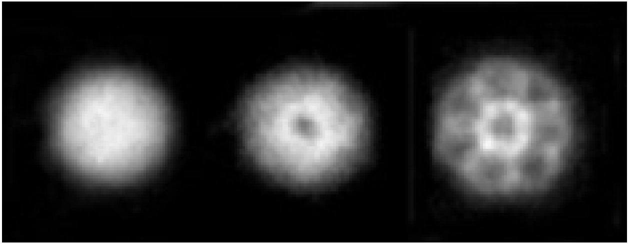
- The functional $\mathcal{E}^{\mathrm{GP}}$ is invariant under rotations around the z-axis.
- If ε is *fixed* and Ω *large*, the GP minimizer is not an eigenfunction of the angular momentum (rotational symmetry breaking), due to the occurrence of isolated vortices.

Vortices

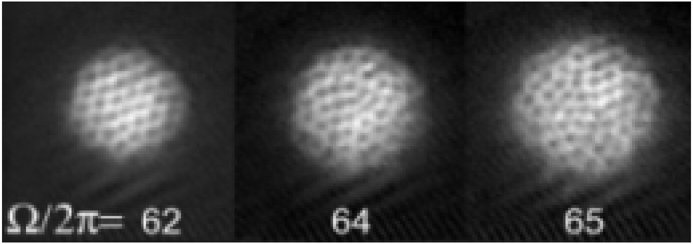
Ψ^{GP} has a vortex at $\vec{r}_0$ ($\zeta_0 = x_0 + iy_0$) with winding number d, if

$$\Psi^{\mathrm{GP}}(\vec{r}_0) = 0, \qquad \text{(locally)} \quad \frac{\Psi^{\mathrm{GP}}}{|\Psi^{\mathrm{GP}}|} \simeq \left[\frac{\zeta - \zeta_0}{|\zeta - \zeta_0|} \right]^d .$$

Vortices in a Rotating BE Condensate [Dalibard *et al* '05]



Formation of quantized vortices in a rotating Rb BE condensate.



Vortices in a fast rotating Rb BE condensate in a quartic+quadratic trap.

Why Are Vortices Energetically Favorable?

$$\mathcal{E}^{\mathrm{GP}}[\Psi] = \int_{\mathcal{B}_1} \mathrm{d}\vec{r} \left\{ |\nabla\Psi|^2 - \Omega \Psi^* L \Psi + \varepsilon^{-2} |\Psi|^4 \right\}$$

Energy Compensation

- For small rotational velocities the condensate is at rest in the inertial frame (superfluidity) $\Longrightarrow |\Psi^{\mathrm{GP}}| = \mathrm{const}$.
- At higher angular velocities vortices start to occur: For small ε, a vortex of degree d at the origin has the form $f(r)\exp\{id\vartheta\}$, with f approx. constant outside $\mathcal{B}_\varepsilon$ ($\Longleftarrow$ nonlinear term).
- Kinetic energy $\displaystyle\int_{\mathcal{B}_1 \setminus \mathcal{B}_\varepsilon} \mathrm{d}\vec{r}\ |\nabla\Psi|^2 \simeq d^2 \int_{\mathcal{B}_1 \setminus \mathcal{B}_\varepsilon} \mathrm{d}\vec{r}\ \frac{1}{r^2} \simeq C d^2 |\log\varepsilon|$.
- Angular momentum $\displaystyle -\Omega \int_{\mathcal{B}_1 \setminus \mathcal{B}_\varepsilon} \mathrm{d}\vec{r}\ \Psi^* L \Psi \simeq -\Omega d$.
- If $\Omega \sim C|\log\varepsilon|$, a vortex can be energetically favorable.

