Vortices in Rotating Bose-Einstein Condensates
A Review of (Recent) Mathematical Results

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Outline

2. Vortices and Rotational Symmetry Breaking.
3. The Thomas-Fermi (TF) Limit of the GP Theory:
   - Harmonic trapping potentials [IM].
   - (Strongly) Anharmonic trapping potentials [CY].

Main References

- Numerics: W. Bao, in *Dynamics in Models of Coarsening, Coagulation, Condensation and Quantization*, 2007.
The GP Theory of a Rotating BE Condensate in a Trap

- 2d BE condensate rotating along the $\hat{z}$-axis with angular velocity $\Omega$.
- External trap given by a potential $V(\vec{r})$ ($V(\vec{r}) \to \infty$ as $r \to \infty$).
- The stationary ground state properties of a rotating BE condensate can be described through minimizers of the GP energy functional (in the non-inertial rotating frame).
- $L = -i(x\partial_y - y\partial_x)$ is the $z$-component of the angular momentum.
- $\varepsilon^{-2}$ is the coupling parameter ($\propto$ scattering length).
- The TF limit is $\varepsilon \to 0$ (large coupling and/or fast rotation).

The GP Energy Functional

$$E^{\text{GP}}[\Psi] = \int_{\mathbb{R}^2} d\vec{r} \left\{ |\nabla \Psi|^2 - \Psi^* \Omega L \Psi + V(\vec{r}) |\Psi|^2 + \frac{|\Psi|^4}{\varepsilon^2} \right\}.$$
Minimization of the GP Functional

$$\mathcal{E}^{\text{GP}}[\psi] = \int_{\mathbb{R}^2} d\vec{r} \left\{ \left| (\nabla - i\vec{A}_\Omega) \psi \right|^2 + \left[ V(\vec{r}) - \frac{\Omega^2 r^2}{4} \right] |\psi|^2 + \frac{|\psi|^4}{\varepsilon^2} \right\}$$

- The vector potential is $\vec{A}_\Omega = \Omega \hat{e}_z \wedge \vec{r}/2$.
- GP ground state energy $E^{\text{GP}} = \inf_{\|\psi\|_2=1} \mathcal{E}^{\text{GP}}[\psi]$.
- $\psi^{\text{GP}}$ stands for any corresponding minimizer.

Boundedness from Below of $\mathcal{E}^{\text{GP}}$

Assume that $V \in L^2_{\text{loc}}(\mathbb{R}^3)$ and either $V(r) \simeq r^2$, $\Omega < 2$, or $V(r) \simeq r^s$, $s > 2$, as $r \to \infty \implies \mathcal{E}^{\text{GP}}$ is bounded from below. Usually one considers

- the harmonic potential $V(r) = r^2$ (upper bound on $\Omega$!),
- (strongly) anharmonic (homogeneous) potentials $V(r) \simeq r^s$, $s > 2$ (no upper bound on $\Omega$!): the simplest example is a confinement to the unitary disc $B_1$ with Neumann boundary conditions ($s = \infty$).
Existence of a Minimizer $\Psi^{GP}$

Under the same hypothesis on $V$, $\exists$ (at least) one minimizer $\Psi^{GP}$.

- $\Psi^{GP}$ solves the GP time-independent equation

$$-\Delta \Psi^{GP} - \Omega L \Psi^{GP} + V \Psi^{GP} + 2\epsilon^{-2} |\Psi^{GP}|^2 \Psi^{GP} = \mu^{GP} \Psi^{GP}. $$

The chemical potential $\mu^{GP}$ is fixed by the $L^2$ normalization:

$$\mu^{GP} = E^{GP} + \epsilon^{-2} \|\Psi^{GP}\|_4^4.$$

- If $V$ is smooth, the same is $\Psi^{GP}$.

Uniqueness of $\Psi^{GP}$

The minimizer is not necessarily unique.

- Non-uniqueness is due to the presence of the angular momentum term. If $\Omega = 0$, $E^{GP}$ is strictly convex in $\rho = |\Psi|^2$.

