Quantized Vortex Stability and Dynamics in Superfluidity and Superconductivity

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Outline

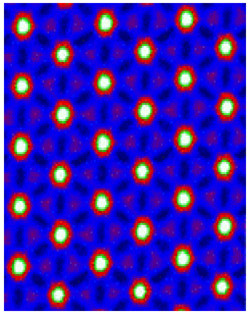
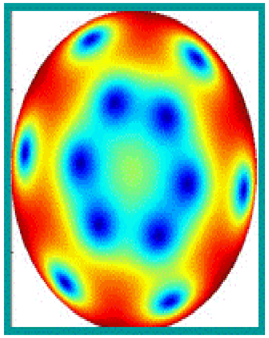
- Motivation
- Mathematical models
- Central vortex states and stability
- Vortex interaction and reduced dynamic laws
- Numerical methods
- Numerical results for vortex interaction
- Conclusions

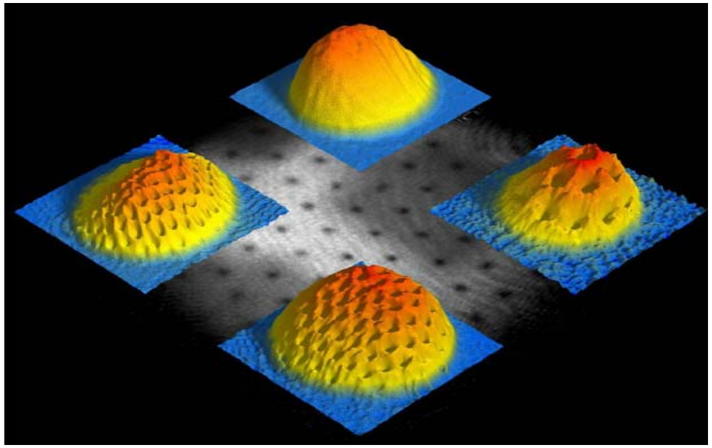
Motivation

- Quantized Vortex: Particle-like (topological) defect
  - Zero of the complex scalar field,
  - localized phase singularities with integer topological charge:

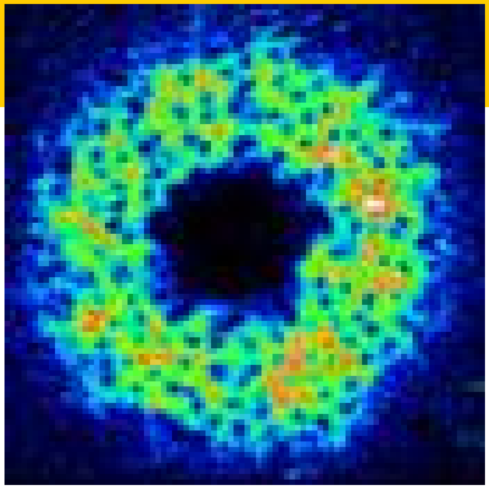
$$\psi(\vec{x}_0)=0,\quad \psi(\vec{x})=\sqrt{\rho}\,e^{i\phi},\quad \int_\Gamma d(\arg\psi)=\int_\Gamma d\phi=2\pi n\neq 0$$

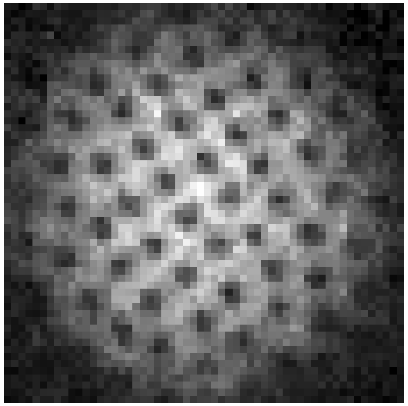
  - Key of superfluidity: ability to support dissipationless flow



Motivation





- Existing in
  - Superconductors:
    - Ginzburg-Landau equations (GLE)
  - Liquid helium:
    - Two-fluid model
    - Gross-Pitaevskii equation (GPE)
  - Bose-Einstein condensation (BEC):
    - Nonlinear Schroedinger equation (NLSE) or GPE
  - Nonlinear optics & propagation of laser beams
    - Nonlinear Schroedinger equation (NLSE)
    - Nonlinear wave equation (NLWE)

Mathematical models

- Ginzburg-Landau equation (GLE): Superconductivity, nonlinear heat flow, etc.

$$\psi_t = \Delta\psi + \frac{1}{\varepsilon^2}\left(V(\vec{x}) - |\psi|^2\right)\psi, \quad \vec{x} \in R^2, \quad t > 0,$$

- Gross-Pitaevskii equation (GPE): nonlinear optics, BEC, superfluidity

$$-i\,\psi_t = \Delta\psi + \frac{1}{\varepsilon^2}\left(V(\vec{x}) - |\psi|^2\right)\psi, \quad \vec{x} \in R^2, \quad t > 0,$$

- Nonlinear wave equation (NLWE): wave motion

$$\psi_{tt} = \Delta\psi + \frac{1}{\varepsilon^2}\left(V(\vec{x}) - |\psi|^2\right)\psi, \quad \vec{x} \in R^2, \quad t > 0,$$

  – Here

ψ : complex-valued wave function,

$\varepsilon > 0$: constant,

$V(\vec{x})$: real-valued external potential

Mathematical models

- `Free Energy' or Lyapunov functional:

$$E(\psi) = \int_{\Re^2} \left[|\nabla \psi|^2 + \frac{1}{2\varepsilon^2}\left(V(\vec{x}) - |\psi|^2\right)^2 \right] d\vec{x},$$

- Dispersive system (GPE): $-i\frac{\partial \psi}{\partial t} = -\frac{\delta E(\psi)}{\delta \psi^*}$
  - Energy conservation: $E(\psi) \equiv E(\psi_0), \quad t \geq 0$
  - Density conservation: $\int_{\Re^2} \left(|\psi(\vec{x},t)|^2 - |\psi_0(\vec{x})|^2 \right) d\vec{x} \equiv 0, \quad t \geq 0$
  - Admits particle like solutions: solitons, kinks & vortices
- Dissipative system (GLE): $\frac{\partial \psi}{\partial t} = -\frac{\delta E(\psi)}{\delta \psi^*}$
  - Energy diminishing, etc.

Two kinds of vortices

- `Bright-tail' vortex: GLE, GPE & NLWE (`order parameter')

$$V(\vec{x}) \to 1, \quad \text{when } |\vec{x}| \to \infty, \quad \Rightarrow |\psi(\vec{x},t)| \to 1, \qquad |\vec{x}| \to \infty, \quad (e.g.\ \psi \to e^{im\theta}),$$

  - Time independent case (*Neu, 90'*): $\varepsilon = 1, \quad V(\vec{x}) \equiv 1$
  - Vortex solutions: $\psi(\vec{x}) = \phi_n(\vec{x}) = f_n(r)\, e^{in\theta}$

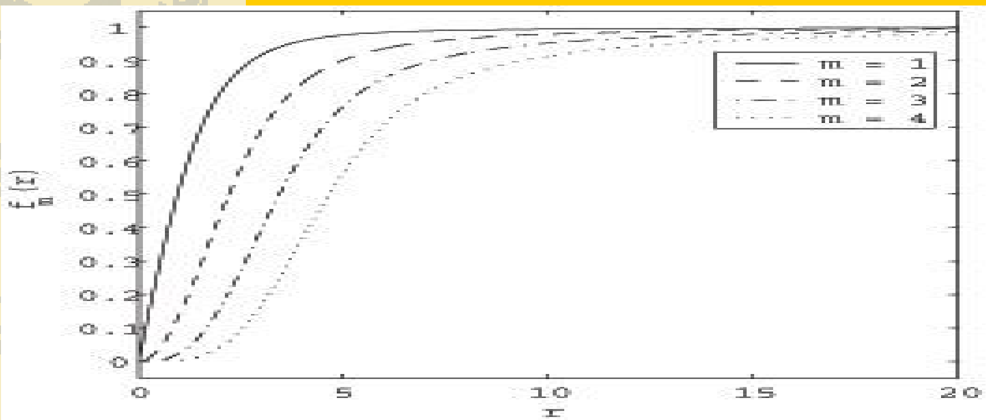
$$f_n''(r) + \frac{1}{r} f_n'(r) - \frac{n^2}{r^2} f_n(r) + (1 - f_n^2(r))\, f_n(r) = 0, \quad 0 < r < \infty,$$

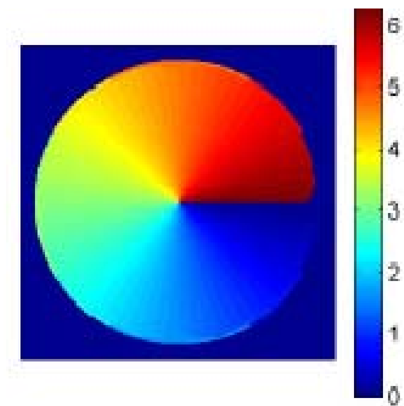
$$f_n(0) = 0, \qquad f_n(\infty) = 1.$$

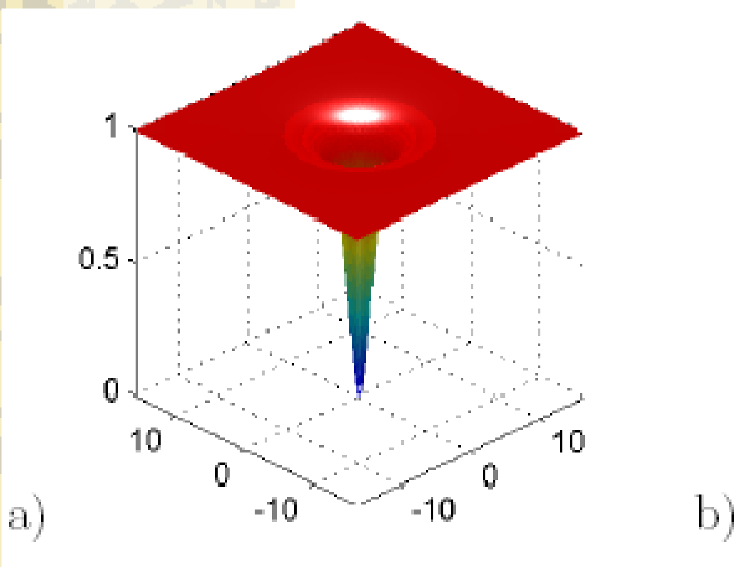
  - Asymptotic results

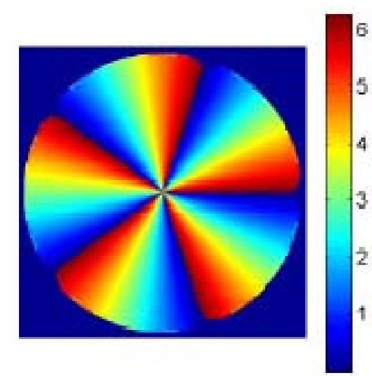
$$f_n(r) \approx \begin{cases} a r^{|n|} + O(r^{|n|+2}), & r \to 0, \\ 1 - n^2/2r^2 + O(1/r^4), & r \to \infty. \end{cases}$$

`Bright-tail' vortex









Two kinds of vortices

- `Dark-tail' vortex: BEC (wave function)

$$i\,\psi_t = -\Delta\psi + \frac{|\vec{x}|^2}{2}\psi + \beta\,|\psi|^2\,\psi, \quad \vec{x}\in R^2, \quad t>0, \quad \|\psi\|^2 = \int |\psi|^2\,d\vec{x} = 1$$

  – Center vortex states: $\psi(\vec{x},t) = e^{-i\mu_n t}\phi_n(\vec{x}) = e^{-i\mu_n t} f_n(r)\,e^{in\theta}$