Why/When More Vortices?

$$\mathcal{E}^{\mathrm{GP}}[\Psi] = \int_{\mathcal{B}_1} \mathrm{d}\vec{r} \left\{ |\nabla\Psi|^2 - \Omega\Psi^* L\Psi + \varepsilon^{-2} |\Psi|^4 \right\}$$

Energy Optimization

- So far we have not justified the breaking of the rotational symmetry, since a vortex at the origin is an eigenfunction of L!
- Ψ^{GP} can contain more than one vortex ($\Longleftarrow$ nonlinearity).
- Ω fixes the total winding number d (angular momentum) of Ψ^{GP} and, if Ω is sufficiently large, $d > 1$.
- Suppose $d = 2$: The kinetic energy is $\propto d^2 \Longrightarrow 2$ vortices of winding number 1 have a smaller kinetic energy ($1 + 1 = 2$) than 1 vortex of winding number 2 ($2^2 = 4$), but *almost* the same angular momentum.
- If $\varepsilon \ll 1$ the vortex cores are small ($\sim \varepsilon$) and the interaction energy can be neglected $\Longrightarrow$ many vortices can be energetically favorable.

Rotational Symmetry Breaking

Theorem (Symmetry Breaking [MC,Rindler-Daller,Yngvason '07])

As $\varepsilon \to 0$, no minimizer of $\mathcal{E}^{\mathrm{GP}}[\Psi]$ is an eigenfunction of the angular momentum, if

$$6|\log\varepsilon| + 3 < \Omega \lesssim \frac{C}{\varepsilon}$$

for any constant $C \in \mathbb{R}^+$.

- Symmetry breaking is due to occurrence of isolated vortices outside of the origin.
- The GP minimizer in no longer unique (∞ degeneracy).
- The estimate of the symmetry breaking threshold is not optimal (it is expected to be $2|\log\varepsilon|$ [Aftalion,Du '01]).
- The rotational symmetry is expected to be broken also for $\Omega \gg \varepsilon^{-1}$.

Nucleation of Vortices (Harmonic Traps) [Ignat, Millot '06]

$$\mathcal{E}^{\mathrm{GP}}[\Psi] = \int_{\mathbb{R}^2} \mathrm{d}\vec{r} \left\{ |\nabla\Psi|^2 - \Psi^* \Omega L \Psi + \varepsilon^{-2} V(x,y)|\Psi|^2 + \varepsilon^{-2}|\Psi|^4 \right\}$$

- Harmonic trapping potential (rescaled): $V(x,y) = (x^2 + \Lambda^2 y^2)$. $0 < \Lambda \leq 1$ measures the asymmetry of the potential.
- The coefficient ε^{-2} of V is chosen so that the first critical velocity is $\mathcal{O}(|\log\varepsilon|)$ (it is equivalent to rescale all lengths):
$$\Omega_1 = \frac{\sqrt{\pi}(\Lambda^2+1)}{\sqrt{2\Lambda}} |\log\varepsilon|.$$
- The vortex free profile η_ε is the (unique L^2 normalized) minimizer of
$$\tilde{\mathcal{E}}^{\mathrm{GP}}_\varepsilon[\phi] = \int_{\mathbb{R}^2} \mathrm{d}\vec{r} \left\{ |\nabla\phi|^2 + \varepsilon^{-2} V|\phi|^2 + \varepsilon^{-2}|\phi|^4 \right\}.$$
- η_ε is *real* and *positive*. If $\Lambda = 1$, it is also *radial*.
- As $\varepsilon \to 0$, $\eta_\varepsilon^2 \longrightarrow \rho^{\mathrm{TF}}(x,y) = \frac{1}{2}[\mu - V(x,y)]_+$ in $L^\infty(\mathcal{D})$.

Theorem (Absence of Vortices below Ω_1 [Ignat,Millot '06])

For any $\delta > 0$, *if* $\Omega \leq \Omega_1 - \delta \log|\log\varepsilon|$, *then*

- $\left|\Psi^{\mathrm{GP}}\right| \underset{\varepsilon\to 0}{\longrightarrow} \sqrt{\rho^{\mathrm{TF}}(x,y)}$ *in* $L^\infty_{\mathrm{loc}}(\mathbb{R}^2 \setminus \partial\mathcal{D})$,
- $E^{\mathrm{GP}} = \mathcal{E}^{\mathrm{GP}}\left[\eta_\varepsilon e^{iS\Omega}\right] + o(1)$, *where* $S(x,y) = \frac{\Lambda^2-1}{\Lambda^2+1}xy$,
- *Up to a subsequence (and a global phase factor* α, $|\alpha| = 1$*)*
$$\Psi^{\mathrm{GP}} \underset{\varepsilon\to 0}{\longrightarrow} \alpha\sqrt{\rho^{\mathrm{TF}}(x,y)}e^{iS\Omega},$$
in $H^1_{\mathrm{loc}}(\mathcal{D}) \Longrightarrow$ *no vortices, i.e., for any* $R_0 < \sqrt{\mu}$ *and* ε *sufficiently small,* Ψ^{GP} *does not vanish inside the region where* $x^2 + \Lambda^2 y^2 < R_0^2$.