- Non-uniqueness is strictly related to the occurrence of isolated vortices $\iff$ rotational symmetry breaking.
Ginzburg-Landau vs. Gross-Pitaevskii

[GL] Minimize $\mathcal{E}^{\text{GL}}$ w.r.t. $u$ and $\vec{A}$ with $h = \text{curl}(\vec{A})$, $h_{\text{ex}} = \Omega$ (uniform magnetic field),

$$
\mathcal{E}^{\text{GL}}[u, \vec{A}] = \int_D d\vec{r} \left\{ \left| \left( \nabla - i\vec{A} \right) u \right|^2 + |h - h_{\text{ex}}|^2 + \varepsilon^{-2} (1 - |u|^2)^2 \right\}.
$$

[GP] Minimize $\mathcal{E}^{\text{GP}}$ w.r.t. $L^2$ normalized $\Psi(\vec{r}) = \sqrt{\rho(\vec{r})} u(\vec{r})$,

$$
\mathcal{E}^{\text{GP}}[\Psi] = \mathcal{E}^{\text{TF}}[\rho] + \int_S d\vec{r} \left\{ \left| \left( \nabla - i\vec{A}_{\Omega} \right) u \right|^2 + \varepsilon^{-2} \rho \left( 1 - |u|^2 \right) \right\} \rho.
$$

Main Differences:

- In GL there is an additional minimization w.r.t. $\vec{A}$, whereas in GP $\vec{A}$ is given $\implies$ major differences in the minimizers but the ground state energies can be close in certain regimes.
- No $L^2$ normalization in GL $\implies$ no chemical potential in GL $\implies$ no density $\rho$ in GL. Not only a technicality since there is some physics behind it!
Vortices and Rotational Symmetry Breaking

\[ \mathcal{E}^{\text{GP}} [\psi] = \int_{\mathbb{R}^2} d\vec{r} \left\{ \left| (\nabla - i\vec{A}\Omega) \psi \right|^2 + \left[ V(r) - \frac{\Omega^2 r^2}{4} \right] |\psi|^2 + \frac{|\psi|^4}{\varepsilon^2} \right\} \]

Rotational Symmetry Breaking

- The functional \( \mathcal{E}^{\text{GP}} \) is invariant under rotations around the \( z \)-axis.
- If \( \varepsilon \) is fixed and \( \Omega \) large, the GP minimizer is not an eigenfunction of the angular momentum (rotational symmetry breaking), due to the occurrence of isolated vortices.

Vortices

\( \psi^{\text{GP}} \) has a vortex at \( \vec{r}_0 \) (\( \zeta_0 = x_0 + iy_0 \)) with winding number \( d \), if

\[ \psi^{\text{GP}}(\vec{r}_0) = 0, \quad (\text{locally}) \quad \frac{\psi^{\text{GP}}}{|\psi^{\text{GP}}|} \approx \left[ \frac{\zeta - \zeta_0}{|\zeta - \zeta_0|} \right]^d. \]
Formation of quantized vortices in a rotating Rb BE condensate.

Vortices in a fast rotating Rb BE condensate in a quartic+quadratic trap.
Why Are Vortices Energetically Favorable?

\[ \mathcal{E}^{GP} [\Psi] = \int_{B_1} d\vec{r} \left\{ |\nabla \Psi|^2 - \Omega \Psi^* L \Psi + \varepsilon^{-2} |\Psi|^4 \right\} \]

**Energy Compensation**

- For small rotational velocities the condensate is at rest in the inertial frame (superfluidity) \( \implies |\Psi^{GP}| = \text{const.} \)

- At higher angular velocities vortices start to occur: For small \( \varepsilon \), a vortex of degree \( d \) at the origin has the form \( f(r) \exp\{id\vartheta\} \), with \( f \) approx. constant outside \( B_\varepsilon \) (\( \leftarrow \) nonlinear term).

- Kinetic energy \( \int_{B_1 \setminus B_\varepsilon} d\vec{r} \ |\nabla \Psi|^2 \approx d^2 \int_{B_1 \setminus B_\varepsilon} d\vec{r} \frac{1}{r^2} \approx Cd^2 |\log \varepsilon| \).

- Angular momentum \( -\Omega \int_{B_1 \setminus B_\varepsilon} d\vec{r} \Psi^* L \Psi \approx -\Omega d \).