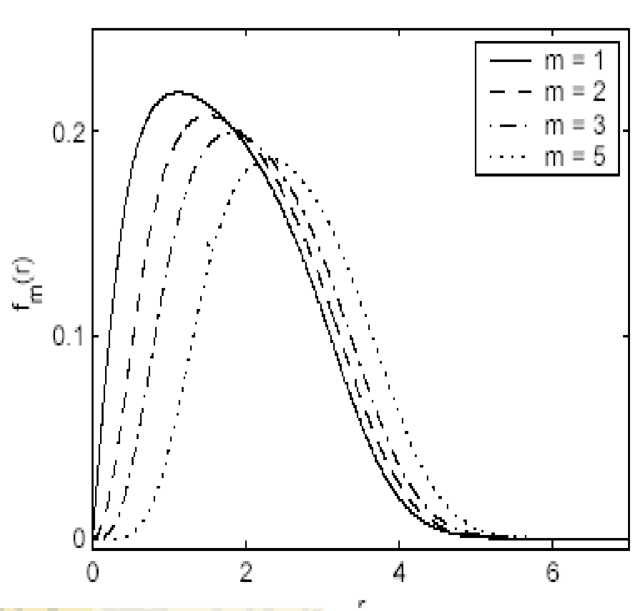
$$\mu_n\; f_n(r) = -\frac{d}{dr}\left(r\frac{df_n(r)}{dr}\right) + \left(\frac{n^2}{r^2}+\frac{r^2}{2}\right) f_n(r) + \beta\, f_n^3(r), \quad 0<r<\infty,$$

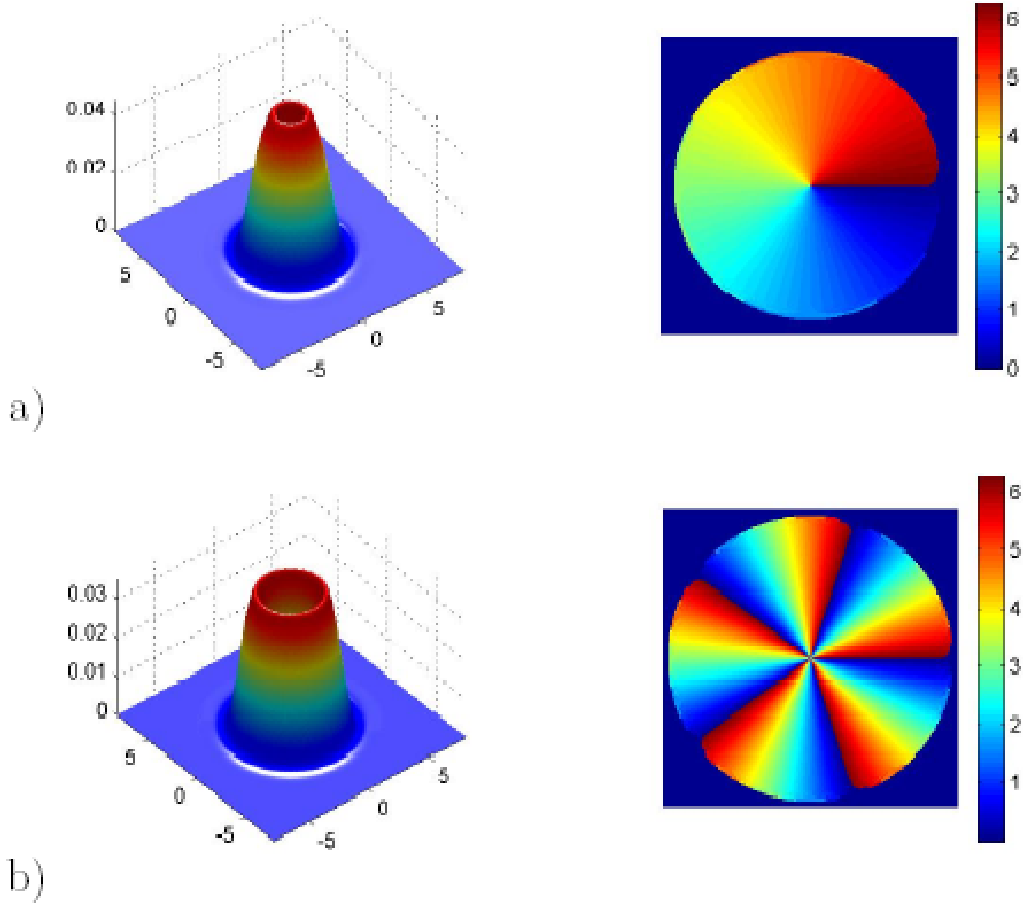
$$f_n(0)=0, \quad f_n(\infty)=0, \quad 2\pi\int_0^\infty f_n^2(r)\,r\,dr = 1$$

  – Asymptotic results

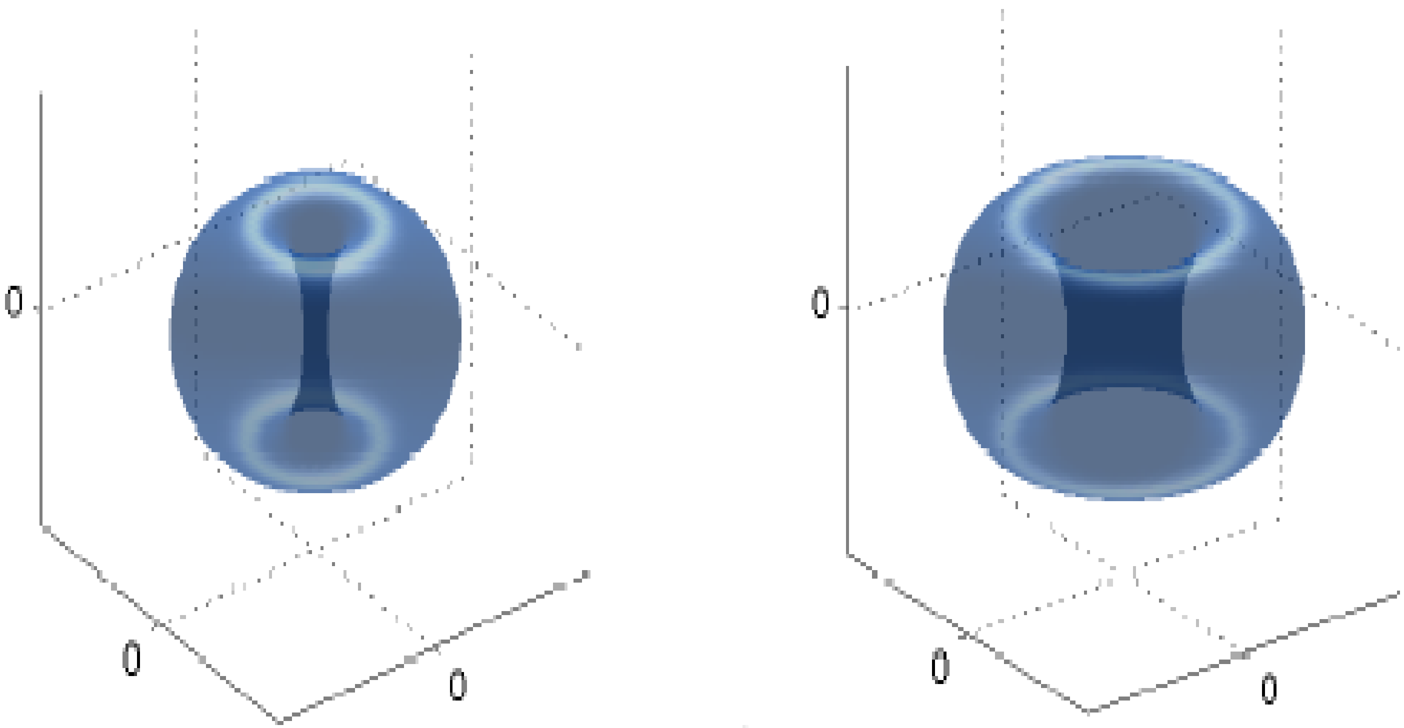
$$f_n(r) \approx \begin{cases} a\,r^{|n|} + O(r^{|n|+2}), & r\to 0, \\ b\,r^{|n|}e^{-r^2}, & r\to\infty. \end{cases}$$

`Dark-tail' vortex





`Dark-tail' vortex line in 3D



b)

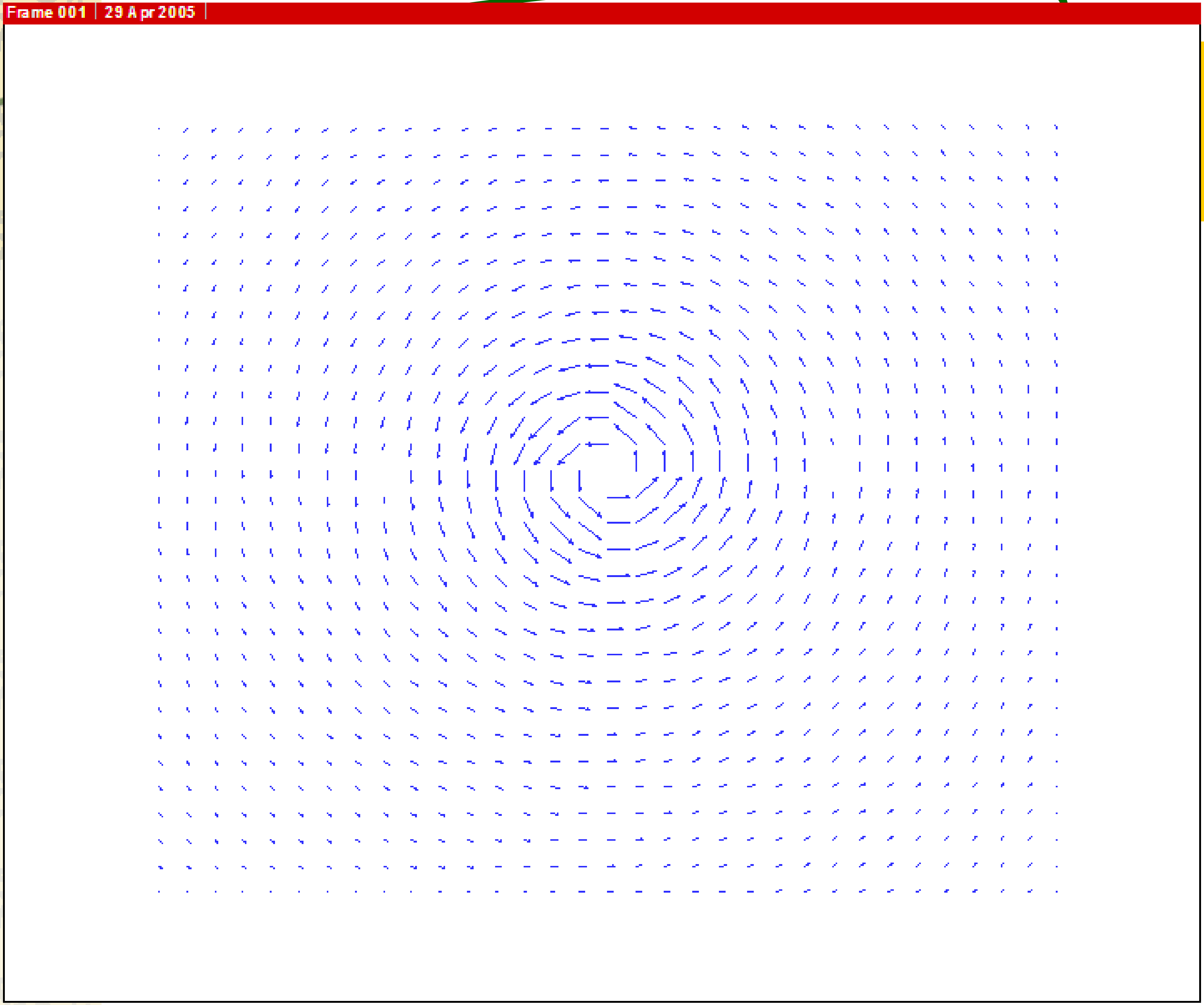
Stability of vortices

- Data chosen

$$\varepsilon = 1, \qquad V(\vec{x},t) \equiv 1 \quad (+W(\vec{x},t)), \qquad \psi_0(\vec{x}) = \phi_n^h(\vec{x}) \quad (+\text{noise})$$

