Spectral methods for dissipative nonlinear Schrödinger equations

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Mathematical Models of Quantum Fluids
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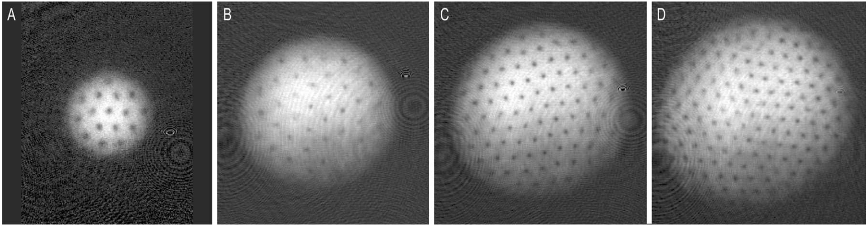
*Joint work with Laura M. Morato and Simone Zuccher

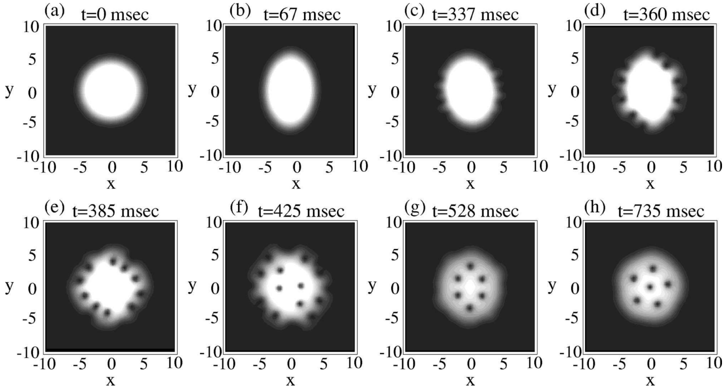
Motivation: observation of vortex lattices

Observation of vortex lattices in Bose–Einstein condensates:



[J. R. Abo-Shaeer, C. Raman, J. M. Vogels, and W. Ketterle, Science 292 (2001)]

Numerical simulation



[K. Kasamatsu, M. Tsubota, and M. Ueda, Phys. Rev. A 67 (2003)]

Gross–Pitaevskii Equation (GPE)

$$\begin{cases} \mathrm{i}\partial_t \Psi(X,t) = \left(-\frac{1}{2}\nabla^2 + V(X) + \theta\left|\Psi(X,t)\right|^2\right)\Psi(X,t), \quad X \in \mathbb{R}^3 \\ \Psi(X,0) = \Psi_0(X), \quad \int_{\mathbb{R}^3} \left|\Psi_0(X)\right|^2 \mathrm{d}X = 1 \end{cases}$$

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Conservation properties:

- particle number

$$\int_{\mathbb{R}^3} \left|\Psi(X,t)\right|^2 \mathrm{d}X = \int_{\mathbb{R}^3} \left|\Psi_0(X)\right|^2 \mathrm{d}X = 1$$

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Conservation properties:

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$$\int_{\mathbb{R}^3} \left|\Psi(X,t)\right|^2 \mathrm{d}X = \int_{\mathbb{R}^3} \left|\Psi_0(X)\right|^2 \mathrm{d}X = 1$$

- energy

$$E(\Psi) = \int_{\mathbb{R}^3} \left(-\frac{1}{2}\nabla^2 + V(X) + \frac{\theta}{2}\left|\Psi(X,t)\right|^2\right)\Psi(X,t)\overline{\Psi(X,t)}\mathrm{d}X = E(\Psi_0)$$

$d = 2$ Rotating asymmetric (harmonic) potential

$$
\begin{aligned}
V(X, Y, t) = {} & \frac{1}{2}(1 + \varepsilon_x)(X \cos \Omega t + Y \sin \Omega t)^2 + \\
& + \frac{1}{2}(1 + \varepsilon_y)(-X \sin \Omega t + Y \cos \Omega t)^2
\end{aligned}
$$

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Change of variables (rotating frame with angular velocity Ω):

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\Omega t & \sin\Omega t \\ -\sin\Omega t & \cos\Omega t \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}$$

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$$\mathrm{i}\partial_t \psi(x,y,t) = \left(-\frac{1}{2}\nabla^2 + V(x,y) + \theta\,|\psi(x,y,t)|^2 - \Omega L_z\right)\psi(x,y,t)$$

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$$

$$
\mathrm{i}\partial_t\psi(x,y,t) = \left(-\frac{1}{2}\nabla^2 + V(x,y) + \theta\left|\psi(x,y,t)\right|^2 - \Omega L_z\right)\psi(x,y,t)
$$

$$
\psi(x,y,t) = \Psi(X,Y,t), \quad V(x,y) = V(x,y,0), \quad L_z = -\mathrm{i}(x\partial_y - y\partial_x)
$$