- If \( \Omega \sim C |\log \varepsilon| \), a vortex can be energetically favorable.
Why/When More Vortices?

\[ \mathcal{E}^{GP} [\psi] = \int_{B_1} d\vec{r} \left\{ |\nabla \psi|^2 - \Omega \psi^* \mathcal{L} \psi + \varepsilon^{-2} |\psi|^4 \right\} \]

Energy Optimization

- So far we have not justified the breaking of the rotational symmetry, since a vortex at the origin is an eigenfunction of \( L \! \! \! . \)
- \( \psi^{GP} \) can contain more than one vortex (\( \Leftarrow \) nonlinearity).
- \( \Omega \) fixes the total winding number \( d \) (angular momentum) of \( \psi^{GP} \) and, if \( \Omega \) is sufficiently large, \( d > 1 \).
- Suppose \( d = 2 \): The kinetic energy is \( \propto d^2 \rightarrow 2 \) vortices of winding number 1 have a smaller kinetic energy (1 + 1 = 2) than 1 vortex of winding number 2 (2^2 = 4), but \textit{almost} the same angular momentum.
- If \( \varepsilon \ll 1 \) the vortex cores are small (\( \sim \varepsilon \)) and the interaction energy can be neglected \( \Rightarrow \) many vortices can be energetically \textit{favorable}. 
Rotational Symmetry Breaking

**Theorem (Symmetry Breaking [MC,Rindler-Daller,Yngvason ’07])**

As \( \varepsilon \to 0 \), *no* minimizer of \( \mathcal{E}_{GP}[\psi] \) is an eigenfunction of the angular momentum, if

\[
6|\log \varepsilon| + 3 < \Omega \lesssim \frac{C}{\varepsilon}
\]

for any constant \( C \in \mathbb{R}^+ \).

- Symmetry breaking is due to occurrence of isolated vortices outside of the origin.
- The GP minimizer is no longer unique (\( \infty \) degeneracy).
- The estimate of the symmetry breaking threshold is not optimal (it is expected to be \( 2|\log \varepsilon| \) [Aftalion,Du ’01]).
- The rotational symmetry is expected to be broken also for \( \Omega \gg \varepsilon^{-1} \).
The Thomas-Fermi Limit of the Gross-Pitaevskii Theory

Nucleation of Vortices (Harmonic Traps) [Ignat, Millot ’06]

\[ \mathcal{E}^{GP} [\psi] = \int_{\mathbb{R}^2} d\vec{r} \left\{ |\nabla \psi|^2 - \psi^* \Omega L \psi + \varepsilon^{-2} V(x, y)|\psi|^2 + \varepsilon^{-2} |\psi|^4 \right\} \]

- Harmonic trapping potential (rescaled): \( V(x, y) = (x^2 + \Lambda^2 y^2) \). 0 < \( \Lambda \leq 1 \) measures the asymmetry of the potential.
- The coefficient \( \varepsilon^{-2} \) of \( V \) is chosen so that the first critical velocity is \( \mathcal{O}(|\log \varepsilon|) \) (it is equivalent to rescale all lengths):
  \[ \Omega_1 = \frac{\sqrt{\pi}(\Lambda^2+1)}{\sqrt{2\Lambda}} |\log \varepsilon|. \]
- The vortex free profile \( \eta_\varepsilon \) is the (unique \( L^2 \) normalized) minimizer of
  \[ \tilde{\mathcal{E}}^{GP}_\varepsilon [\phi] = \int_{\mathbb{R}^2} d\vec{r} \left\{ |\nabla \phi|^2 + \varepsilon^{-2} V|\phi|^2 + \varepsilon^{-2} |\phi|^4 \right\}. \]
- \( \eta_\varepsilon \) is real and positive. If \( \Lambda = 1 \), it is also radial.
- As \( \varepsilon \to 0 \), \( \eta_\varepsilon^2 \to \rho^{TF}(x, y) = \frac{1}{2} [\mu - V(x, y)]_+ \) in \( L^\infty(D) \).
The Thomas-Fermi Limit of the Gross-Pitaevskii Theory

**Theorem (Absence of Vortices below $\Omega_1$ [Ignat, Millot '06])**

For any $\delta > 0$, if $\Omega \leq \Omega_1 - \delta \log |\log \varepsilon|$, then

- $|\Psi_{GP}| \xrightarrow{\varepsilon \to 0} \sqrt{\rho_{TF}(x, y)}$ in $L^\infty_{\text{loc}}(\mathbb{R}^2 \setminus \partial D)$,

- $E^{GP} = \mathcal{E}^{GP} [\eta_{\varepsilon} e^{iS\Omega}] + o(1)$, where $S(x, y) = \frac{\Lambda^2 - 1}{\Lambda^2 + 1} xy$,

- Up to a subsequence (and a global phase factor $\alpha$, $|\alpha| = 1$)

$\Psi_{GP} \xrightarrow{\varepsilon \to 0} \alpha \sqrt{\rho_{TF}(x, y)} e^{iS\Omega}$,

in $H^1_{\text{loc}}(D) \implies$ no vortices, i.e., for any $R_0 < \sqrt{\mu}$ and $\varepsilon$ sufficiently small, $\Psi_{GP}$ does not vanish inside the region where $x^2 + \Lambda^2 y^2 < R_0^2$.