- Results (Bao,Du & Zhang, Eur. J. Appl. Math., 07')
  - For GLE or NLWE with initial data perturbed
    - n=1: velocity density
    - N=3: velocity density
  - For NLSE with external potential perturbed
    - n=1: velocity density
    - N=3: velocity density
- Conclusion

next



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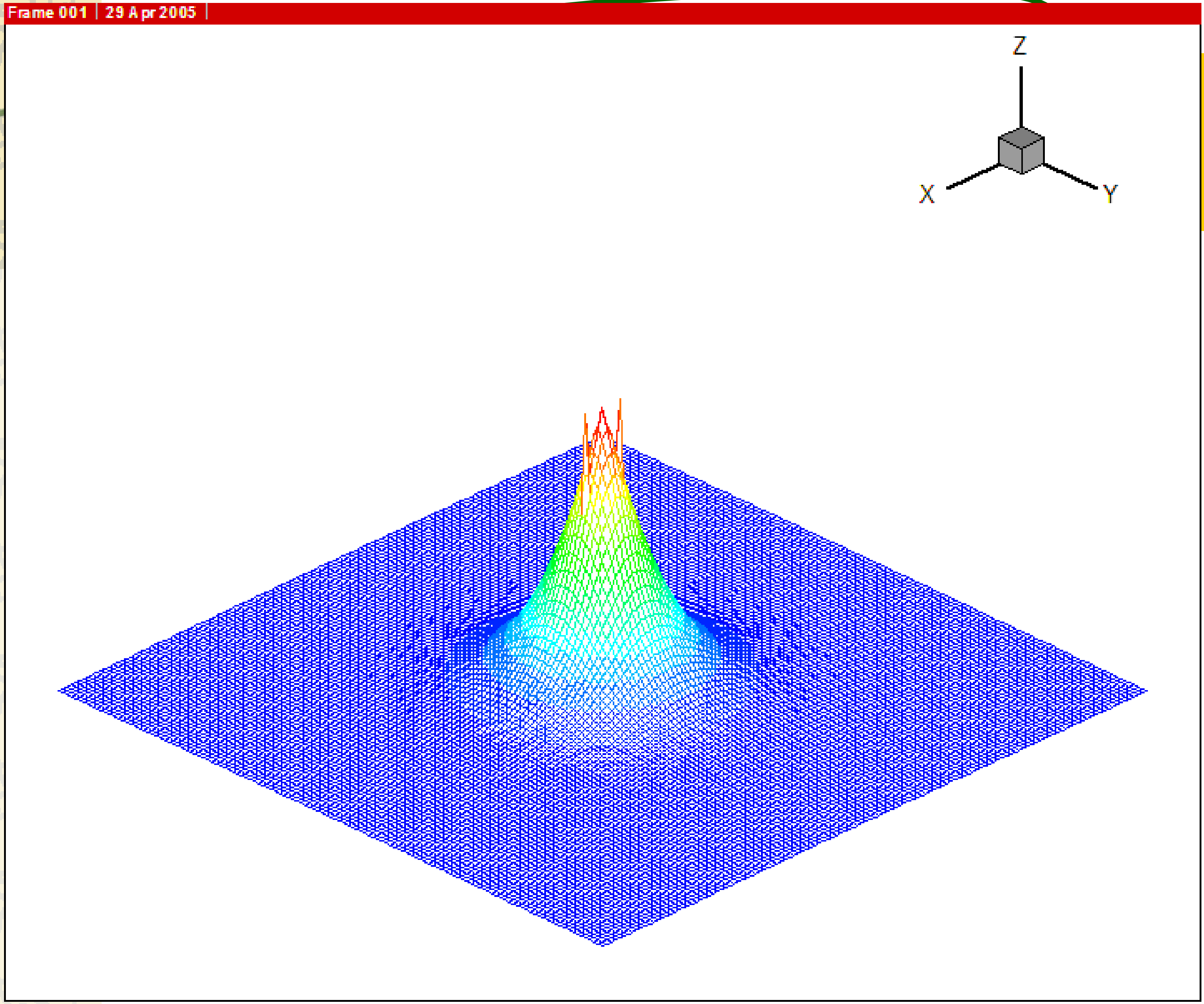

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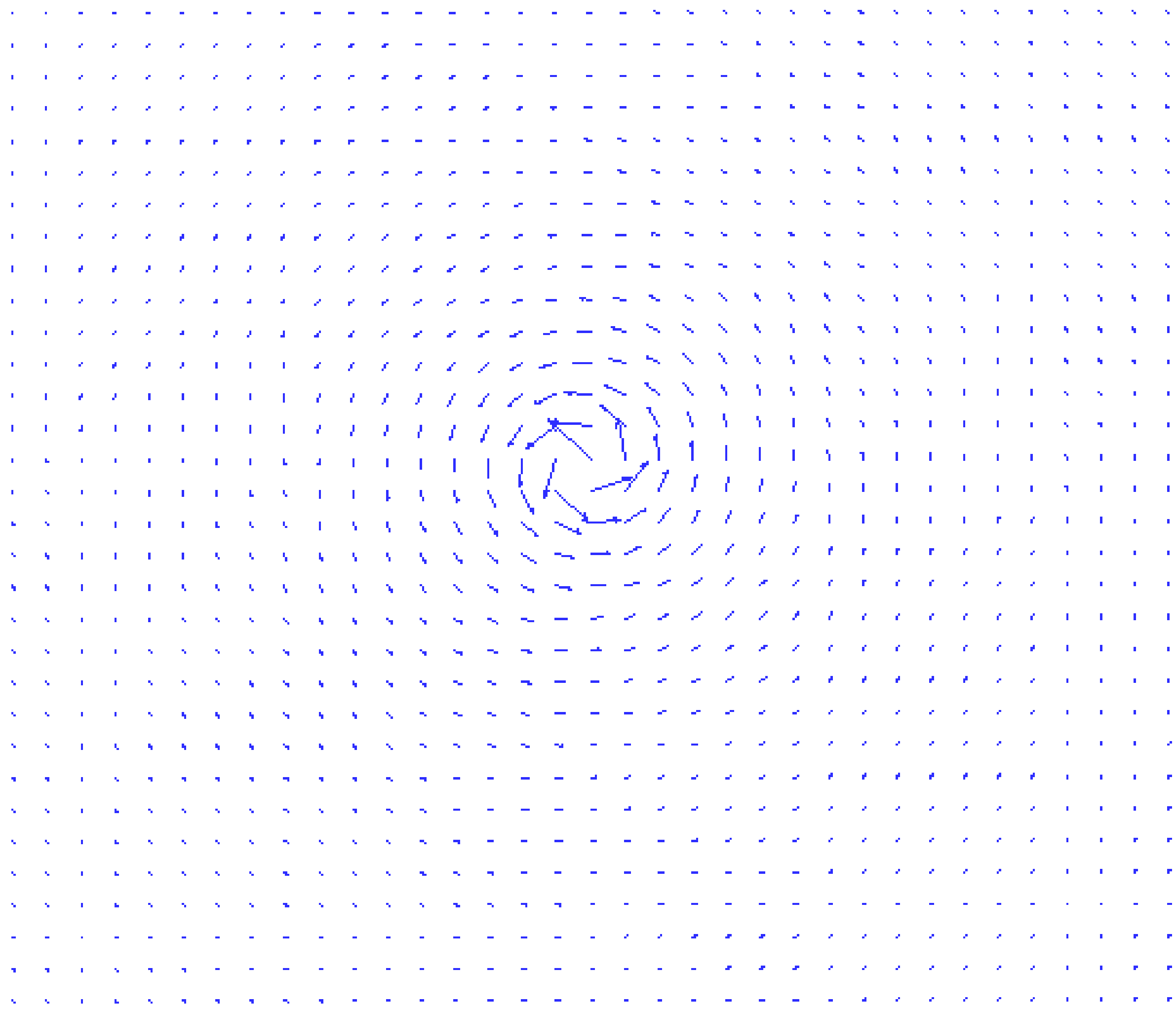

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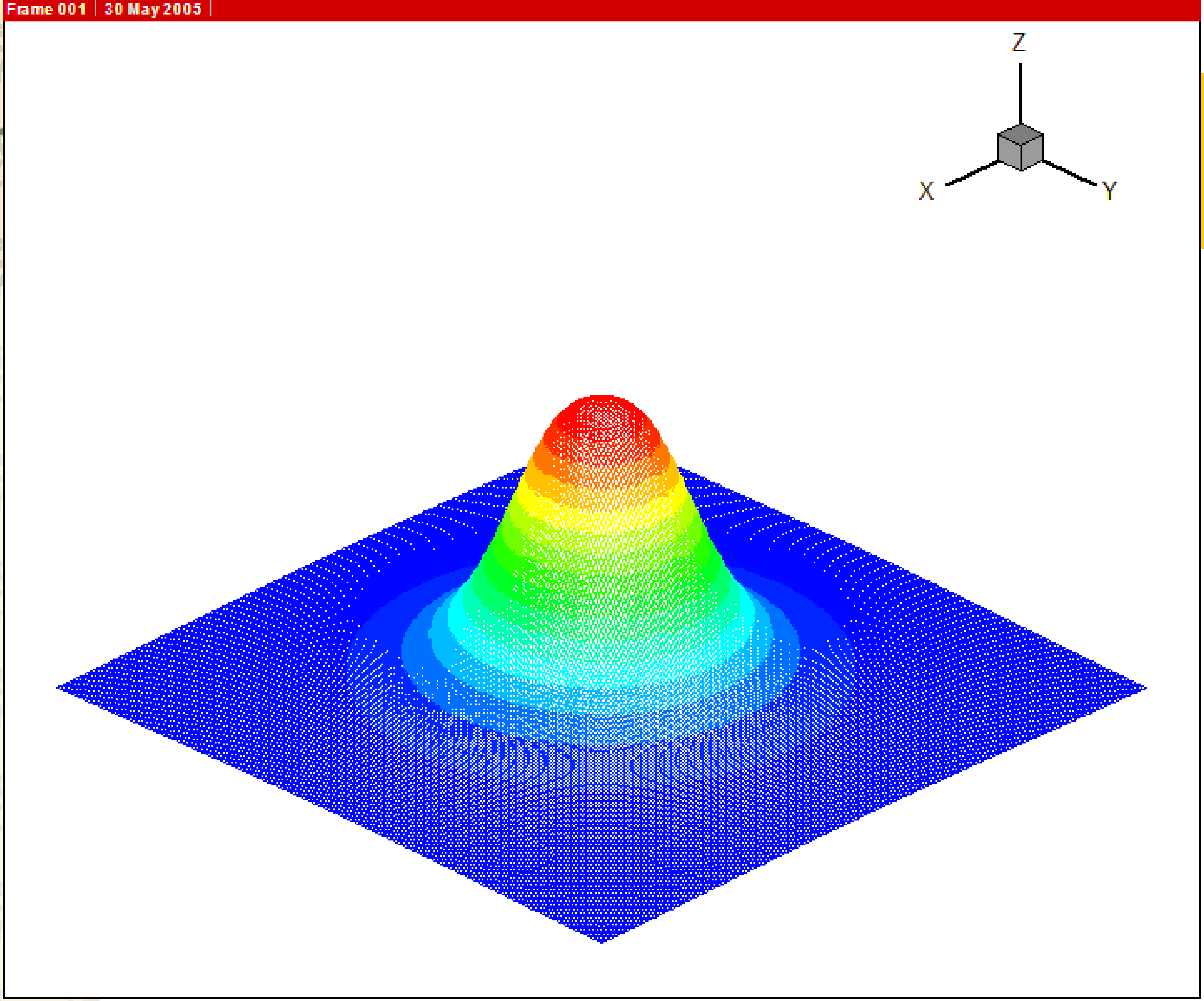