Properties

- the particle number is still preserved:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2}|\psi(x,y,t)|^2\,\mathrm{d}x\mathrm{d}y =$$

$$\int_{\mathbb{R}^2}\partial_t\psi(x,y,t)\overline{\psi(x,y,t)}+\partial_t\overline{\psi(x,y,t)}\psi(x,y,t)\mathrm{d}x\mathrm{d}y =$$

$$=(-\mathrm{i}+\mathrm{i})\int_{\mathbb{R}^2}\frac{1}{2}\left|\nabla\psi(x,y,t)\right|^2+V(x,y)\left|\psi(x,y,t)\right|^2+\theta\left|\psi(x,y,t)\right|^4\mathrm{d}x\mathrm{d}y+$$

$$+\Omega\int_{\mathbb{R}^2}(x\partial y-y\partial x)\left|\psi(x,y,t)\right|^2\mathrm{d}x\mathrm{d}y=0$$

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$$= (-\mathrm{i}+\mathrm{i})\int_{\mathbb{R}^2}\frac{1}{2}|\nabla\psi(x,y,t)|^2+V(x,y)|\psi(x,y,t)|^2+\theta|\psi(x,y,t)|^4\,\mathrm{d}x\mathrm{d}y+$$

$$+\Omega\int_{\mathbb{R}^2}(x\partial y-y\partial x)|\psi(x,y,t)|^2\,\mathrm{d}x\mathrm{d}y = 0$$

- symmetry: if $\psi_0(x,y)=\psi_0(-x,-y)$, then

$$\psi(x,y,t)=\psi(-x,-y,t),\quad \forall t\geq 0$$

Time-independent GPE: ground state solution

The solution

$$\psi(x,y,t) = \phi(x,y)\mathrm{e}^{-\mathrm{i}\mu t}, \quad \phi(x,y) \in \mathbb{R}$$

which minimizes the energy is the ground state solution. Then,

$$\mu\phi(x,y) = \left(-\frac{1}{2}\nabla^2 + V(x,y) + \theta\,|\phi(x,y)|^2 - \Omega L_z\right)\phi(x,y)$$

$(\mu, \phi(x,y))$ is an eigen-pair.

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$(\mu, \phi(x,y))$ is an eigen-pair.

$\phi(x,y)$ is usually called time-independent solution of the GPE and

$$\mu = \int_{\mathbb{R}^2}\left(-\frac{1}{2}\nabla^2 + V(x,y) + \theta\,|\phi(x,y)|^2 - \Omega L_z\right)\phi(x,y)\phi(x,y)\mathrm{d}x\mathrm{d}y$$

is the chemical potential.

Spatial discretization: Hermite functions

(for semplicity, $d = 1$)

$$\psi(x,t) \approx \sum_{j=0}^{N-1} \psi_j(t) H_j(x) \mathrm{e}^{-x^2/2} = \sum_{j=0}^{N-1} \psi_j(t) \mathcal{H}_j(x), \quad \psi_j(t) \in \mathbb{C}$$

$\mathcal{H}_j(x)$ the Hermite function of "degree" j.

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$\mathcal{H}_j(x)$ the Hermite function of "degree" j.They satisfy

$$\int_{\mathbb{R}} \mathcal{H}_j(x) \mathcal{H}_i(x) \mathrm{d}x = \delta_{ij}$$

$$\left(-\frac{1}{2}\nabla^2 + \frac{1}{2}x^2\right) \mathcal{H}_j(x) = \left(j + \frac{1}{2}\right) \mathcal{H}_j(x)$$

So, they are a basis in $L^2(\mathbb{R})$ and eigenfunctions of the (harmonic) Schrödinger operator. They naturally vanish at $x \to \infty$.

Tensor product

Extension to $d = 2$ by tensor product:

$$H_j(x,y) = H_{j_1}(x)H_{j_2}(y)$$
$$\mathcal{H}_j(x,y) = \mathcal{H}_{j_1}(x)\mathcal{H}_{j_2}(y)$$

$$\int_{\mathbb{R}^2} \mathcal{H}_j(x,y)\mathcal{H}_i(x,y)\mathrm{d}x\mathrm{d}y = \delta_{ij}$$

$$\left(-\frac{1}{2}\nabla^2 + \frac{1}{2}(x^2+y^2)\right)\mathcal{H}_j(x,y) = (j_1 + j_2 + 1)\mathcal{H}_j(x,y) = \\ = \lambda_j \mathcal{H}_j(x,y)$$