**Theorem (Occurrence of Vortices above $\Omega_1$ [Ignat, Millot '06])**

For any $\delta > 0$, if $\Omega \geq \Omega_1 + \delta \log |\log \varepsilon|$ and $\varepsilon$ is sufficiently small, then

- $\Psi_{GP}$ has at least one vortex at $\vec{r}_\varepsilon$ such that $\text{dist}(\vec{r}_\varepsilon, \partial D) \geq C$.

- If in addition $\Omega \leq \Omega_1 + O(\log |\log \varepsilon|)$, then the vortex remains close to the origin, i.e., $|\vec{r}_\varepsilon| = o(1)$.  

M. Correggi (CIRM)  
Vortices in Rotating BE Condensates  
Verona 15/09/2009 13 / 35
Critical Velocities [Ignat, Millot ’06]

\[ \Omega_d = \frac{\sqrt{\pi}(1+\Lambda^2)}{\sqrt{2\Lambda}} \left( |\log \varepsilon| + (d - 1) \log |\log \varepsilon| \right), \quad d \in \mathbb{N} \]

Theorem (Number and Distribution of Vortices [Ignat, Millot ’06])

For any \(0 < \delta \ll 1\), if \(\Omega_d + \delta \log |\log \varepsilon| \leq \Omega \leq \Omega_{d+1} - \delta \log |\log \varepsilon|\), then

- For any \(R_0 < \sqrt{\mu}\) and \(\varepsilon\) sufficiently small, \(\Psi^{\text{GP}}\) has exactly \(d\) vortices of winding number 1 at \(\vec{r}_{i,\varepsilon}, i = 1, \ldots, d\) inside \(x^2 + \Lambda^2 y^2 < R_0^2\),

- Vortices remain close to the origin and close one another, i.e.,
  \[|\vec{r}_i| \leq C \Omega^{-1/2}, \quad |\vec{r}_i - \vec{r}_j| \leq C \Omega^{-1/2}.\]

- Setting \(\vec{\xi}_i = \sqrt{\Omega} \vec{r}_i\), the configuration \((\vec{\xi}_1, \ldots, \vec{\xi}_d)\) minimizes the renormalized energy

\[
W \left( \vec{\xi}_1, \ldots, \vec{\xi}_d \right) = -\pi \sqrt{\mu} \sum_{i \neq j} \log |\vec{\xi}_i - \vec{\xi}_j| + \frac{\pi \sqrt{\mu}}{1 + \Lambda^2} \sum_{i=1}^{d} \left( x_i^2 + \Lambda^2 y_i^2 \right).
\]
### Distribution of Vortices [Gueron, Shafrir ’00]

The distribution of vortices is determined by the minimization of $W$:

- If $\Lambda = 1$ and $d \leq 6$, regular polygons and stars ($d - 1$ side regular polygon plus the origin) centered at the origin are (local) minimizing configurations for the renormalized energy $W$.
- If $d \geq 11$ neither regular polygons nor stars are local minimizers of $W$. As $d$ increases the minimizers approach a triangular lattice.

### Larger Angular Velocities [Baldo, Jerrard, Orlandi, Soner ’08]

- If $\Omega \gg \Omega_d$ for any $d$ but $\Omega = \mathcal{O}(|\log \varepsilon|)$, the number of vortices is not uniformly bounded in $\varepsilon$.
- The vortex distribution minimizes a (rescaled) free boundary problem.