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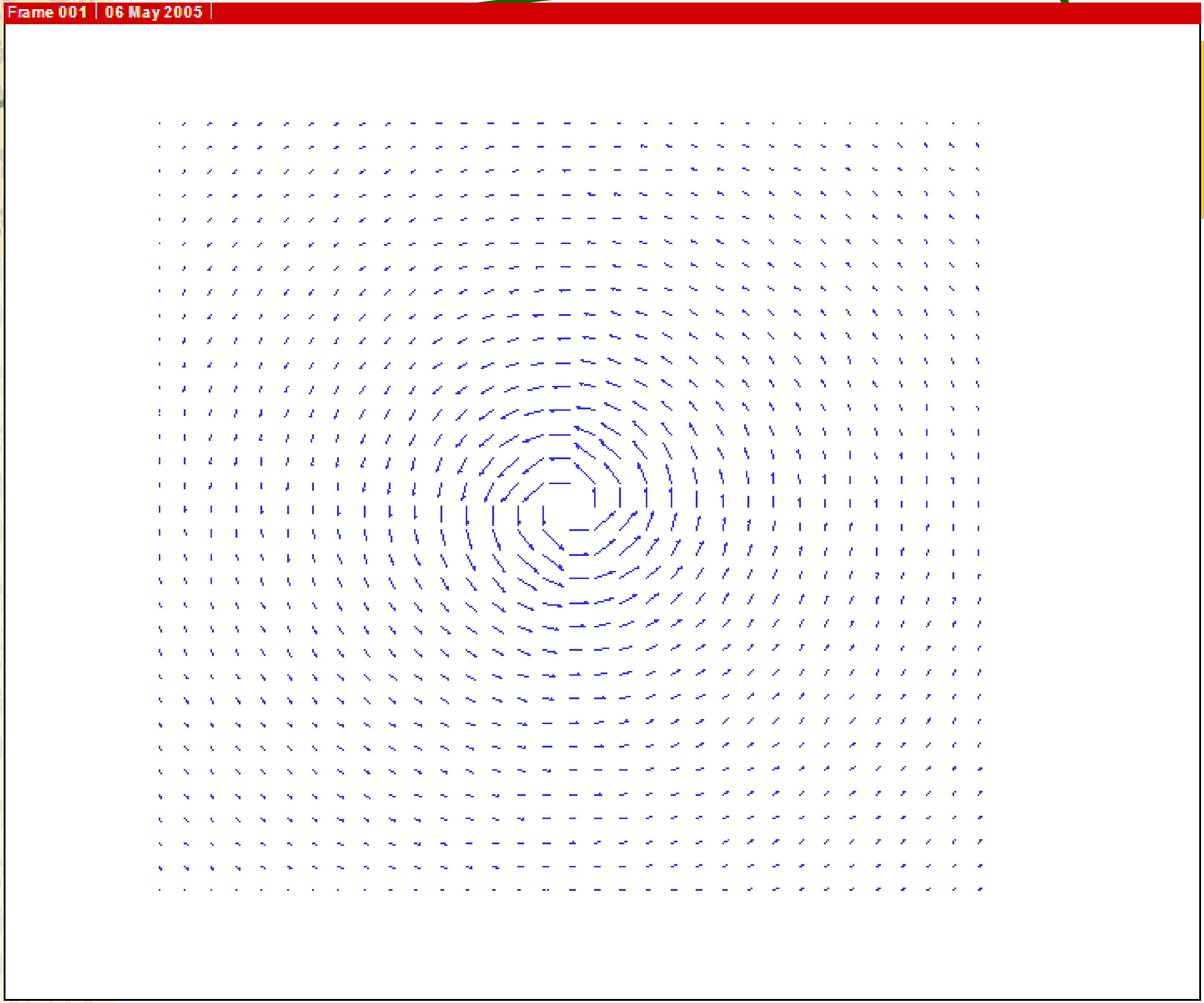

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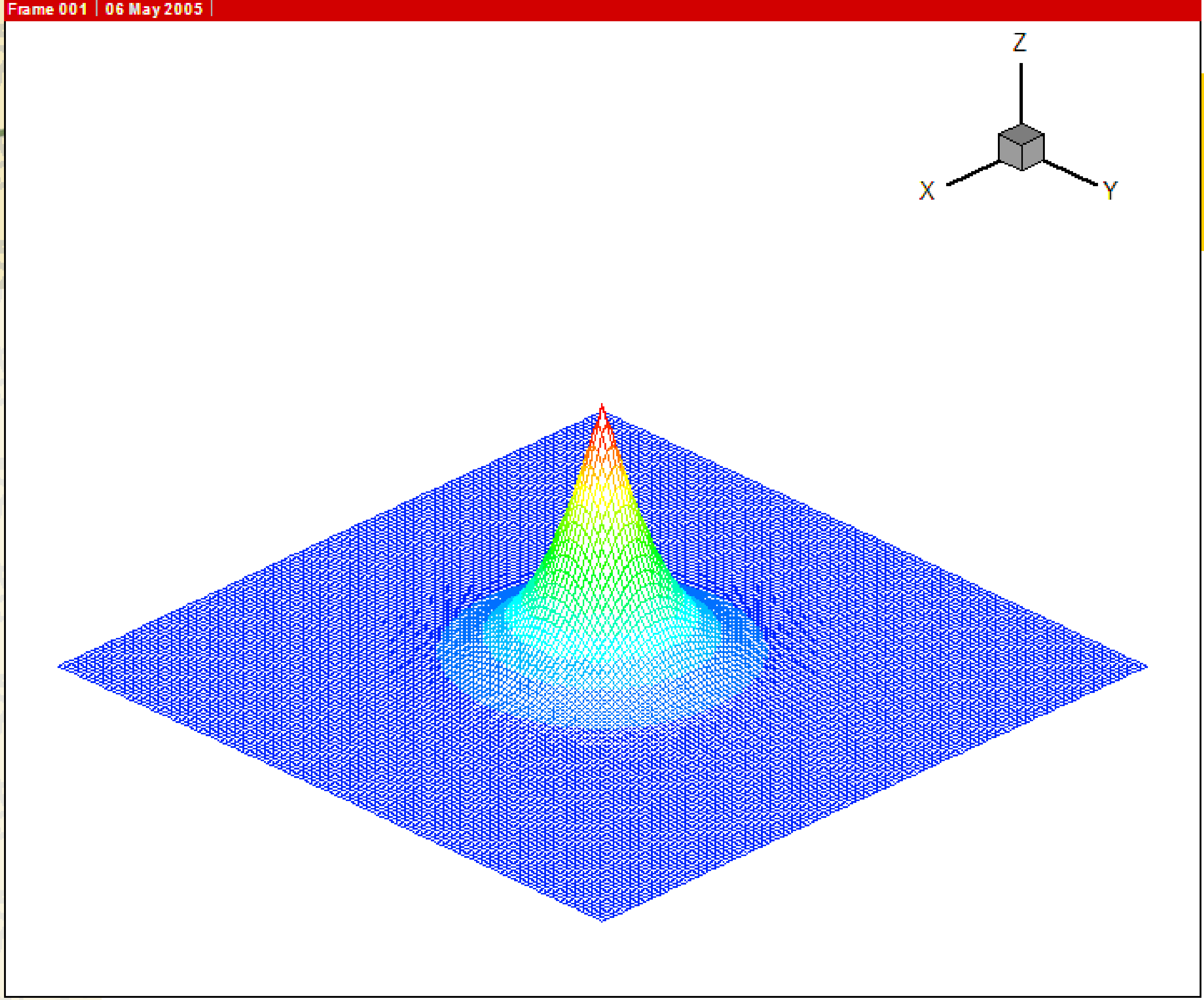

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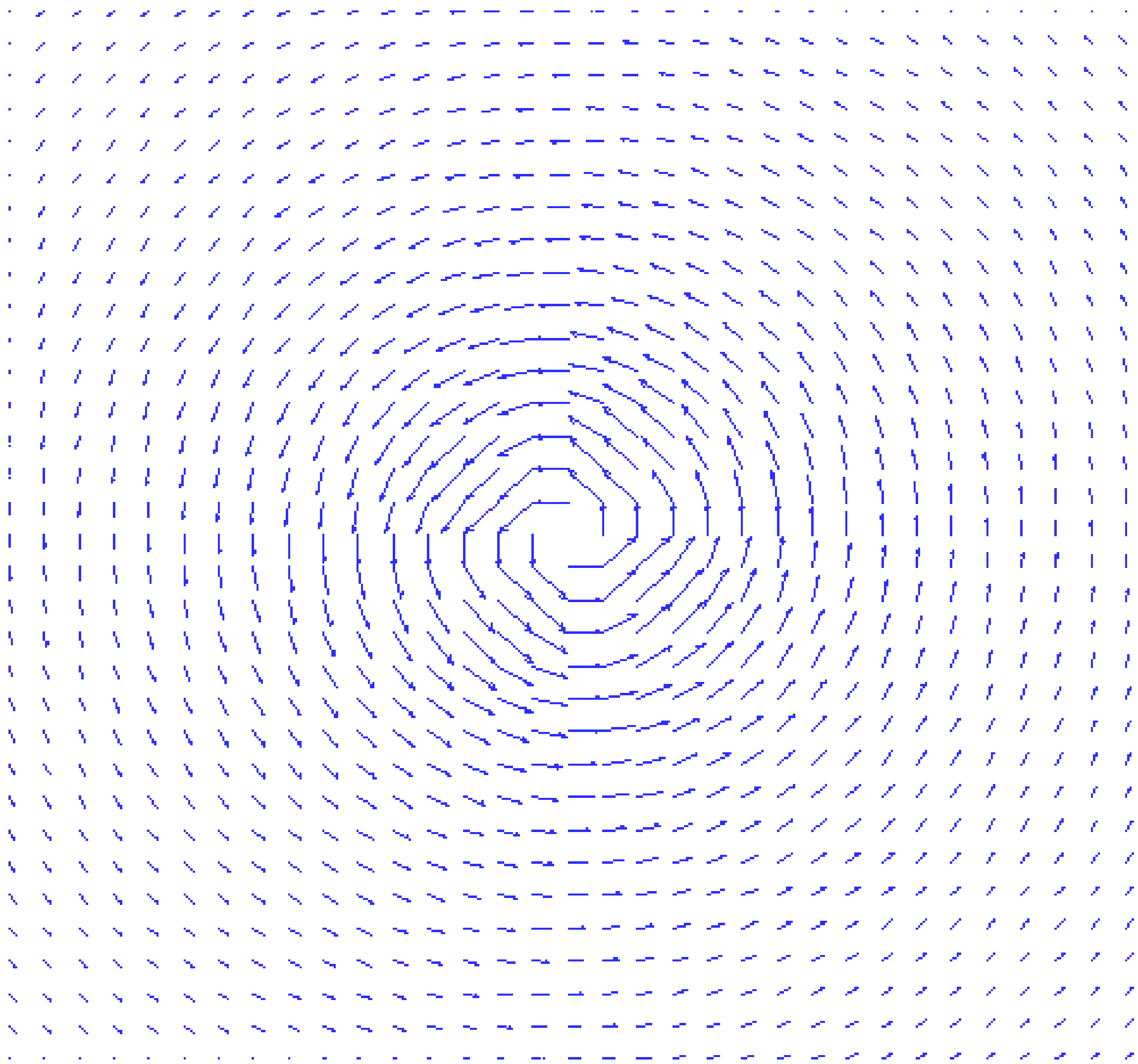