Galerkin

$$
\begin{aligned}
&\mathrm{i}\int_{\mathbb{R}^2}\Bigg(\sum_{|j|=0}^{N-1}\psi_j'(t)\mathcal{H}_j(x,y)\Bigg)\mathcal{H}_i(x,y)\mathrm{d}x\mathrm{d}y = \\
&=\int_{\mathbb{R}^2}\left(-\frac{1}{2}\nabla^2+\frac{1}{2}(x^2+y^2)\right)\left(\sum_{|j|=0}^{N-1}\psi_j(t)\mathcal{H}_j(x,y)\right)\mathcal{H}_i(x,y)\mathrm{d}x\mathrm{d}y+ \\
&+\int_{\mathbb{R}^2}\left(-\frac{1}{2}(x^2+y^2)+V(x,y)\right)\Bigg(\sum_{|j|=0}^{N-1}\psi_j(t)\mathcal{H}_j(x,y)\Bigg)\mathcal{H}_i(x,y)\mathrm{d}x\mathrm{d}y+ \\
&+\theta\int_{\mathbb{R}^2}\left|\sum_{|j|=0}^{N-1}\psi_j(t)\mathcal{H}_j(x,y)\right|^2\Bigg(\sum_{|j|=0}^{N-1}\psi_j(t)\mathcal{H}_j(x,y)\Bigg)\mathcal{H}_i(x,y)\mathrm{d}x\mathrm{d}y+ \\
&-\Omega\int_{\mathbb{R}^2}L_z\Bigg(\sum_{|j|=0}^{N-1}\psi_j(t)\mathcal{H}_j(x,y)\Bigg)\mathcal{H}_i(x,y)\mathrm{d}x\mathrm{d}y
\end{aligned}
$$

Galerkin: orthonormality

$$
\begin{aligned}
&\mathrm{i}\sum_{|j|=0}^{N-1}\psi_j'(t)\int_{\mathbb{R}^2}\mathcal{H}_j(x,y)\mathcal{H}_i(x,y)\mathrm{d}x\mathrm{d}y = \\
&= \int_{\mathbb{R}^2}\left(-\frac{1}{2}\nabla^2+\frac{1}{2}(x^2+y^2)\right)\left(\sum_{|j|=0}^{N-1}\psi_j(t)\mathcal{H}_j(x,y)\right)\mathcal{H}_i(x,y)\mathrm{d}x\mathrm{d}y+ \\
&+\int_{\mathbb{R}^2}\left(-\frac{1}{2}(x^2+y^2)+V(x,y)\right)\left(\sum_{|j|=0}^{N-1}\psi_j(t)\mathcal{H}_j(x,y)\right)\mathcal{H}_i(x,y)\mathrm{d}x\mathrm{d}y+ \\
&+\theta\int_{\mathbb{R}^2}\left|\sum_{|j|=0}^{N-1}\psi_j(t)\mathcal{H}_j(x,y)\right|^2\left(\sum_{|j|=0}^{N-1}\psi_j(t)\mathcal{H}_j(x,y)\right)\mathcal{H}_i(x,y)\mathrm{d}x\mathrm{d}y+ \\
&-\Omega\int_{\mathbb{R}^2}L_z\left(\sum_{|j|=0}^{N-1}\psi_j(t)\mathcal{H}_j(x,y)\right)\mathcal{H}_i(x,y)\mathrm{d}x\mathrm{d}y
\end{aligned}
$$

Galerkin

$$\mathrm{i}\psi_i'(t) =$$

$$= \int_{\mathbb{R}^2} \left(-\frac{1}{2}\nabla^2 + \frac{1}{2}(x^2+y^2) \right) \left(\sum_{|j|=0}^{N-1} \psi_j(t)\mathcal{H}_j(x,y) \right) \mathcal{H}_i(x,y)\mathrm{d}x\mathrm{d}y +$$

$$+ \int_{\mathbb{R}^2} \left(-\frac{1}{2}(x^2+y^2) + V(x,y) \right) \left(\sum_{|j|=0}^{N-1} \psi_j(t)\mathcal{H}_j(x,y) \right) \mathcal{H}_i(x,y)\mathrm{d}x\mathrm{d}y +$$

$$+\theta \int_{\mathbb{R}^2} \left| \sum_{|j|=0}^{N-1} \psi_j(t)\mathcal{H}_j(x,y) \right|^2 \left(\sum_{|j|=0}^{N-1} \psi_j(t)\mathcal{H}_j(x,y) \right) \mathcal{H}_i(x,y)\mathrm{d}x\mathrm{d}y +$$

$$-\Omega \int_{\mathbb{R}^2} L_z \left(\sum_{|j|=0}^{N-1} \psi_j(t)\mathcal{H}_j(x,y) \right) \mathcal{H}_i(x,y)\mathrm{d}x\mathrm{d}y$$