### Landau Regime [Aftalion et al ’06]

When $\Omega \approx \varepsilon^{-1}$, the occupation of Landau levels become relevant...
Rotating BE Condensates in Anharmonic Traps
[MC,Rindler-Daller,Yngvason ’07]

\[ \mathcal{E}^{\text{GP}} [\psi] = \int_{B_1} d\vec{r} \left\{ \left| \left[ \nabla - i \vec{A}_\Omega \right] \psi \right|^2 - \frac{\Omega^2 r^2 |\psi|^2}{4} + \frac{|\psi|^4}{\varepsilon^2} \right\} \]

Motivations

- There is no upper bound on the angular velocity \( \Omega \), i.e., the condensate is confined for any \( \Omega \) \( \Rightarrow \) one can explore regimes of very fast rotation.
- The unitary disc is the strongest anharmonic trap one can think of since it can be formally obtained as the limit \( s \to \infty \) of a homogeneous potential \( V(r) = r^s \).
- Any homogeneous potential \( V(r) = cr^s, s > 2 \) can be mapped to the above model by means of a suitable rescaling of all lengths [MC,Rindler-Daller,Yngvason ’07].
Extraction of the TF Density

\[ E^{\text{GP}} [\Psi] = \int_{B_1} d\vec{r} \left\{ \left| \left[ \nabla - i \widetilde{A}_\Omega \right] \Psi \right|^2 - \frac{\Omega^2 r^2 |\Psi|^2}{4} + \frac{|\Psi|^4}{\varepsilon^2} \right\} \]

- If \( \Omega \lesssim \varepsilon^{-2} \), the kinetic energy gives a smaller order correction.
- In anharmonic traps the second part of \( E^{\text{GP}} \) depends on \( \Omega \).

TF Energy Functional

\[ E^{\text{TF}} [\rho] = \int_{B_1} d\vec{r} \left\{ \frac{\rho^2}{\varepsilon^2} - \frac{\Omega^2 r^2 \rho}{4} \right\}, \]

with ground state energy \( E^{\text{TF}} = \inf_{\|\rho\|_1=1} E^{\text{TF}} [\rho] \) and \( (L^1 \text{ normalized}) \) minimizer

\[ \rho^{\text{TF}} (r) = \frac{1}{2} \left[ \mu^{\text{TF}} + \frac{\varepsilon^2 \Omega^2 r^2}{4} \right]_+ . \]
Asymptotics of the TF Functional

\[ E_{\text{TF}}[\rho] = \int_{B_1} d\vec{r} \left\{ \frac{\rho^2}{\varepsilon^2} - \frac{\Omega^2 r^2 \rho}{4} \right\} \]

Slow Rotation (\( \Omega \ll \varepsilon^{-1} \))
- \( E_{\text{TF}} = \pi^{-1} \varepsilon^{-2} + \mathcal{O}(\Omega^2) \) and \( \rho_{\text{TF}}(r) = \pi^{-1} + \mathcal{O}(\varepsilon^2 \Omega^2) \).

Rapid Rotation (\( \Omega \sim \varepsilon^{-1} \))
- \( E_{\text{TF}} = \mathcal{O}(\varepsilon^{-2}) \) and \( \rho_{\text{TF}}(r) \approx [C_1 + C_2 r^2]_+ \).
- If \( \Omega > 4/(\sqrt{\pi} \varepsilon) \), \( \rho_{\text{TF}}(r) = 0 \) for any \( r \leq R_{\text{in}} \equiv \sqrt{1 - \frac{4}{\sqrt{\pi} \varepsilon \Omega}} \) (hole).

Ultrarapid Rotation (\( \Omega \gg \varepsilon^{-1} \))
- \( E_{\text{TF}} = -\Omega^2/4 + \mathcal{O}(\Omega) \) and \( \rho_{\text{TF}}(r) = \frac{\varepsilon^2 \Omega^2}{8} \left[ r^2 - R_{\text{in}}^2 \right]_+ \).
- \( R_{\text{in}} = 1 - \mathcal{O}(\varepsilon^{-1} \Omega^{-1}) \implies \rho_{\text{TF}} \) approaches \( \delta(1 - r) \).
Experimental Observations [Engels et al ’03]

Condensate Density

“Giant Vortex” (Hole) formation in a rotating $^{87}\text{Rb}$ BE condensate (induced by a laser beam).
The Thomas-Fermi Limit of the Gross-Pitaevskii Theory

Slow Rotation ($\Omega \ll \varepsilon^{-1}$)

**Theorem (GP Asymptotics [MC, Rindler-Daller, Yngvason ’07])**

*If* $\varepsilon \Omega \to 0$ *as* $\varepsilon \to 0$,

$$E_{GP} = \frac{1}{\pi \varepsilon^2} + O(\Omega^2), \quad \left\| |\Psi_{GP}|^2 - \frac{1}{\pi} \right\|_{L^1(B_1)} = O(\varepsilon \Omega).$$