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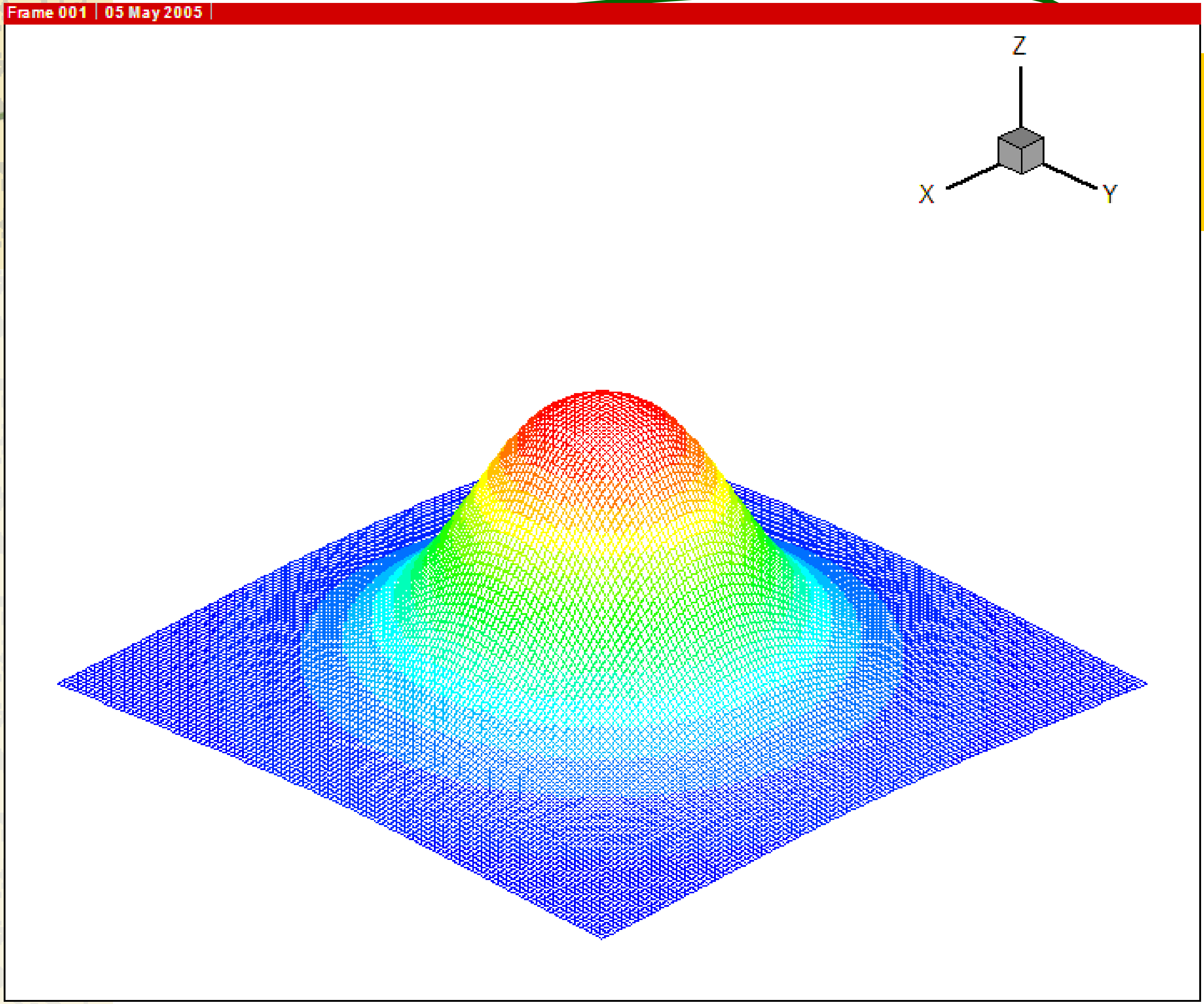

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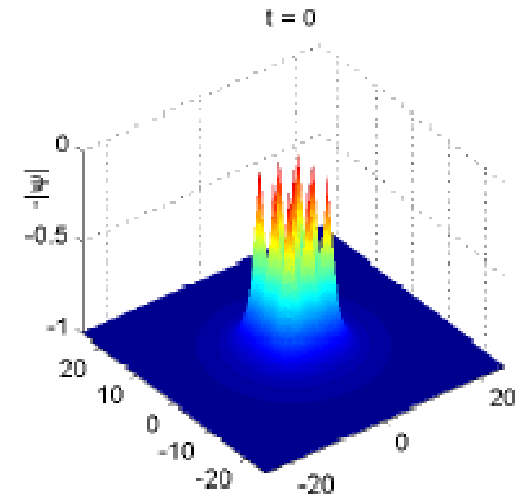
Conclusion of Stability

- For GLE or NLWE:
  - under perturbation either in initial data or in external potential
    - $|n|=1$: dynamically stable,
    - $|n|>1$: dynamically unstable. Splitting pattern depends on perturbation, when $t \gg 1$, it becomes n well-separated vortices with index 1 or –1.
- For NLSE or GPE:
  - under perturbation in initial data
    - dynamically stable: Angular momentum expectation is conserved!!
  - under perturbation in external potential
    - $|n|=1$: dynamically stable, back
    - $|n|>1$: dynamically unstable. Vortex centers can NOT move out of core size.

Vortex interaction

- For `bright-tail' vortex:

$$\psi_0(\vec{x}) = \prod_{j=1}^{N} \phi_{m_j}(\vec{x} - \vec{x}_j^0) = \prod_{j=1}^{N} \phi_{m_j}(x - x_j^0, y - y_j^0)$$

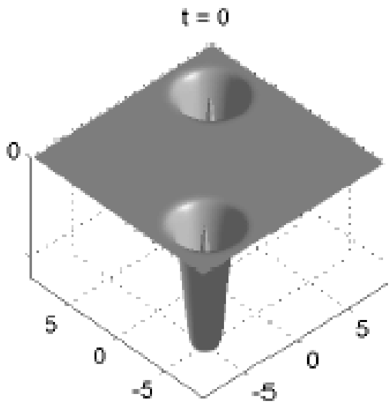


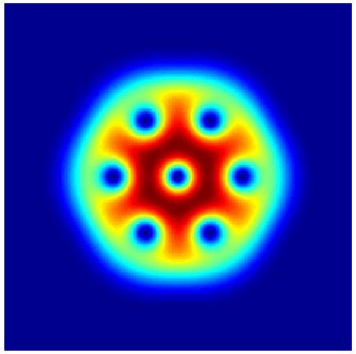
- For `dark-tail' vortex
  - Well-separate

$$\psi_0(x,y) = \sum_{j=1}^{N} \phi_{n_j}(x - x_j, y - y_j) \quad / \|\bullet\|$$

  - Overlapped

$$\psi(\mathbf{r}, t=0) = \alpha \psi_{gs}(\mathbf{r}) \prod_{j=1}^{n} p_{q_j}(\mathbf{r} - \mathbf{r}_j),$$





Reduced dynamic laws

- For `bright-tail' vortex
  - For GLE

$$\kappa \vec{v}_j(t) := \kappa \frac{d\,\vec{x}_j(t)}{dt} = 2m_j \sum_{l=1, l\neq j}^{N} m_l \frac{\vec{x}_j(t) - \vec{x}_l(t)}{|\vec{x}_j(t) - \vec{x}_l(t)|^2}, \qquad t \geq 0$$

$$\vec{x}_j(0) = \vec{x}_j^0, \qquad 1 \leq j \leq N.$$

  - For GPE

$$\vec{v}_j(t) := \frac{d\,\vec{x}_j(t)}{dt} = 2 \sum_{l=1, l\neq j}^{N} m_l \frac{J(\vec{x}_j(t) - \vec{x}_l(t))}{|\vec{x}_j(t) - \vec{x}_l(t)|^2} \qquad t \geq 0,$$

$$\vec{x}_j(0) = \vec{x}_j^0 \qquad 1 \leq j \leq N; \qquad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

  - For NLWE

$$\kappa \frac{d^2\,\vec{x}_j(t)}{dt^2} = 2m_j \sum_{l=1, l\neq j}^{N} m_l \frac{\vec{x}_j(t) - \vec{x}_l(t)}{|\vec{x}_j(t) - \vec{x}_l(t)|^2}, \qquad t \geq 0$$

$$\vec{x}_j(0) = \vec{x}_j^0, \quad \vec{x}_j^{'}(0) = \vec{v}_j^0, \quad 1 \leq j \leq N.$$

Existing mathematical results

- For GLE (or NLHE):
  - Pairs of vortices with like (opposite) index undergo a replusive (attractive) interaction: *Neu 90*, *Pismen & Rubinstein 91, W. E, 94,* Bethual, etc.
  - *Energy concentrated at vortices in 2D & filaments in 3D: Lin 95--*
  - *Vortices are attracted by impurities: Chapman & Richardson 97, Jian 01*
- For NLSE:
  - Vortices behave like point vortices in ideal fluid: *Neu 90*
  - Obeys classical Kirchhoff law for fluid point vortices: *Lin & Xin 98*
  - Equations for vortex dynamics & radiation: *Ovchinnikov& Sigal 98--; Jerrard, Bethual, etc.*

Conservation laws

- Mass center $\overline{x}(t) := \frac{1}{N}\sum_{j=1}^{N} \vec{x}_j(t)$

Lemma The mass center of the N vortices for GLE is conserved

$$\overline{x}(t) := \frac{1}{N}\sum_{j=1}^{N} \vec{x}_j(t) \equiv \overline{x}(0) := \frac{1}{N}\sum_{j=1}^{N} \vec{x}_j(0) = \frac{1}{N}\sum_{j=1}^{N} \vec{x}_j^0$$

Lemma The mass center of the N vortices for NLWE is conserved if initial velocity is zero and it moves linearly if initial velocity is nonzero

$$\overline{\mathbf{x}}(t) := \frac{1}{N}\sum_{j=1}^{N} \mathbf{x}_j(t) = \overline{\mathbf{x}}(0) + t\,\overline{\mathbf{v}}(0) = \frac{1}{N}\sum_{j=1}^{N} \mathbf{x}_j^0 + t\left(\frac{1}{N}\sum_{j=1}^{N} \mathbf{x}_j^1\right), \quad t \geq 0,$$

Conservation laws

- Signed mass center $\tilde{x}(t) := \frac{1}{N}\sum_{j=1}^{N} m_j\, \vec{x}_j(t)$

Lemma The signed mass center of the N vortices for NLSE is conserved

$$\tilde{x}(t) := \frac{1}{N}\sum_{j=1}^{N} m_j\, \vec{x}_j(t) \equiv \tilde{x}(0) := \frac{1}{N}\sum_{j=1}^{N} m_j\, \vec{x}_j(0) = \frac{1}{N}\sum_{j=1}^{N} m_j\, \vec{x}_j^{\,0}$$

Analytical Solutions

- $N\,(\geq 2)$ like vortices on a circle (Bao, Du & Zhang, SIAP, 07')

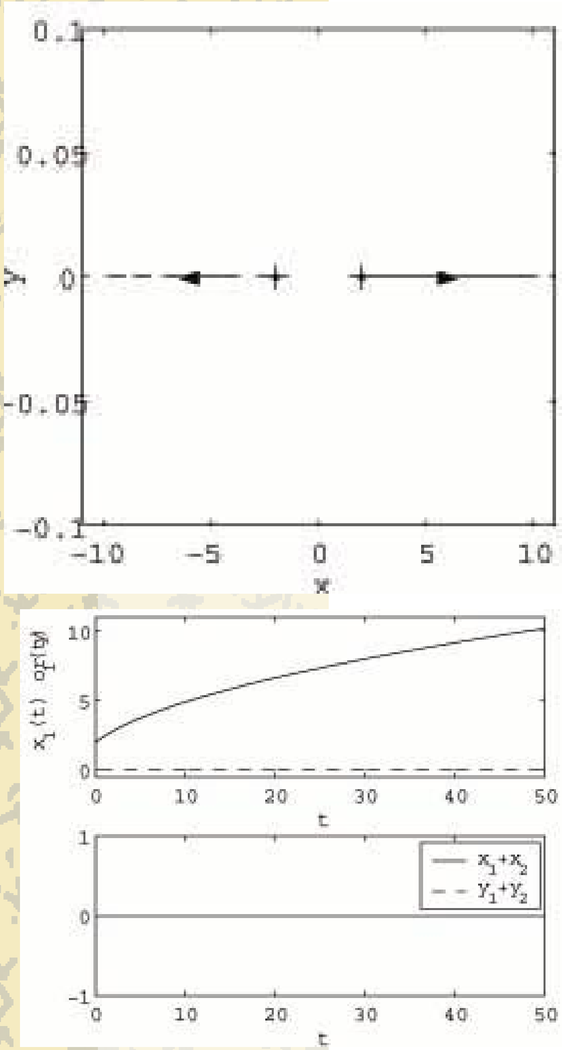
$$\vec{x}_j^0 = a\left(\cos\left(\frac{2j\pi}{N}\right), \quad \sin\left(\frac{2j\pi}{N}\right)\right), \qquad m_j = m_0 = \pm 1, \qquad 1 \leq j \leq N$$