Galerkin: eigenfunctions

$$\mathrm{i}\psi_i'(t) =$$

$$= \sum_{|j|=0}^{N-1} \psi_j(t) \int_{\mathbb{R}^2} \left(-\frac{1}{2}\nabla^2 + \frac{1}{2}(x^2+y^2) \right) \mathcal{H}_j(x,y)\mathcal{H}_i(x,y)\mathrm{d}x\mathrm{d}y +$$

$$+ \int_{\mathbb{R}^2} \left(-\frac{1}{2}(x^2+y^2) + V(x,y) \right) \left(\sum_{|j|=0}^{N-1} \psi_j(t)\mathcal{H}_j(x,y) \right) \mathcal{H}_i(x,y)\mathrm{d}x\mathrm{d}y +$$

$$+\theta \int_{\mathbb{R}^2} \left| \sum_{|j|=0}^{N-1} \psi_j(t)\mathcal{H}_j(x,y) \right|^2 \left(\sum_{|j|=0}^{N-1} \psi_j(t)\mathcal{H}_j(x,y) \right) \mathcal{H}_i(x,y)\mathrm{d}x\mathrm{d}y +$$

$$-\Omega \int_{\mathbb{R}^2} L_z \left(\sum_{|j|=0}^{N-1} \psi_j(t)\mathcal{H}_j(x,y) \right) \mathcal{H}_i(x,y)\mathrm{d}x\mathrm{d}y$$

Galerkin: orthonormality

$$
\begin{aligned}
&\mathrm{i}\psi_i'(t) = \\
&= \sum_{|j|=0}^{N-1} \psi_j(t) \int_{\mathbb{R}^2} \lambda_j \mathcal{H}_j(x,y)\mathcal{H}_i(x,y)\mathrm{d}x\mathrm{d}y + \\
&+ \int_{\mathbb{R}^2} \left(-\frac{1}{2}(x^2+y^2)+V(x,y)\right)\left(\sum_{|j|=0}^{N-1}\psi_j(t)\mathcal{H}_j(x,y)\right)\mathcal{H}_i(x,y)\mathrm{d}x\mathrm{d}y + \\
&+\theta \int_{\mathbb{R}^2} \left|\sum_{|j|=0}^{N-1}\psi_j(t)\mathcal{H}_j(x,y)\right|^2 \left(\sum_{|j|=0}^{N-1}\psi_j(t)\mathcal{H}_j(x,y)\right)\mathcal{H}_i(x,y)\mathrm{d}x\mathrm{d}y + \\
&-\Omega \int_{\mathbb{R}^2} L_z\left(\sum_{|j|=0}^{N-1}\psi_j(t)\mathcal{H}_j(x,y)\right)\mathcal{H}_i(x,y)\mathrm{d}x\mathrm{d}y
\end{aligned}
$$

Galerkin: Gauss–Hermite quadrature

$$\mathrm{i}\psi_i'(t) =$$

$$\lambda_i\psi_i(t)+$$

$$+\int_{\mathbb{R}^2}\left(-\frac{1}{2}(x^2+y^2)+V(x,y)\right)\left(\sum_{|j|=0}^{N-1}\psi_j(t)\mathcal{H}_j(x,y)\right)\mathcal{H}_i(x,y)\mathrm{d}x\mathrm{d}y+$$

$$+\theta\int_{\mathbb{R}^2}\left|\sum_{|j|=0}^{N-1}\psi_j(t)\mathcal{H}_j(x,y)\right|^2\left(\sum_{|j|=0}^{N-1}\psi_j(t)\mathcal{H}_j(x,y)\right)\mathcal{H}_i(x,y)\mathrm{d}x\mathrm{d}y+$$

$$-\Omega\int_{\mathbb{R}^2}L_z\left(\sum_{|j|=0}^{N-1}\psi_j(t)\mathcal{H}_j(x,y)\right)\mathcal{H}_i(x,y)\mathrm{d}x\mathrm{d}y$$