Theorem (Occurence of Vortices above Ω_1 [Ignat,Millot '06])

For any $\delta > 0$, *if* $\Omega \geq \Omega_1 + \delta \log|\log\varepsilon|$ *and* ε *is sufficiently small, then*

- Ψ^{GP} *has at least one vortex at* $\vec{r}_\varepsilon$ *such that* $\mathrm{dist}(\vec{r}_\varepsilon, \partial\mathcal{D}) \geq C$.
- *If in addition* $\Omega \leq \Omega_1 + \mathcal{O}(\log|\log\varepsilon|)$, *then the vortex remains close to the origin, i.e.,* $|\vec{r}_\varepsilon| = o(1)$.

Critical Velocities [Ignat,Millot '06]

$$\Omega_d = \frac{\sqrt{\pi}(1+\Lambda^2)}{\sqrt{2\Lambda}} \left(|\log\varepsilon| + (d-1)\log|\log\varepsilon| \right), \qquad d \in \mathbb{N}$$

Theorem (Number and Distribution of Vortices [Ignat,Millot '06])

For any $0 < \delta \ll 1$, *if* $\Omega_d + \delta \log|\log\varepsilon| \leq \Omega \leq \Omega_{d+1} - \delta\log|\log\varepsilon|$, *then*

- *For any* $R_0 < \sqrt{\mu}$ *and* ε *sufficiently small,* Ψ^{GP} *has exactly* d *vortices of winding number 1 at* $\vec{r}_{i,\varepsilon}$, $i = 1, \ldots, d$ *inside* $x^2 + \Lambda^2 y^2 < R_0^2$,
- *Vortices remain close to the origin and close one another, i.e.,*

$$|\vec{r}_i| \leq C\Omega^{-1/2}, \qquad |\vec{r}_i - \vec{r}_j| \leq C\Omega^{-1/2}.$$

- *Setting* $\vec{\xi}_i = \sqrt{\Omega}\vec{r}_i$, *the configuration* $(\vec{\xi}_1, \ldots, \vec{\xi}_d)$ *minimizes the renormalized energy*

$$W\left(\vec{\xi}_1, \ldots, \vec{\xi}_d\right) = -\pi\sqrt{\mu}\sum_{i\neq j}\log\left|\vec{\xi}_i - \vec{\xi}_j\right| + \frac{\pi\sqrt{\mu}}{1+\Lambda^2}\sum_{i=1}^{d}\left(x_i^2 + \Lambda^2 y_i^2\right).$$

Distribution of Vortices [Gueron,Shafrir '00]

The distribution of vortices is determined by the minimization of W:

- If $\Lambda = 1$ and $d \leq 6$, regular polygons and stars ($d-1$ side regular polygon plus the origin) centered at the origin are (local) minimizing configurations for the renormalized energy W.
- If $d \geq 11$ *neither* regular polygons *nor* stars are local minimizers of W. As d increases the minimizers approach a triangular lattice.

Larger Angular Velocities [Baldo,Jerrard,Orlandi,Soner '08]

- If $\Omega \gg \Omega_d$ for any d but $\Omega = \mathcal{O}(|\log\varepsilon|)$, the number of vortices is *not* uniformly bounded in ε.
- The vortex distribution minimizes a (rescaled) free boundary problem.

Landau Regime [Aftalion *et al* '06]

When $\Omega \simeq \varepsilon^{-1}$, the occupation of Landau levels become relevant...