- $\pi^{-1}$ is the GS density of the GP functional without rotation (with energy $\pi^{-1} \varepsilon^{-2}$) $\Rightarrow$ the rotation has no leading order effect on the GS asymptotics.
- If $\Omega \ll |\log \varepsilon|$, the GP minimizer is unique and strictly positive. In this case $\Psi_{GP} \to \pi^{-1}$ as $\varepsilon \to 0$ in $H^1(B_1)$ (no vortex).
- If $\Omega \sim |\log \varepsilon|$, vortices start to occur in $\Psi_{GP} \Rightarrow$ rotational symmetry breaking.
Occurrence and Distribution of Vortices ($\Omega \ll \varepsilon^{-1}$)

1. $\Omega \ll 2|\log \varepsilon|$
   - The GP minimizer is a *unique* and a *strictly positive* radial function.

2. $\Omega \sim 2|\log \varepsilon| + \mathcal{O}(\log |\log \varepsilon|)$
   - *Uniformly bounded* (in $\varepsilon$) number of vortices at $\vec{r}_i$, $i = 1, \ldots, n$.
   - $n$ is fixed by the *remainder* in the angular velocity asymptotics (coefficient of $\log |\log \varepsilon|$).
   - Vortices are very *close* to the origin: $|\vec{r}_i| \sim |\log \varepsilon|^{-1/2}$ and $|\vec{r}_i - \vec{r}_j| \sim |\log \varepsilon|^{-1/2}$. The vortex core is a ball of radius $\sim \varepsilon^n$.
   - Vortices arrange in *regular polygons* centered at the origin to minimize the interaction energy.

3. $\Omega \sim C|\log \varepsilon|$, $C > 2$
   - The number of vortices is *no longer* uniformly bounded.
   - Vortices are confined to a *subset* of $B_1$ (free boundary problem).

4. $2|\log \varepsilon| \ll \Omega \ll \varepsilon^{-1}$
   - The number of vortices is $\sim \Omega/2$.
   - Vortices with winding number 1 are *uniformly* distributed over $B_1$. 
Vortex Energy Contribution ($\Omega \ll \varepsilon^{-1}$)

**Theorem (Improved Energy Asymptotics [MC,Yngvason '08])**

For any $|\log \varepsilon| \ll \Omega \ll \varepsilon^{-1}$,

$$E_{GP} = E_{TF} + \frac{\Omega |\log(\varepsilon^2 \Omega)|}{2} (1 + o(1)).$$

- Since $\Omega \ll \varepsilon^{-1}$, $E_{TF} = \pi^{-1} \varepsilon^{-2} (1 + o(1))$.
- Vortices of winding number 1 are uniformly distributed over $B_1$, their number is $\sim \Omega/2$ and their core is $\sim \varepsilon$.
- Each vortex gives a kinetic contribution of order $|\log(\varepsilon^2 \Omega)|$:
  $$2\pi \int_{\varepsilon}^{\Omega^{-1/2}} dr \ r^{-1} \approx \pi |\log(\varepsilon^2 \Omega)|.$$
- No proof of the existence of isolated vortices but only of a uniform distribution of vorticity satisfying the above properties.
Rapid Rotation ($\Omega \sim \varepsilon^{-1}$): Numerical Simulations

[ Kasamatsu et al. '02 ]

Condensate Density

The GP minimizer is exponentially small in a disc centered at the origin (hole) and vortices cover the whole trap.
The Thomas-Fermi Limit of the Gross-Pitaevskii Theory

Rapid Rotation \((\Omega \sim \varepsilon^{-1})\)

**Theorem (GP Asymptotics [MC,Yngvason '08])**

For any \(\Omega \sim \varepsilon^{-1}\),

\[
E^{\text{GP}} = E^{\text{TF}} + \frac{\Omega |\log \varepsilon|}{2}(1 + o(1)),
\]

\[
\|\psi^{\text{GP}}|^2 - \rho^{\text{TF}}\|_{L^1(B_1)} = O \left( \sqrt{\varepsilon |\log \varepsilon|} \right).
\]

- \(\psi^{\text{GP}}\) contains a number \(\sim \Omega/2\) of vortices with winding number 1 uniformly distributed over a regular lattice with spacing \(\sim \sqrt{\varepsilon}\).
- The vortex core is a ball of radius \(\sim \varepsilon\).
- Each vortex gives an kinetic contribution of order \(|\log \varepsilon|\):

\[
2\pi \int_{\sqrt{\varepsilon}}^{\varepsilon} dr \ r^{-1} \simeq \pi |\log \varepsilon|.
\]
Vortex Energy Contribution ($\Omega \sim \varepsilon^{-1}$)