Galerkin

$$\mathrm{i}\psi_i'(t) =$$

$$\lambda_i\psi_i(t) +$$

$$\sum_{|m|=0}^{M-1}\left[\left(-\frac{1}{2}(x_{m_1}^2 + y_{m_2}^2) + V(x_{m_1}, y_{m_2})\right)\mathrm{e}^{(x_{m_1}^2 + y_{m_2}^2)} + \right.$$

$$\left. +\theta\left|\sum_{|j|=0}^{N-1}\psi_j(t)H_j(x_{m_1}, y_{m_2})\right|^2\right]\left(\sum_{|j|=0}^{N-1}\psi_j(t)H_j(x_{m_1}, y_{m_2})\right)H_i(x_{m_1}, y_{m_2})w_{m_1}w_{m_2} +$$

$$-\Omega\int_{\mathbb{R}^2} L_z\left(\sum_{|j|=0}^{N-1}\psi_j(t)\mathcal{H}_j(x, y)\right)\mathcal{H}_i(x, y)\mathrm{d}x\mathrm{d}y$$

Galerkin: Spectral derivatives

$$\mathrm{i}\psi_i'(t) =$$

$$\lambda_i\psi_i(t) +$$

$$\sum_{|m|=0}^{M-1}\left[\left(-\frac{1}{2}(x_{m_1}^2+y_{m_2}^2)+V(x_{m_1},y_{m_2})\right)\mathrm{e}^{(x_{m_1}^2+y_{m_2}^2)}+\right.$$

$$\left.+\theta\left|\sum_{|j|=0}^{N-1}\psi_j(t)H_j(x_{m_1},y_{m_2})\right|^2\right]\left(\sum_{|j|=0}^{N-1}\psi_j(t)H_j(x_{m_1},y_{m_2})\right)H_i(x_{m_1},y_{m_2})w_{m_1}w_{m_2}+$$

$$-\Omega\int_{\mathbb{R}^2}\left(\sum_{|j|=0}^{N-1}\psi_j(t)L_z\mathcal{H}_j(x,y)\right)\mathcal{H}_i(x,y)\mathrm{d}x\mathrm{d}y$$

Galerkin: Gauss–Hermite quadrature

$$\mathrm{i}\psi_i'(t) =$$

$$\lambda_i\psi_i(t)+$$

$$\sum_{|m|=0}^{M-1}\left[\left(-\frac{1}{2}(x_{m_1}^2+y_{m_2}^2)+V(x_{m_1},y_{m_2})\right)\mathrm{e}^{(x_{m_1}^2+y_{m_2}^2)}+\right.$$
$$\left.+\theta\left|\sum_{|j|=0}^{N-1}\psi_j(t)H_j(x_{m_1},y_{m_2})\right|^2\right]\left(\sum_{|j|=0}^{N-1}\psi_j(t)H_j(x_{m_1},y_{m_2})\right)H_i(x_{m_1},y_{m_2})w_{m_1}w_{m_2}+$$
$$+\mathrm{i}\Omega\int_{\mathbb{R}^2}\left(\sum_{|j|=0}^{N-1}\psi_j(t)\left(x\frac{\partial}{\partial y}\mathcal{H}_j(x,y)-y\frac{\partial}{\partial x}\mathcal{H}_j(x,y)\right)\right)\mathcal{H}_i(x,y)\mathrm{d}x\mathrm{d}y$$

Galerkin: done!

$$\mathrm{i}\psi_i'(t) =$$

$$\lambda_i\psi_i(t) +$$

$$\sum_{|m|=0}^{M-1}\left[\left(-\frac{1}{2}(x_{m_1}^2+y_{m_2}^2)+V(x_{m_1},y_{m_2})\right)\mathrm{e}^{(x_{m_1}^2+y_{m_2}^2)}+\right.$$
$$\left.+\theta\left|\sum_{|j|=0}^{N-1}\psi_j(t)H_j(x_{m_1},y_{m_2})\right|^2\right]\left(\sum_{|j|=0}^{N-1}\psi_j(t)H_j(x_{m_1},y_{m_2})\right)H_i(x_{m_1},y_{m_2})w_{m_1}w_{m_2}+$$
$$+\mathrm{i}\Omega\sum_{|m|=0}^{M-1}\left(\sum_{|j|=0}^{N-1}\psi_j(t)\left(x_{m_1}\frac{\partial}{\partial y}\mathcal{H}_j\Big|_{(x_{m_1},y_{m_2})}-y_{m_2}\frac{\partial}{\partial x}\mathcal{H}_j\Big|_{(x_{m_1},y_{m_2})}\right)\mathrm{e}^{(x_{m_1}^2+y_{m_2}^2)/2}\right)\times$$
$$\times\mathrm{e}^{(x_{m_1}^2+y_{m_2}^2)}\mathcal{H}_i(x_{m_1},y_{m_2})w_{m_1}w_{m_2}$$

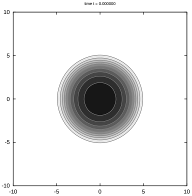
Time discretization

$$\begin{cases} \mathrm{i}\psi'(t) = F(\psi(t)) \\ \psi(0) = \psi_0 \end{cases}$$

Explicit Runge–Kutta of order 4: not the optimal choice, but easy and already used in literature (e.g., [C. M. Dion and E. Cancès, Phys. Rev. E 67 (2003)]).