- Analytical solutions for GLE figure

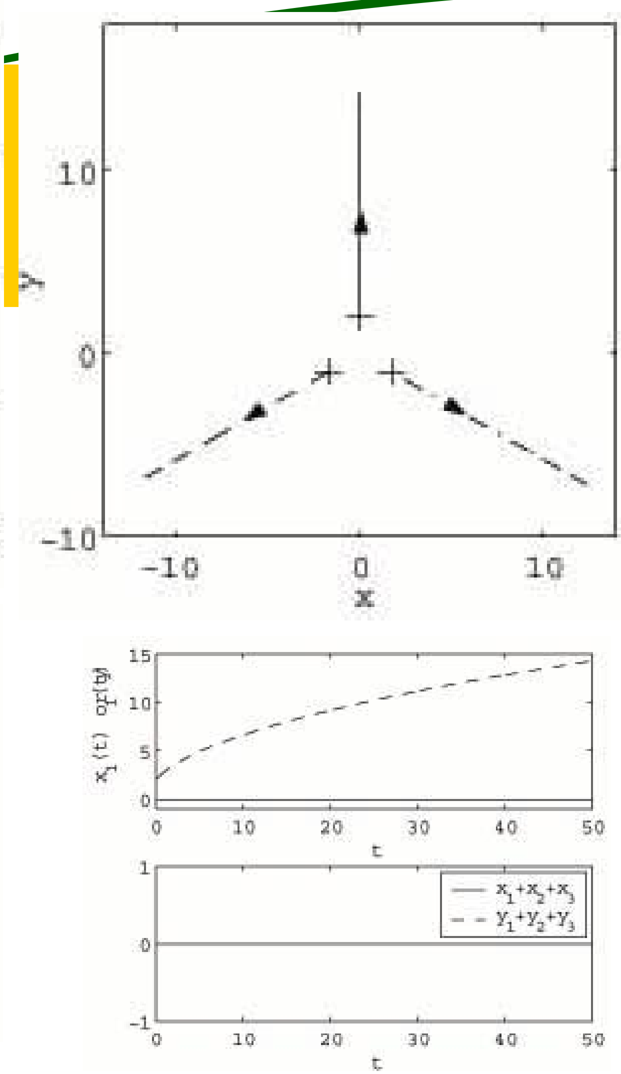
$$\vec{x}_j(t) = \sqrt{a^2 + \frac{2(N-1)}{\kappa}t}\;\left(\cos\left(\frac{2j\pi}{N}\right), \quad \sin\left(\frac{2j\pi}{N}\right)\right)$$

- Analytical solutions for NLSE figure next

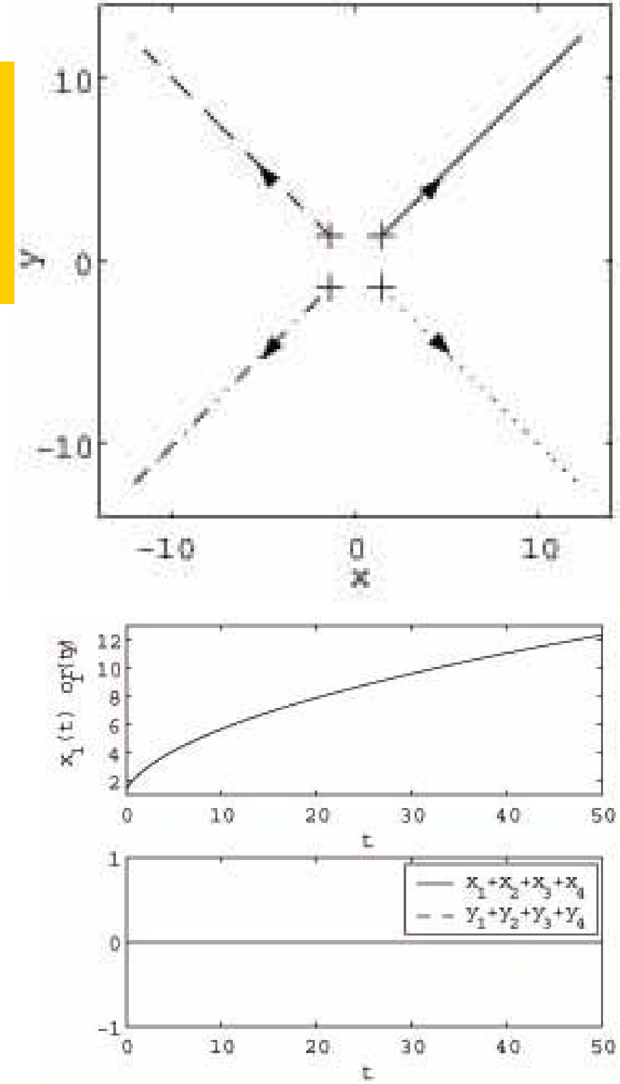
$$\vec{x}_j(t) = a\left(\cos\left(\frac{2j\pi}{N} + \frac{N-1}{a^2}t\right), \quad \sin\left(\frac{2j\pi}{N} + \frac{N-1}{a^2}t\right)\right)$$


x
y
t
x1(t) or(y)
x1+x2
y1+y2

N=2

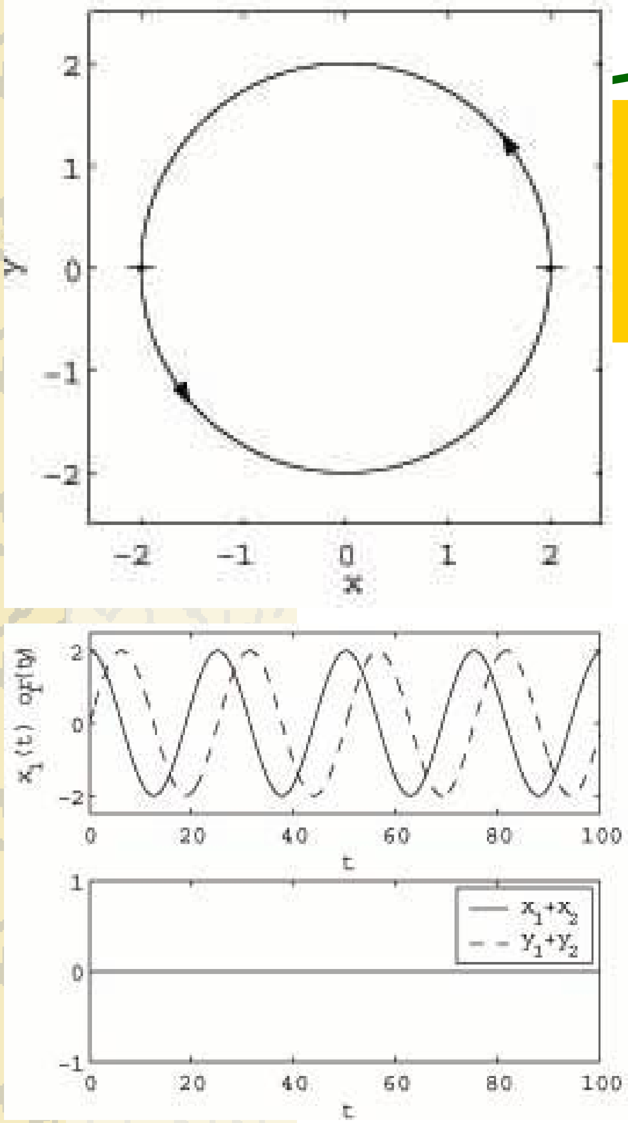

x
y
t
x1(t) or(y)
x1+x2+x3
y1+y2+y3

N=3

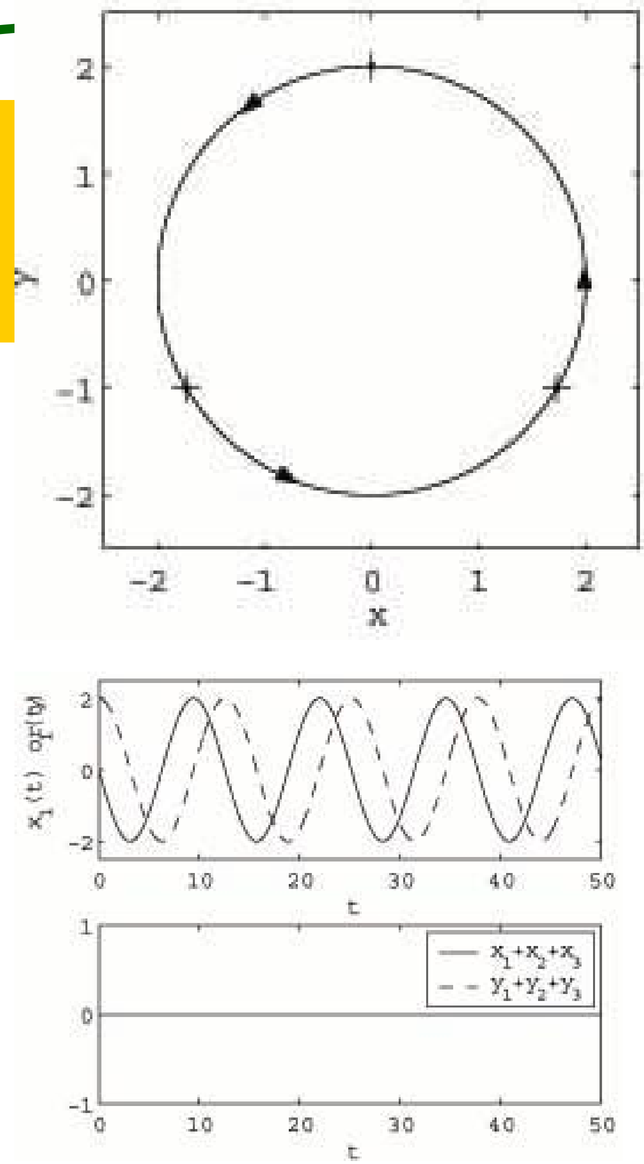

x
y
t
x1(t) or(y)
x1+x2+x3+x4
y1+y2+y3+y4

N=4

back



N=2



N=3

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Analytical Solutions

- $N\ (\geq 3)$ like vortices on a circle and its center (Bao,Du&Zhang, SIAP, 07')

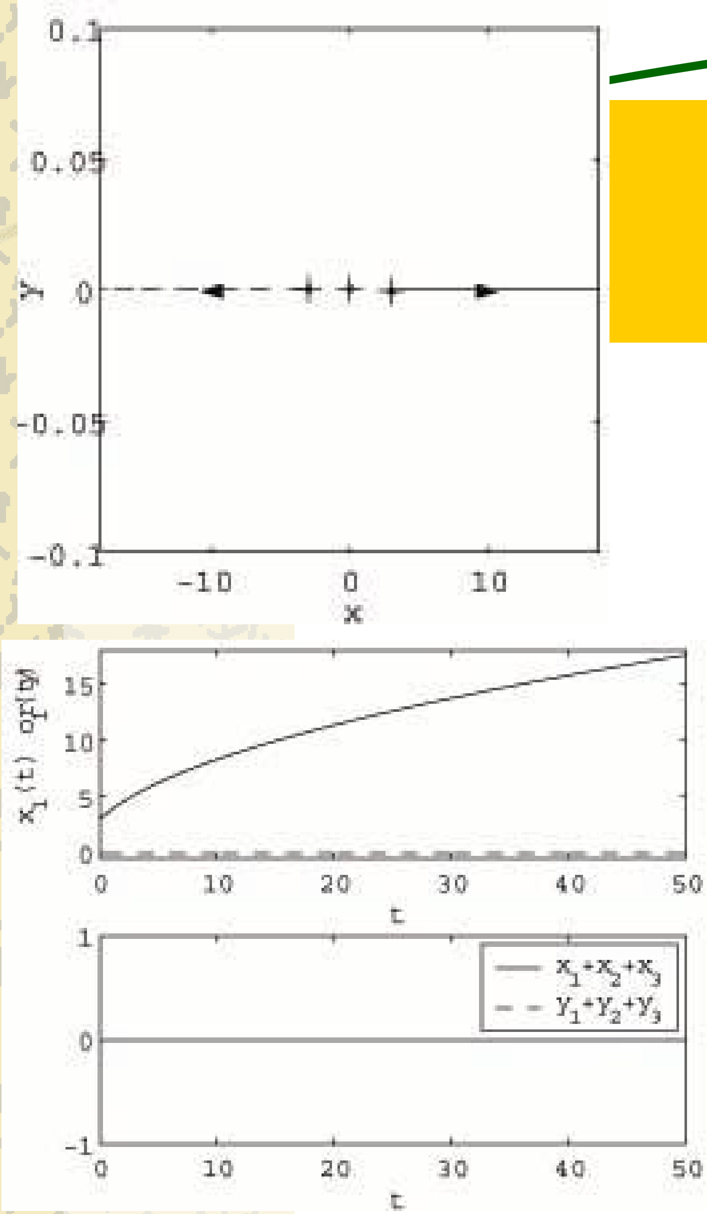
$$\vec{x}_N^0 = \vec{0}; \qquad \vec{x}_j^0 = a\left(\cos\left(\frac{2j\pi}{N-1}\right), \ \sin\left(\frac{2j\pi}{N-1}\right)\right), \qquad 1 \leq j \leq N-1$$