Rotating BE Condensates in Anharmonic Traps [MC,Rindler-Daller,Yngvason '07]

$$\mathcal{E}^{\mathrm{GP}}[\Psi] = \int_{\mathcal{B}_1} \mathrm{d}\vec{r} \left\{ \left| \left[\nabla - i\vec{A}_\Omega \right] \Psi \right|^2 - \frac{\Omega^2 r^2 |\Psi|^2}{4} + \frac{|\Psi|^4}{\varepsilon^2} \right\}$$

Motivations

- There is *no* upper bound on the angular velocity Ω, i.e., the condensate is confined for *any* $\Omega \Longrightarrow$ one can explore regimes of very fast rotation.
- The unitary disc is the *strongest* anharmonic trap one can think of since it can be formally obtained as the limit $s \to \infty$ of a homogeneous potential $V(r) = r^s$.
- Any homoheneous potential $V(r) = cr^s$, $s > 2$ can be mapped to the above model by means of a suitable *rescaling* of all lenghts [MC,Rindler-Daller,Yngvason '07].

Extraction of the TF Density

$$\mathcal{E}^{\mathrm{GP}}[\Psi] = \int_{\mathcal{B}_1} \mathrm{d}\vec{r} \left\{ \left| \left[\nabla - i\vec{A}_{\Omega} \right] \Psi \right|^2 - \frac{\Omega^2 r^2 |\Psi|^2}{4} + \frac{|\Psi|^4}{\varepsilon^2} \right\}$$

- If $\Omega \lesssim \varepsilon^{-2}$, the kinetic energy gives a *smaller order* correction.
- In anharmonic traps the second part of $\mathcal{E}^{\mathrm{GP}}$ *depends* on Ω.

TF Energy Functional

$$\mathcal{E}^{\mathrm{TF}}[\rho] = \int_{\mathcal{B}_1} \mathrm{d}\vec{r} \left\{ \frac{\rho^2}{\varepsilon^2} - \frac{\Omega^2 r^2 \rho}{4} \right\},$$

with ground state energy $E^{\mathrm{TF}} = \inf_{\|\rho\|_1 = 1} \mathcal{E}^{\mathrm{TF}}[\rho]$ and (L^1 normalized) minimizer

$$\rho^{\mathrm{TF}}(r) = \tfrac{1}{2} \left[\mu^{\mathrm{TF}} + \tfrac{\varepsilon^2 \Omega^2 r^2}{4} \right]_+ .$$

Asymptotics of the TF Functional

$$\mathcal{E}^{\mathrm{TF}}[\rho] = \int_{\mathcal{B}_1} \mathrm{d}\vec{r} \left\{ \frac{\rho^2}{\varepsilon^2} - \frac{\Omega^2 r^2 \rho}{4} \right\}$$

Slow Rotation ($\Omega \ll \varepsilon^{-1}$)

- $E^{\mathrm{TF}} = \pi^{-1}\varepsilon^{-2} + \mathcal{O}(\Omega^2)$ and $\rho^{\mathrm{TF}}(r) = \pi^{-1} + \mathcal{O}(\varepsilon^2\Omega^2)$.

Rapid Rotation ($\Omega \sim \varepsilon^{-1}$)

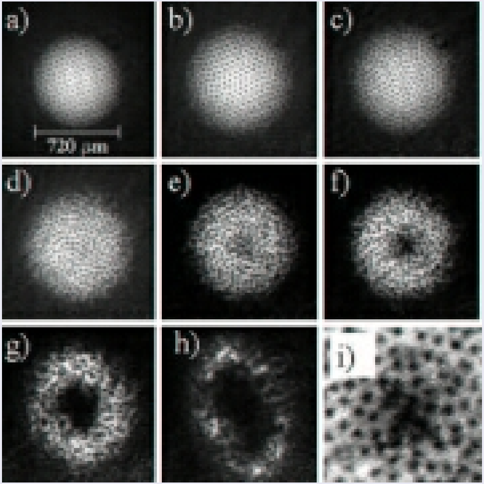
- $E^{\mathrm{TF}} = \mathcal{O}(\varepsilon^{-2})$ and $\rho^{\mathrm{TF}}(r) \simeq [C_1 + C_2 r^2]_+$.
- If $\Omega > 4/(\sqrt{\pi}\varepsilon)$, $\rho^{\mathrm{TF}}(r) = 0$ for any $r \leq R_{\mathrm{in}} \equiv \sqrt{1 - \frac{4}{\sqrt{\pi}\varepsilon\Omega}}$ (hole).