First simulation

$\theta = \frac{500}{\sqrt{2}}$, $\varepsilon_x = 0.09$, $\varepsilon_x = 0.03$, $\Omega = 0.7$, final $T = 360$:



$$|\psi(x, y, t)|^2$$

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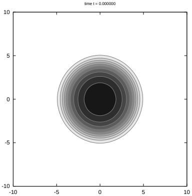

$|\psi(x, y, t)|^2$

$N = 16$ Hermite functions, time step-size $h = 1/100$.

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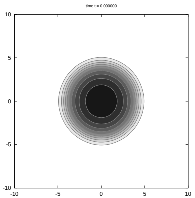


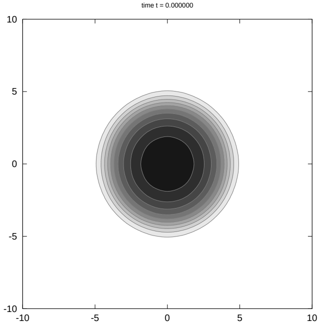
$|\psi(x, y, t)|^2$

$N = 16$ Hermite functions, time step-size $h = 1/100$.

There are artificial "vortices" due to the poor space resolution [Private communication with M. Tsubota at nlqugas08, Toledo]. A similar experiment was published: [***, *Vortex Lattice Formation in Bose–Einstein Condensates*, Phys. Rev. Lett. 92, (2004)]

Increasing the space resolution

Same experiment, with $N = 64$ Hermite functions:



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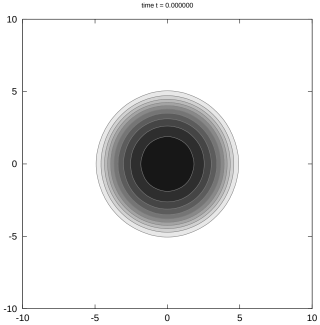

No vortex!

Dissipative GPE

A numerical simulation showing vortices was based (almost) on

$$(i-\gamma)\partial_t\psi(x,y,t)=\left(-\frac{1}{2}\nabla^2+V(x,y)+\theta\left|\psi(x,y,t)\right|^2-\Omega L_z\right)\psi(x,y,t)$$

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$$\gamma\geq 0$$

"The term with γ introduces the dissipation. Although the detailed mechanism of the dissipation is yet to be understood [...] the observation of the vortex lattices implies the presence of dissipation."

[K. Kasamatsu, M. Tsubota, and M. Ueda, Phys. Rev. A 67 (2003)]

Meaning of γ

Let's rewrite the equation in the laboratory frame:

$$(\mathrm{i}-\gamma)\partial_t\Psi(X,Y,t) = \Big[-\frac{1}{2}\nabla^2 + V(X,Y,t) + \theta\,|\Psi(X,Y,t)|^2 + \\ + \gamma\Omega\,(X\partial_Y - Y\partial_X)\Big]\Psi(X,Y,t)$$

"When the condensate is grown from a rotating vapor cloud the growth process itself is dissipative, and in the experiments [...] dissipation can arise by transfer of atoms between the thermal cloud and the condensate. [...] A dissipative term arises from collisions between atoms in the thermal cloud trapped by the same potential as the condensate, in which one of the colliding atoms enters the condensate after the collision."

[A. A. Penckwitt and R. J. Ballagh, Phys. Rev. Lett. 89, 260402 (2002)]

Particle number conservation

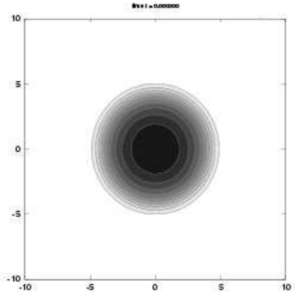
The particle number is no more preserved: let's subtract the chemical potential μ

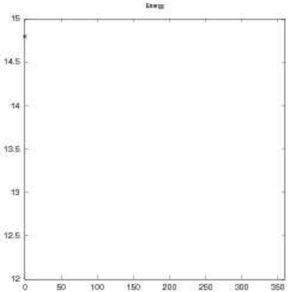
$$(i-\gamma)\partial_t\psi(x,y,t) = \left(-\frac{1}{2}\nabla^2 + V(x,y) + \theta\left|\psi(x,y,t)\right|^2 - \Omega L_z - \mu\right)\psi(x,y,t)$$

The chemical potential depends on ψ: so, it is computed after each time step and inserted into the equation.