- $\Psi^{GP} \approx \sqrt{\rho_{TF}(r)} e^{i\phi_{\varepsilon}}$, where the phase $\phi_{\varepsilon}$ minimizes the kinetic energy and contains the vortices:
  
  $$ e^{i\phi_{\varepsilon}} = \prod \frac{\zeta - \zeta_i}{|\zeta - \zeta_i|}, \quad \phi_{\varepsilon} = \sum \arctan \left[ \frac{y - y_i}{x - x_i} \right]. $$

- Kinetic energy $\left\| \nabla \phi_{\varepsilon} - \vec{A}_\Omega \right\|_2^2 = \left\| \nabla \tilde{\phi}_{\varepsilon} - |\vec{A}_\Omega| \hat{r} \right\|_2^2$, where we have used the rotation $\phi_{\varepsilon} \rightarrow \tilde{\phi}_{\varepsilon} = \sum \log |\zeta - \zeta_i|$, $\vec{A}_\Omega \rightarrow |\vec{A}_\Omega| \hat{r}$.

Electrostatic Analogy

- Vortex at $\vec{r}_i$ (vortex core of size $\sim \varepsilon^\eta$) $\iff$ Unitary Point Charge + 1 at $\vec{r}_i$ (smeared over a ball of size $\sim \varepsilon^\eta$)
- Vector Potential $\vec{A}_\Omega$ $\iff$ Uniform Charge Density $- \frac{\Omega}{2\pi}$
- Vortex Kinetic Energy $\iff$ Energy of the Electric Field
Finding the optimal distribution of vortices is *equivalent* to minimize the electrostatic energy of a charge distribution given by positive point charges and a negative uniform background.

As long as $\rho^{\text{TF}}$ varies on a scale of order 1, the optimal distribution is uniform: Vortices on a regular lattice with fundamental cell $Q_\varepsilon$ and spacing $\ell_\varepsilon \sim \sqrt{|Q_\varepsilon|}$ covering $B_1$ (triangular, square or hexagonal lattice).

The cell *volume* is chosen so that the cell is neutral, i.e., $|Q_\varepsilon| = 2\pi/\Omega$.

The dipole associated with any cell $Q^i_\varepsilon$ vanishes because of the cell symmetry $\implies$ the electric field $\vec{E}_i$ generated by $Q^i_\varepsilon$ decays very fast (at least $\sim r^{-3}$) outside $Q^i_\varepsilon$ $\implies$ the leading order contribution is given by the self-energy inside $Q^i_\varepsilon$ (*minimized* by the triangular lattice).

Self-energy inside $Q^i_\varepsilon$: $\int_{Q^i_\varepsilon \setminus B^i_\varepsilon} d\vec{r} \left| \nabla \phi_\varepsilon - \vec{A}_\Omega \right|^2 = \pi |\log \varepsilon| + O(1)$.
The Thomas-Fermi Limit of the Gross-Pitaevskii Theory

Slow and Rapid Rotation \((Ω \lesssim \varepsilon^{-1})\): Vorticity

**Theorem (Uniform Distribution of Vorticity [MC,Yngvason ’08])**

Let \(\varepsilon > 0\) be sufficiently small and \(|\log \varepsilon| \ll Ω \lesssim \varepsilon^{-1}\). Then there exists a finite family of disjoint balls \(\{B^i_\varepsilon\} \subset \text{supp}(\rho^{\text{TF}})\) such that

1. the radius of any ball is at most of order \(1/\sqrt{Ω}\),
2. the sum of all the radii is at most of order \(\sqrt{Ω}\),
3. \(|\Psi^{\text{GP}}| \geq C|\log(\varepsilon^2Ω)|^{-1}\) on \(\partial B^i_\varepsilon\), for some \(C > 0\),

and, denoting by \(\vec{r}_{i,\varepsilon}\) the center of each ball \(B^i_\varepsilon\) and by \(d_{i,\varepsilon}\) the winding number of \(|\Psi^{\text{GP}}|^{-1}\Psi^{\text{GP}}\) on \(\partial B^i_\varepsilon\),

\[
\frac{2\pi}{Ω} \sum d_{i,\varepsilon} \delta (\vec{r} - \vec{r}_{i,\varepsilon}) \xrightarrow{\varepsilon \to 0} \chi^{\text{TF}}(\vec{r}) \, d\vec{r}.
\]

If one could prove that \(\Psi^{\text{GP}}\) has only isolated vortices, then the Theorem would yield their number and uniform distribution.
Ultrarapid Rotation \((\Omega \gg \varepsilon^{-1})\)

- The rotational energy dominates:
  \[ E_{TF} = -\frac{\Omega^2}{4} \{1 + O(\varepsilon^{-1}\Omega^{-1})\} \]
  and \(\rho_{TF}\) tends to a distribution supported on \(\partial B_1\).