Ultrarapid Rotation ($\Omega \gg \varepsilon^{-1}$)

- $E^{\mathrm{TF}} = -\Omega^2/4 + \mathcal{O}(\Omega)$ and $\rho^{\mathrm{TF}}(r) = \frac{\varepsilon^2\Omega^2}{8}\left[r^2 - R_{\mathrm{in}}^2\right]_+$.
- $R_{\mathrm{in}} = 1 - \mathcal{O}(\varepsilon^{-1}\Omega^{-1}) \Longrightarrow \rho^{\mathrm{TF}}$ approaches $\delta(1-r)$.

Experimental Observations [Engels *et al* '03]

Condensate Density



"Giant Vortex" (Hole) formation in a rotating $^{87}\mathrm{Rb}$ BE condensate (induced by a laser beam).

Slow Rotation ($\Omega \ll \varepsilon^{-1}$)

Theorem (GP Asymptotics [MC,Rindler-Daller,Yngvason '07])

If $\varepsilon\Omega \to 0$ *as* $\varepsilon \to 0$,

$$E^{\mathrm{GP}} = \frac{1}{\pi\varepsilon^2} + \mathcal{O}(\Omega^2), \qquad \left\| |\Psi^{\mathrm{GP}}|^2 - \frac{1}{\pi} \right\|_{L^1(\mathcal{B}_1)} = \mathcal{O}\left(\varepsilon\Omega\right).$$

- π^{-1} is the GS density of the GP functional without rotation (with energy $\pi^{-1}\varepsilon^{-2}$) $\Longrightarrow$ the rotation has no leading order effect on the GS asymptotics.
- If $\Omega \ll |\log\varepsilon|$, the GP minimizer is unique and strictly positive. In this case $\Psi^{\mathrm{GP}} \to \pi^{-1}$ as $\varepsilon \to 0$ in $H^1(\mathcal{B}_1)$ (no vortex).
- If $\Omega \sim |\log\varepsilon|$, vortices start to occur in $\Psi^{\mathrm{GP}} \Longrightarrow$ rotational symmetry breaking.

Occurrence and Distribution of Vortices ($\Omega \ll \varepsilon^{-1}$)

1. $\Omega \ll 2|\log \varepsilon|$
   - The GP minimizer is a *unique* and a *strictly positive* radial function.
2. $\Omega \sim 2|\log \varepsilon| + \mathcal{O}(\log|\log \varepsilon|)$
   - *Uniformly bounded* (in ε) number of vortices at $\vec{r}_i$, $i = 1, \ldots, n$.
   - n is fixed by the *remainder* in the angular velocity asymptotics (coefficient of $\log|\log \varepsilon|$).
   - Vortices are very *close* to the origin: $|\vec{r}_i| \sim |\log \varepsilon|^{-1/2}$ and $|\vec{r}_i - \vec{r}_j| \sim |\log \varepsilon|^{-1/2}$. The vortex core is a ball of radius $\sim \varepsilon^{\eta}$.
   - Vortices arrange in *regular polygons* centered at the origin to minimize the interaction energy.
3. $\Omega \sim C|\log \varepsilon|$, $C > 2$
   - The number of vortices is *no* longer uniformly bounded.
   - Vortices are confined to a *subset* of $\mathcal{B}_1$ (free boundary problem).
4. $2|\log \varepsilon| \ll \Omega \ll \varepsilon^{-1}$
   - The number of vortices is $\sim \Omega/2$.
   - Vortices with winding number 1 are *uniformly* distributed over $\mathcal{B}_1$.