Vortices

$\gamma = 0.03$, $\theta = \frac{500}{\sqrt{2}}$, $\varepsilon_x = 0.09$, $\varepsilon_x = 0.03$, $\Omega = 0.7$, final $T = 360$:





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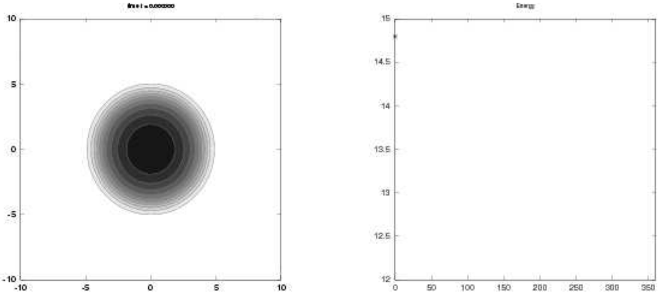
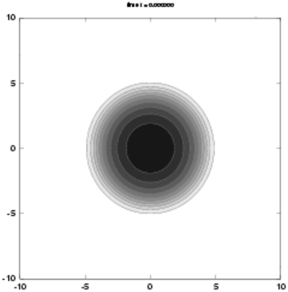


$N = 64$ Hermite functions (vanishing boundary conditions), time step-size $h = 1/100$.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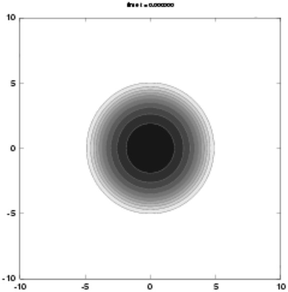
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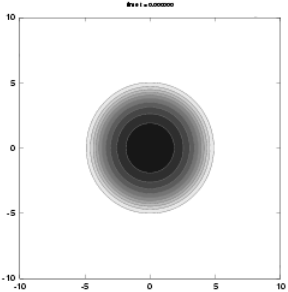
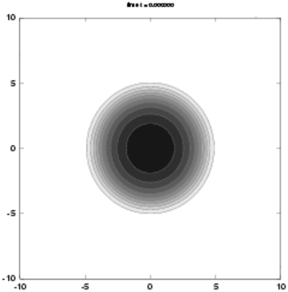
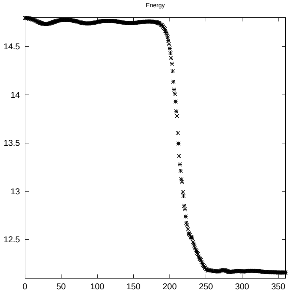
Symmetry $\psi(x, y, t) = \psi(-x, -y, t)$?



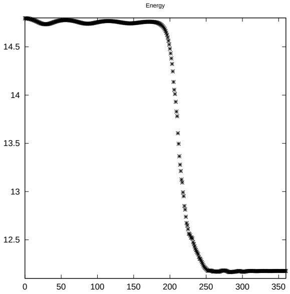


$N = 100$ Hermite functions, time step-size $h = 1/500$, (right, using only polynomials of total degree even).

Energies



$E(T) \approx 12.16$



$E(T) \approx 12.18$

Madelung fluid

The Schrödinger Equation (SE) can be written as

$$\begin{cases} \partial_t v(X,t) + (v(X,t)\cdot\nabla)v(X,t) - \nabla\left(\dfrac{\nabla^2\sqrt{\rho(X,t)}}{\sqrt{\rho(X,t)}}\right) + \nabla V(X) = 0 \\ \partial_t \rho(X,t) + \nabla\cdot(\rho(X,t)v(X,t)) = 0 \end{cases} \qquad X \in \mathbb{R}^3$$

with

$$\Psi(X,t) = \rho(X,t)^{\frac{1}{2}} \mathrm{e}^{\mathrm{i}S(X,t)}$$
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There are excited stationary solutions in ρ, v with $\rho(0,t) = 0$ and $\nabla \wedge v(X,t) = 0$ for $X \neq 0$ and $\nabla \wedge v(0,t) \to \infty$ (vortex) with definite energy levels $E_1, E_2, \ldots$ (e.g. $v = \left[-\frac{X}{X^2+Y^2}, \frac{Y}{X^2+Y^2}, 0\right]$)

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But if we consider a dissipative SE...