**Theorem (Energy and Density Asymptotics [MC,Yngvason ’08])**

For any \(\varepsilon^{-1} \lesssim \Omega \ll \frac{1}{\varepsilon^2|\log \varepsilon|}\),

\[ E_{GP} = E_{TF} + \frac{\Omega|\log \varepsilon|}{2}(1 + o(1)). \]

The density \(|\psi_{GP}|^2\) is exponentially small in \(\varepsilon\) almost everywhere, except for a thin layer of width \(\sim \varepsilon^{-1}\Omega^{-1}\) around the boundary \(\partial B_1\).

- If \(\Omega \sim \varepsilon^{-2}\) the occupation of Landau levels becomes relevant (Landau regime). Why the upper bound \(\Omega \ll \frac{1}{\varepsilon^2|\log \varepsilon|}\)?
Ultrarapid Rotation ($\Omega \gg \varepsilon^{-1}$): Emergence of the Giant Vortex

- As long as $\Omega \ll \frac{1}{\varepsilon^2 |\log \varepsilon|}$ there are vortices in the support of $\rho^{\text{TF}}$ even though it is very thin.
- If $\Omega \gtrsim \frac{1}{\varepsilon^2 |\log \varepsilon|}$ one can concentrate the whole vorticity in the center and lower the energy (giant vortex).

Heuristic Comparison

- For $\Omega \gg \varepsilon^{-1}$ the number of cells inside $\text{supp}(\rho^{\text{TF}})$ is $\sim \varepsilon^{-1} \implies$ the mutual interaction is $\propto \# \text{ of pairs} \sim \varepsilon^{-2}$.
- Vortices (one inside each cell) neutralize the mutual interaction but have an energy cost $\sim \Omega |\log \varepsilon|$.
- The vortex energy is of the same order of the mutual energy if $\Omega \gtrsim \varepsilon^{-2} |\log \varepsilon|^{-1} \implies$ a transition takes place at that threshold.
The GP minimizer is concentrated in a thin annulus near $\partial B_1$ (giant vortex) and exponentially small everywhere else but the essential support of $\psi_{\text{GP}}$ contains no vortices.
The Thomas-Fermi Limit of the Gross-Pitaevskii Theory

Ultrarapid Rotation ($\Omega \gg \varepsilon^{-1}$): The Giant Vortex

Rigorous Comparison [MC, Rougerie, Yngvason ’09]

- The upper bound $E_{\text{GP}} \leq E_{\text{TF}} + \frac{\Omega |\log \varepsilon|}{2} (1 + o(1))$ holds for any $|\log \varepsilon| \ll \Omega \ll \varepsilon^{-2}$.

- A trial function of the form $\psi_{\text{giant}}(r) \approx \sqrt{\rho_{\text{TF}}(r)} e^{i N \Omega \vartheta}$, with winding number $N_{\Omega} \sim \Omega/2$ yields
  \[ E_{\text{GP}}[\psi_{\text{giant}}] = E_{\text{TF}} + O(\varepsilon^{-2}) + O(\varepsilon^{2} \Omega^{2} |\log \varepsilon|) \]

  *but* $\varepsilon^{-2} \lesssim \Omega |\log \varepsilon|$ in this regime $\implies \psi_{\text{giant}}$ lowers the energy.

- If $\Omega > \Omega_{c} = O(\varepsilon^{-2} |\log \varepsilon|^{-1})$, vortices are expelled from the essential support of $\psi_{\text{GP}}$.

- Inside the *hole* the GP minimizer is exponentially small in $\varepsilon$ and contains a very large number of vortices, i.e., $\psi_{\text{GP}} \sim \psi_{\text{giant}}$ only inside its essential support.
References (Physics)


M.C., N. Rougerie, J. Yngvason, in preparation.


References (Math)