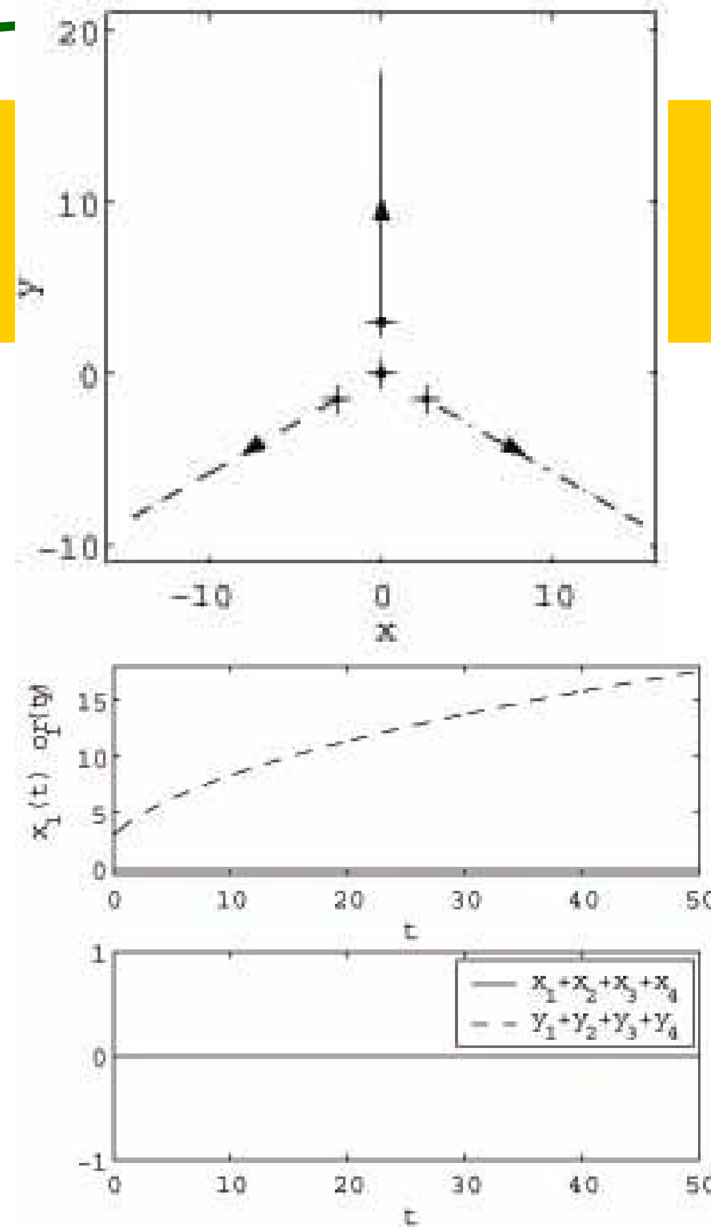
- Analytical solutions for GLE figure

$$\vec{x}_N(t) = \vec{0}; \qquad \vec{x}_j(t) = \sqrt{a^2 + \frac{2N}{\kappa}t}\ \left(\cos\left(\frac{2j\pi}{N-1}\right), \ \sin\left(\frac{2j\pi}{N-1}\right)\right), \quad 1 \leq j \leq N-1$$

- Analytical solutions for NLSE figure next

$$\vec{x}_N(t) = \vec{0}; \qquad \vec{x}_j(t) = a\left(\cos\left(\frac{2j\pi}{N-1} + \frac{N-2}{a^2}t\right), \ \sin\left(\frac{2j\pi}{N-1} + \frac{N-2}{a^2}t\right)\right)$$

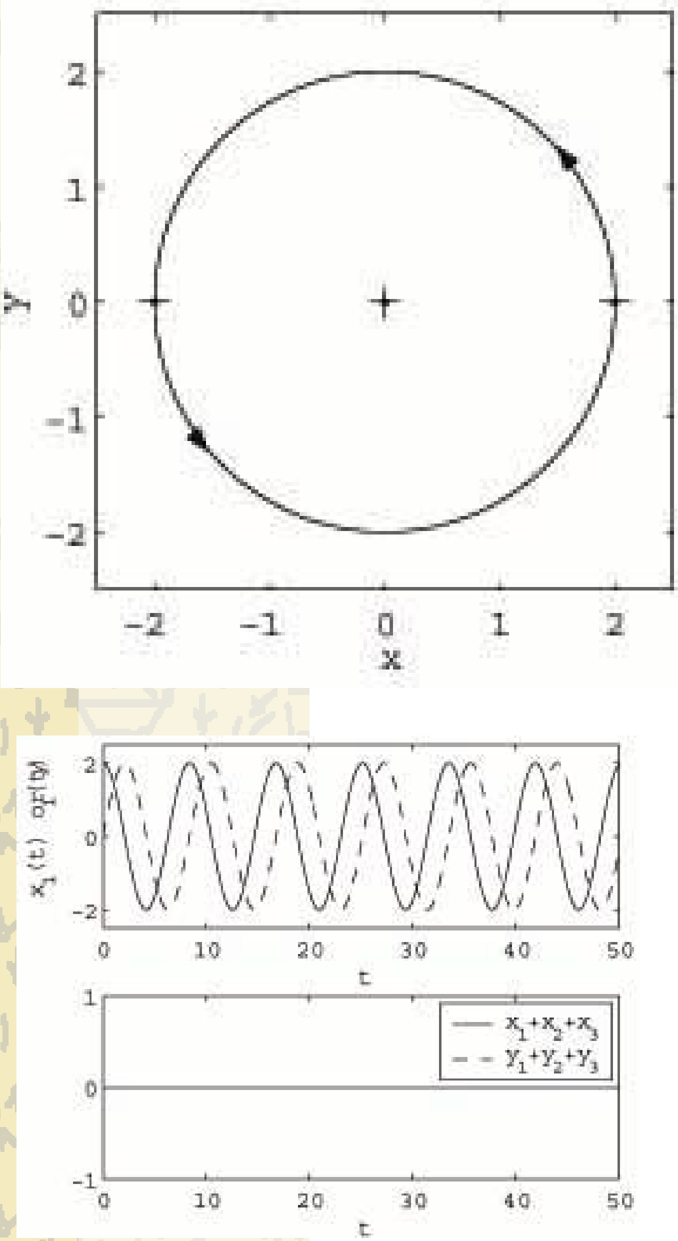

0.1
0.05
0
-0.05
-0.1
y
-10
0
10
x
$x_1(t)$ or (y)
t
$x_1+x_2+x_3$
$y_1+y_2+y_3$

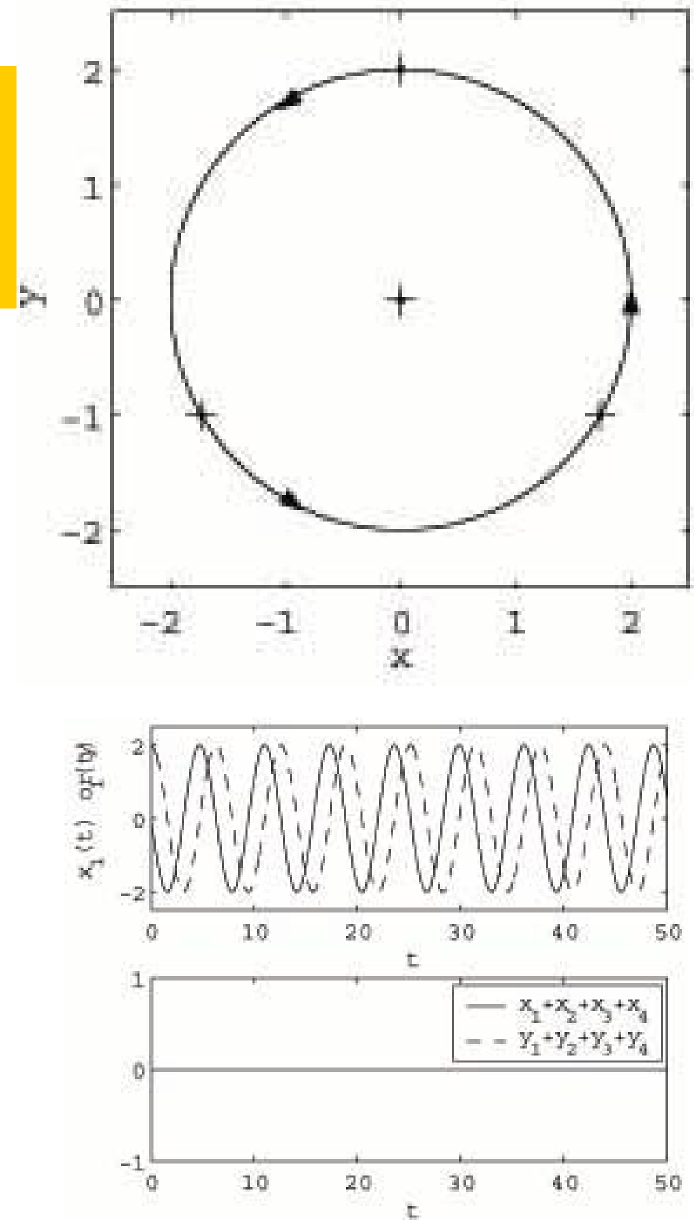

20
10
0
-10
y
x
$x_1(t)$ or (y)
t
$x_1+x_2+x_3+x_4$
$y_1+y_2+y_3+y_4$

N=3

back

N=4





N=3

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N=4

Analytical Solutions

- Two opposite vortices (Bao, Du & Zhang, SIAP, 07')

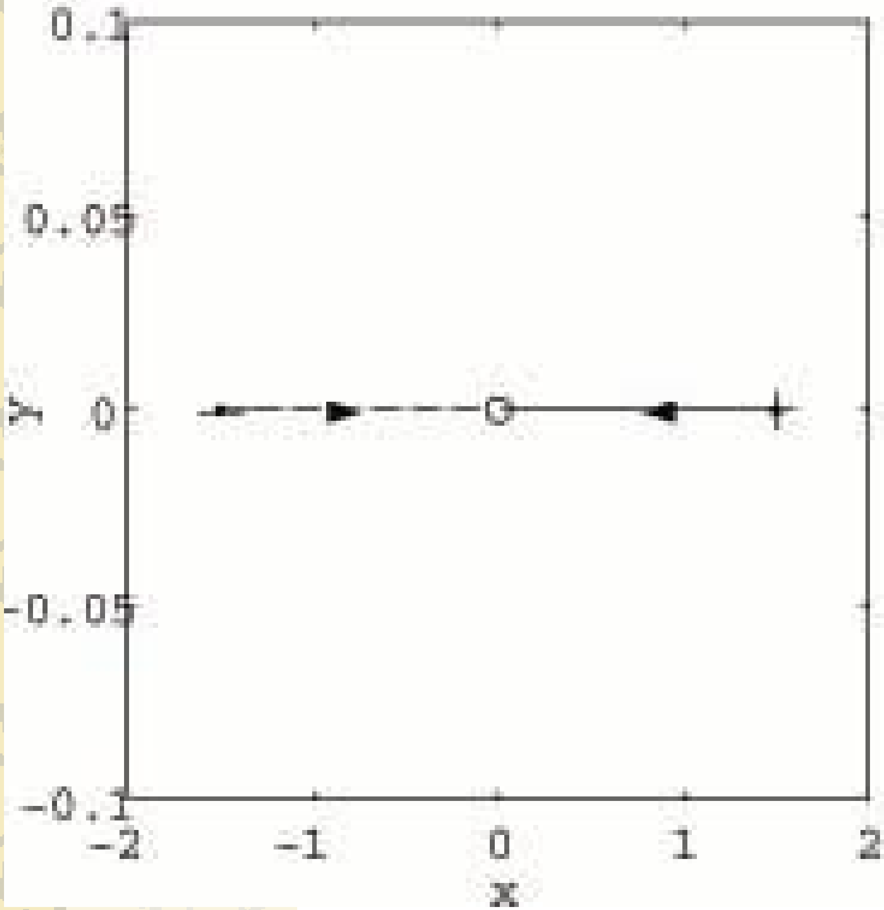
$$\vec{x}_1^0 = -\vec{x}_2^0 = a\left(\cos(\theta_0) \quad , \quad \sin(\theta_0)\right), \qquad m_1 = -m_2 = 1$$