Vortex Energy Contribution ($\Omega \ll \varepsilon^{-1}$)

Theorem (Improved Energy Asymptotics [MC,Yngvason '08])

For any $|\log \varepsilon| \ll \Omega \ll \varepsilon^{-1}$,

$$E^{\mathrm{GP}} = E^{\mathrm{TF}} + \frac{\Omega |\log(\varepsilon^2 \Omega)|}{2}(1 + o(1)).$$

- Since $\Omega \ll \varepsilon^{-1}$, $E^{\mathrm{TF}} = \pi^{-1}\varepsilon^{-2}(1 + o(1))$.
- Vortices of winding number 1 are uniformly distributed over $\mathcal{B}_1$, their number is $\sim \Omega/2$ and their core is $\sim \varepsilon$.
- Each vortex gives a kinetic contribution of order $|\log(\varepsilon^2\Omega)|$:

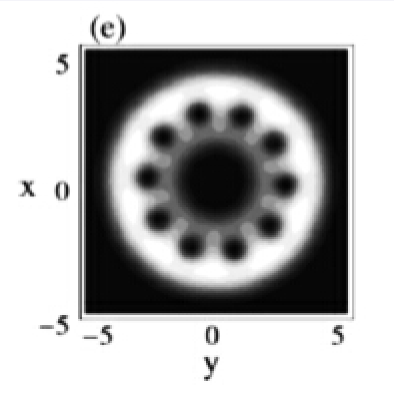
$$2\pi \int_{\varepsilon}^{\Omega^{-1/2}} \mathrm{d}r \, r^{-1} \simeq \pi |\log(\varepsilon^2 \Omega)|.$$

- No proof of the existence of isolated vortices but only of a uniform distribution of vorticity satisfying the above properties.

Rapid Rotation ($\Omega \sim \varepsilon^{-1}$): Numerical Simulations [Kasamatsu *et al* '02]

Condensate Density

Hole



The GP minimizer is exponentially small in a disc centered at the origin (hole) and vortices cover the whole trap.

Rapid Rotation ($\Omega \sim \varepsilon^{-1}$)

Theorem (GP Asymptotics [MC,Yngvason '08])

For any $\Omega \sim \varepsilon^{-1}$,

$$E^{\mathrm{GP}} = E^{\mathrm{TF}} + \frac{\Omega |\log \varepsilon|}{2}(1 + o(1)),$$

$$\left\| |\Psi^{\mathrm{GP}}|^2 - \rho^{\mathrm{TF}} \right\|_{L^1(\mathcal{B}_1)} = \mathcal{O}\left(\sqrt{\varepsilon |\log \varepsilon|} \right).$$

- Ψ^{GP} contains a number $\sim \Omega/2$ of vortices with winding number 1 uniformly distributed over a regular lattice with spacing $\sim \sqrt{\varepsilon}$.
- The vortex core is a ball of radius $\sim \varepsilon$.
- Each vortex gives an kinetic contribution of order $|\log \varepsilon|$:

$$2\pi \int_{\varepsilon}^{\sqrt{\varepsilon}} \mathrm{d}r \; r^{-1} \simeq \pi |\log \varepsilon|.$$

Vortex Energy Contribution ($\Omega \sim \varepsilon^{-1}$)

- $\Psi^{\mathrm{GP}} \simeq \sqrt{\rho^{\mathrm{TF}}(r)} e^{i\phi_\varepsilon}$, where the phase ϕ_ε minimizes the kinetic energy and contains the vortices:

$$e^{i\phi_\varepsilon} = \prod \frac{\zeta - \zeta_i}{|\zeta - \zeta_i|}, \qquad \phi_\varepsilon = \sum \arctan\left[\frac{y - y_i}{x - x_i}\right].$$

- Kinetic energy $\left\| \nabla \phi_\varepsilon - \vec{A}_\Omega \right\|_2^2 = \left\| \nabla \tilde{\phi}_\varepsilon - |\vec{A}_\Omega| \hat{r} \right\|_2^2$, where we have used the rotation $\phi_\varepsilon \to \tilde{\phi}_\varepsilon = \sum \log|\zeta - \zeta_i|$, $\vec{A}_\Omega \to |\vec{A}_\Omega| \hat{r}$.