Schrödinger Equation with vorticity

$$
\begin{cases}
\mathrm{i}\partial_t \Psi(X,t) = \left[\frac{1}{2}(\mathrm{i}\nabla + A(X,t))^2 + V(X) + C\left|\Psi(X,t)\right|^2\right]\Psi(X,t) \\
\partial_t A(X,t) = \left[b_*(X,t) - \frac{1}{2}\nabla\right] \wedge (\nabla \wedge A(X,t))
\end{cases}
$$

where

$$
\begin{aligned}
A(\cdot,t)\colon \mathbb{R}^3 &\to \mathbb{R}^3 \\
b_*(X,t) &= \nabla S(X,t) - A(X,t) - \frac{1}{2}\nabla \ln \rho(X,t) \\
\Psi(X,t) &= \rho(X,t)^{\frac{1}{2}} \mathrm{e}^{\mathrm{i}S(X,t)} \\
v(X,t) &= \nabla S(X,t) - A(X,t)
\end{aligned}
$$

(proposed in [M. C., M. Loffredo, L. M. Morato, and S. Zuccher, New J. Phys. (2008)], derived by path-wise stochastic variational principle for $C = 0$ in [L. M. Morato, Phys. Rev. D 31 (1985), M. Loffredo and L. M. Morato, J. Math. Phys. 30 (1989)])

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$$E(t) = \int_{\mathbb{R}^3} \left[\frac{1}{2}(\mathrm{i}\nabla + A(X,t))^2 + V(X) + \frac{C}{2}\left|\Psi(X,t)\right|^2\right]\left|\Psi(X,t)\right|^2 \mathrm{d}X$$

satisfies

$$\frac{\mathrm{d}E(t)}{\mathrm{d}t} = -\frac{1}{2}\int_{\mathbb{R}^3} \left|\nabla \wedge v(X,t)\right|^2 \left|\Psi(X,t)\right|^2 \mathrm{d}X \leq 0$$

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(cfr. Round table on vortex formation)

Spatial discretization?

For $C = 0$

$$\begin{cases} \mathrm{i}\partial_t \Psi = \left[\frac{1}{2}(\mathrm{i}\nabla + A)^2 + V\right]\Psi \\ \partial_t A = \left(b_* - \frac{1}{2}\nabla\right) \wedge (\nabla \wedge A) \end{cases}$$

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$A\Psi \in L^2(\mathbb{R}^3 \to \mathbb{C})$: let's introduce $\Gamma = A\Psi$ and rewrite the equation in Ψ and Γ.

RSE in ψ, Γ

$$
\begin{cases}
i\overline{\Psi}\partial_t\Psi = \dfrac{1}{2}(-\overline{\Psi}\nabla^2\Psi + \overline{\Gamma}\cdot\Gamma + i\overline{\Psi}\nabla\cdot\Gamma + i\overline{\Gamma}\cdot\nabla\Psi) + \overline{\Psi}\Phi\Psi \\
\Psi\overline{\Psi}\partial_t\Gamma = -\dfrac{i}{2}\nabla\overline{\Psi}\wedge(\nabla\Psi\wedge\Gamma) + \dfrac{1}{2}\nabla\wedge[\overline{\Psi}(\nabla\Psi\wedge\Gamma - \Psi\nabla\wedge\Gamma)] + \\
\qquad + \overline{\Psi}\left[\left(\dfrac{1}{2}\nabla\Psi - \Gamma\right)\wedge(\nabla\wedge\Gamma)\right] + \dfrac{i}{2}(\Psi\nabla\overline{\Psi} - \overline{\Psi}\nabla\Psi)\wedge(\nabla\wedge\Gamma) + \\
\qquad + \left[\dfrac{1}{2}(i-1)\nabla\Psi + \Gamma\right]\wedge(\nabla\Psi\wedge\overline{\Gamma}) + \Gamma\overline{\Psi}\partial_t\Psi
\end{cases}
$$

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Multiplying the first equation by Ψ and using the Galerkin approach with spectral basis $\{H_j(x)\mathrm{e}^{-x^2/4}\}_j$, we have the same real mass matrix.

Reference solution (Gaussian-linear)

Given

$$\begin{cases} \Psi_0(X,Y) = \sqrt{\dfrac{A_0}{\pi}} e^{-A_0(X^2+Y^2)/2} \\ v_0(X,Y) = a_0 R\hat{R} - \Omega_0 R\hat{\theta} \quad (R = \sqrt{X^2+Y^2}) \end{cases}$$

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then

$$\begin{cases} \dot{A}(t) = -2A(t)a(t) \\ \dot{a}(t) = A(t)^2 - a(t)^2 + \Omega(t)^2 - 1 \\ \dot{\Omega}(t) = -2(A(t)+a(t))\Omega(t) \end{cases}$$