Electrostatic Analogy

Vortex at $\vec{r}_i$ (vortex core of size $\sim \varepsilon^\eta$)	$\Longleftrightarrow$	Unitary Point Charge $+1$ at $\vec{r}_i$ (smeared over a ball of size $\sim \varepsilon^\eta$)
Vector Potential $\vec{A}_\Omega$	$\Longleftrightarrow$	Uniform Charge Density $-\frac{\Omega}{2\pi}$
Vortex Kinetic Energy	$\Longleftrightarrow$	Energy of the Electric Field

An Electrostatic Problem ($\Omega \sim \varepsilon^{-1}$) [MC,Yngvason '08]

- Finding the optimal distribution of vortices is *equivalent* to minimize the electrostatic energy of a charge distribution given by *positive* point charges and a *negative uniform* background.
- As long as ρ^{TF} varies on a *scale* of order 1, the optimal distribution is uniform: Vortices on a regular lattice with fundamental cell $\mathcal{Q}_\varepsilon$ and spacing $\ell_\varepsilon \sim \sqrt{|\mathcal{Q}_\varepsilon|}$ covering $\mathcal{B}_1$ (triangular, square or hexagonal lattice).
- The *cell volume* is chosen so that the cell is neutral, i.e., $|\mathcal{Q}_\varepsilon| = 2\pi/\Omega$.
- The dipole associated with any cell $\mathcal{Q}_\varepsilon^i$ vanishes because of the cell symmetry $\Longrightarrow$ the electric field $\vec{E}_i$ generated by $\mathcal{Q}_\varepsilon^i$ decays very fast (at least $\sim r^{-3}$) outside $\mathcal{Q}_\varepsilon^i$ $\Longrightarrow$ the leading order contribution is given by the self-energy inside $\mathcal{Q}_\varepsilon^i$ (*minimized* by the triangular lattice).
- Self-energy inside $\mathcal{Q}_\varepsilon^i$: $\displaystyle\int_{\mathcal{Q}_\varepsilon^i \setminus \mathcal{B}_\varepsilon^i} \mathrm{d}\vec{r} \left| \nabla \phi_\varepsilon - \vec{A}_\Omega \right|^2 = \pi |\log \varepsilon| + \mathcal{O}(1)$.

Slow and Rapid Rotation ($\Omega \lesssim \varepsilon^{-1}$): Vorticity

Theorem (Uniform Distribution of Vorticity [MC,Yngvason '08])

Let $\varepsilon > 0$ *be sufficiently small and* $|\log \varepsilon| \ll \Omega \lesssim \varepsilon^{-1}$. *Then there exists a finite family of disjoint balls* $\{\mathcal{B}^i_\varepsilon\} \subset \mathrm{supp}(\rho^{\mathrm{TF}})$ *such that*

1. *the radius of any ball is at most of order* $1/\sqrt{\Omega}$,
2. *the sum of all the radii is at most of order* $\sqrt{\Omega}$,
3. $\left|\Psi^{\mathrm{GP}}\right| \geq C|\log(\varepsilon^2\Omega)|^{-1}$ *on* $\partial\mathcal{B}^i_\varepsilon$, *for some* $C > 0$,

and, denoting by $\vec{r}_{i,\varepsilon}$ *the center of each ball* $\mathcal{B}^i_\varepsilon$ *and by* $d_{i,\varepsilon}$ *the winding number of* $|\Psi^{\mathrm{GP}}|^{-1}\Psi^{\mathrm{GP}}$ *on* $\partial\mathcal{B}^i_\varepsilon$,

$$\frac{2\pi}{\Omega}\sum d_{i,\varepsilon}\delta\left(\vec{r} - \vec{r}_{i,\varepsilon}\right) \xrightarrow[\varepsilon\to 0]{\mathrm{w}} \chi^{\mathrm{TF}}(\vec{r})\,\mathrm{d}\vec{r}.$$

- If one could prove that Ψ^{GP} has *only* isolated vortices, then the Theorem would yield their number and uniform distribution.

Ultrarapid Rotation ($\Omega \gg \varepsilon^{-1}$)

- The rotational energy dominates: $E^{\mathrm{TF}} = -\frac{\Omega^2}{4}\left\{1 + \mathcal{O}(\varepsilon^{-1}\Omega^{-1})\right\}$ and ρ^{TF} tends to a distribution supported on $\partial\mathcal{B}_1$.

Theorem (Energy and Density Asymptotics [MC,Yngvason '08])

For any $\varepsilon^{-1} \lesssim \Omega \ll \frac{1}{\varepsilon^2 |\log\varepsilon|}$,

$$E^{\mathrm{GP}} = E^{\mathrm{TF}} + \frac{\Omega|\log\varepsilon|}{2}(1 + o(1)).$$

The density $|\Psi^{\mathrm{GP}}|^2$ *is exponentially small in* ε *almost everywhere, except for a thin layer of width* $\sim \varepsilon^{-1}\Omega^{-1}$ *around the boundary* $\partial\mathcal{B}_1$.

- If $\Omega \sim \varepsilon^{-2}$ the occupation of Landau levels becomes relevant (Landau regime). Why the upper bound $\Omega \ll \frac{1}{\varepsilon^2|\log\varepsilon|}$?

Ultrarapid Rotation ($\Omega \gg \varepsilon^{-1}$): Emergence of the Giant Vortex

- As long as $\Omega \ll \dfrac{1}{\varepsilon^2|\log\varepsilon|}$ there are vortices in the support of ρ^{TF} even though it is very thin.
- If $\Omega \gtrsim \dfrac{1}{\varepsilon^2|\log\varepsilon|}$ one can concentrate the whole vorticity in the center and lower the energy (giant vortex).

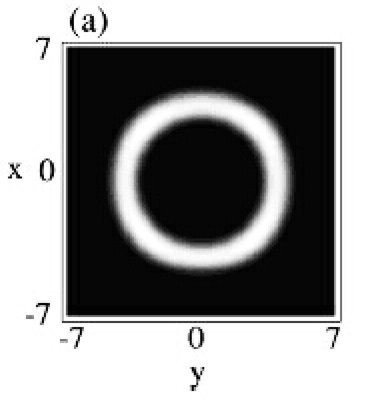
Heuristic Comparison

- For $\Omega \gg \varepsilon^{-1}$ the number of cells inside $\mathrm{supp}(\rho^{\mathrm{TF}})$ is $\sim \varepsilon^{-1} \Longrightarrow$ the mutual interaction is $\propto \#$ of pairs $\sim \varepsilon^{-2}$.
- Vortices (one inside each cell) neutralize the mutual interaction but have an energy cost $\sim \Omega|\log\varepsilon|$.
- The vortex energy is of the same order of the mutual energy if $\Omega \gtrsim \varepsilon^{-2}|\log\varepsilon|^{-1} \Longrightarrow$ a transition takes place at that threshold.

Numerical Simulations [Kasamatsu *et al* '02]

Condensate Density

Giant Vortex



The GP minimizer is concentrated in a thin annulus near $\partial\mathcal{B}_1$ (giant vortex) and exponentially small everywhere else but the essential support of Ψ^{GP} contains *no* vortices.

Ultrarapid Rotation ($\Omega \gg \varepsilon^{-1}$): The Giant Vortex

Rigorous Comparison [MC,Rougerie,Yngvason '09]

- The upper bound $E^{\mathrm{GP}} \leq E^{\mathrm{TF}} + \frac{\Omega|\log\varepsilon|}{2}(1+o(1))$ holds for any $|\log\varepsilon| \ll \Omega \ll \varepsilon^{-2}$.
- A trial function of the form $\Psi_{\mathrm{giant}}(\vec{r}) \simeq \sqrt{\rho^{\mathrm{TF}}(r)}e^{iN_\Omega\vartheta}$, with winding number $N_\Omega \sim \Omega/2$ yields
$$\mathcal{E}^{\mathrm{GP}}[\Psi_{\mathrm{giant}}] = E^{\mathrm{TF}} + \mathcal{O}(\varepsilon^{-2}) + \mathcal{O}(\varepsilon^2\Omega^2|\log\varepsilon|)$$
but $\varepsilon^{-2} \lesssim \Omega|\log\varepsilon|$ in this regime $\Longrightarrow \Psi_{\mathrm{giant}}$ lowers the energy.
- If $\Omega > \Omega_c = \mathcal{O}(\varepsilon^{-2}|\log\varepsilon|^{-1})$, vortices are expelled from the *essential* support of Ψ^{GP}.
- Inside the *hole* the GP minimizer is exponentially small in ε and contains a very large number of *vortices*, i.e., $\Psi^{\mathrm{GP}} \sim \Psi_{\mathrm{giant}}$ only inside its *essential* support

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