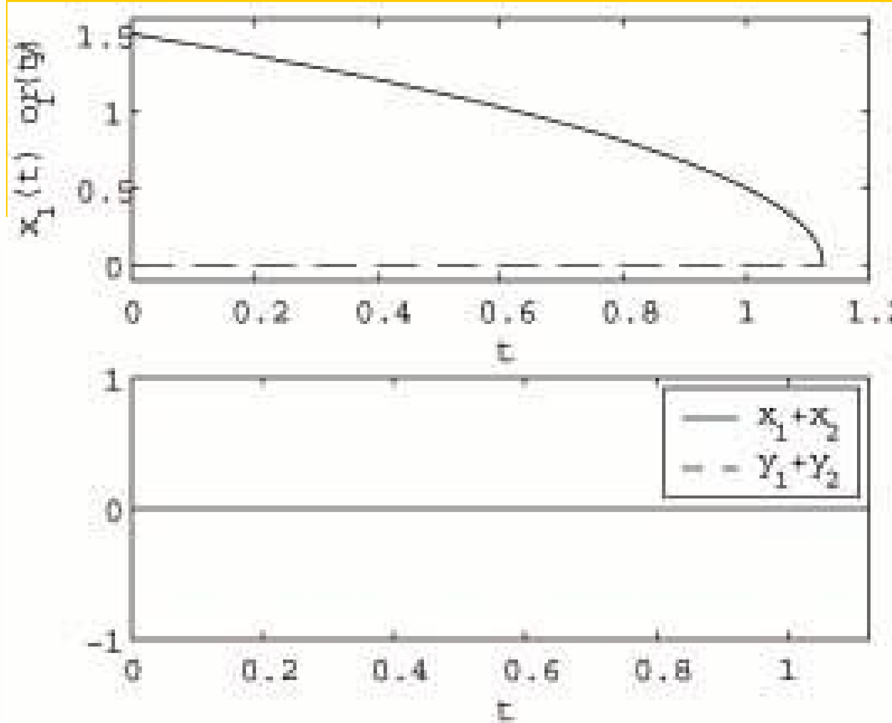
- Analytical solutions for GLE figure

$$\vec{x}_1(t) = -\vec{x}_2(t) = \sqrt{a^2 - \frac{2}{\kappa}t}\ \left(\cos(\theta_0) \quad , \quad \sin(\theta_0)\right)$$

- Analytical solutions for NLSE figure next

$$\vec{x}_j(t) = \vec{x}_j^0 + \frac{t}{a}\left(-\sin(\theta_0) \quad , \quad \cos(\theta_0)\right), \qquad j = 1, 2$$





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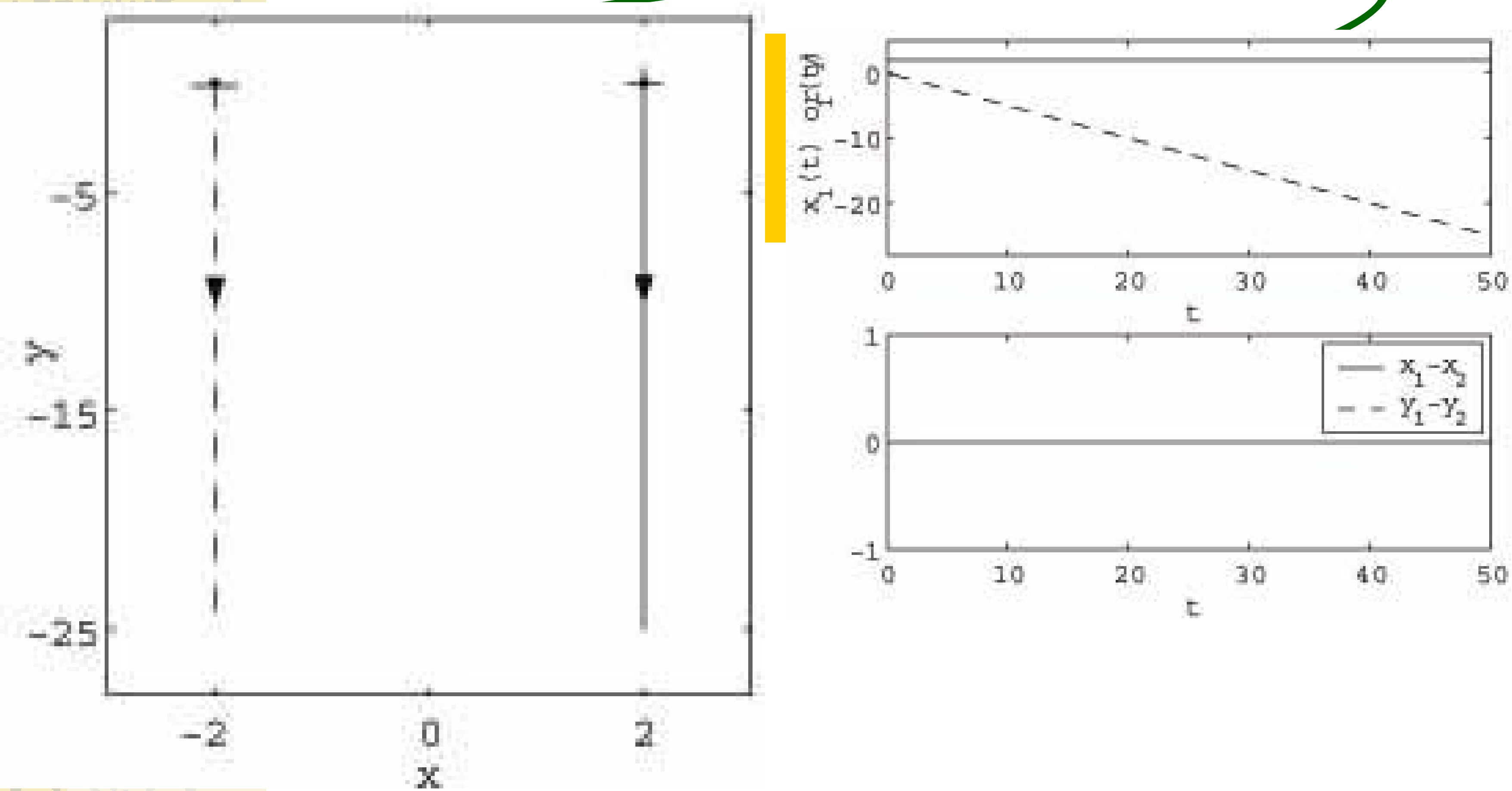


-5
-15
-25
y
-2
0
2
x
$x_1(t)$ or y_1
0
-10
-20
0
10
20
30
40
50
t
1
0
-1
x_1-x_2
y_1-y_2
0
10
20
30
40
50
t

back

Analytical Solutions

- $N\ (\geq 3)$ opposite vortices on a circle and its center

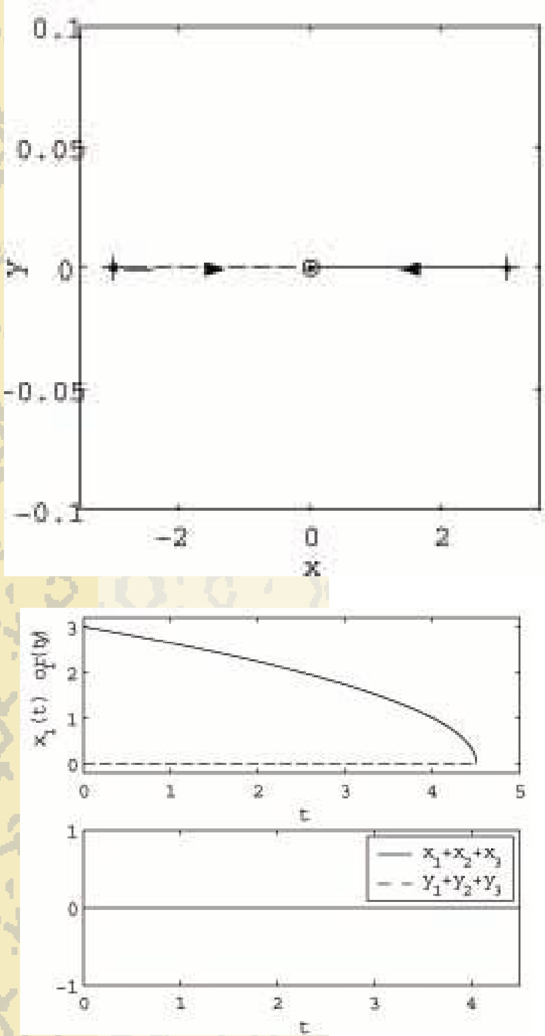
$$\vec{x}_N^0 = \vec{0} \quad (-); \quad \vec{x}_j^0 = a\left(\cos\left(\frac{2j\pi}{N-1}\right), \ \sin\left(\frac{2j\pi}{N-1}\right)\right) \quad (+), \qquad 1 \leq j \leq N-1$$

- Analytical solutions for GLE   figure

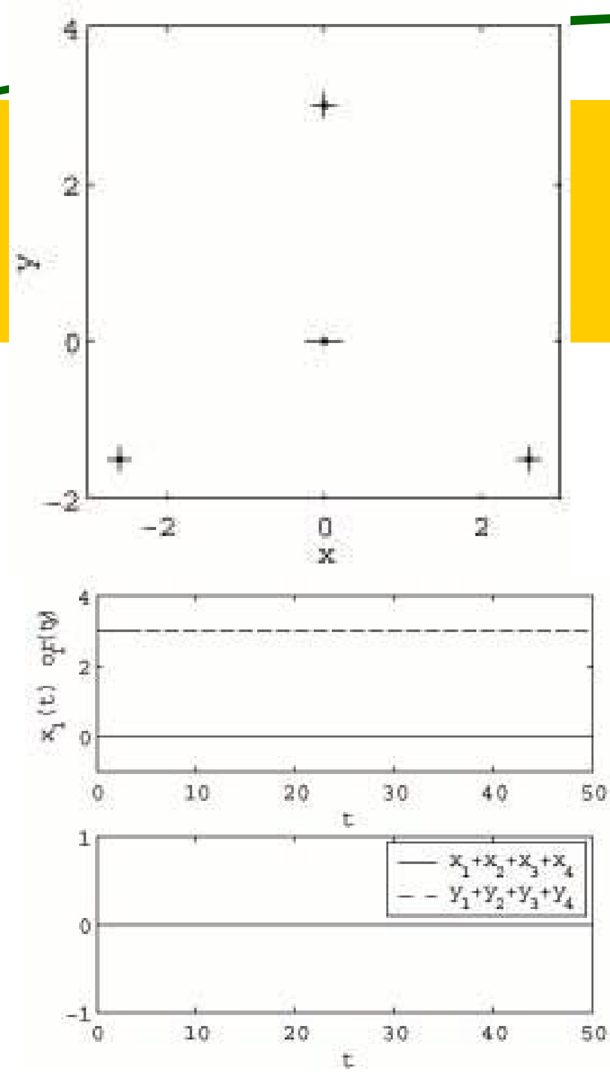
$$\vec{x}_N(t) = \vec{0}; \qquad \vec{x}_j(t) = \sqrt{a^2 + \frac{2(N-4)}{\kappa}t}\ \left(\cos\left(\frac{2j\pi}{N-1}\right), \ \sin\left(\frac{2j\pi}{N-1}\right)\right), \quad 1 \leq j \leq N-1$$

- Analytical solutions for NLSE   figure   next

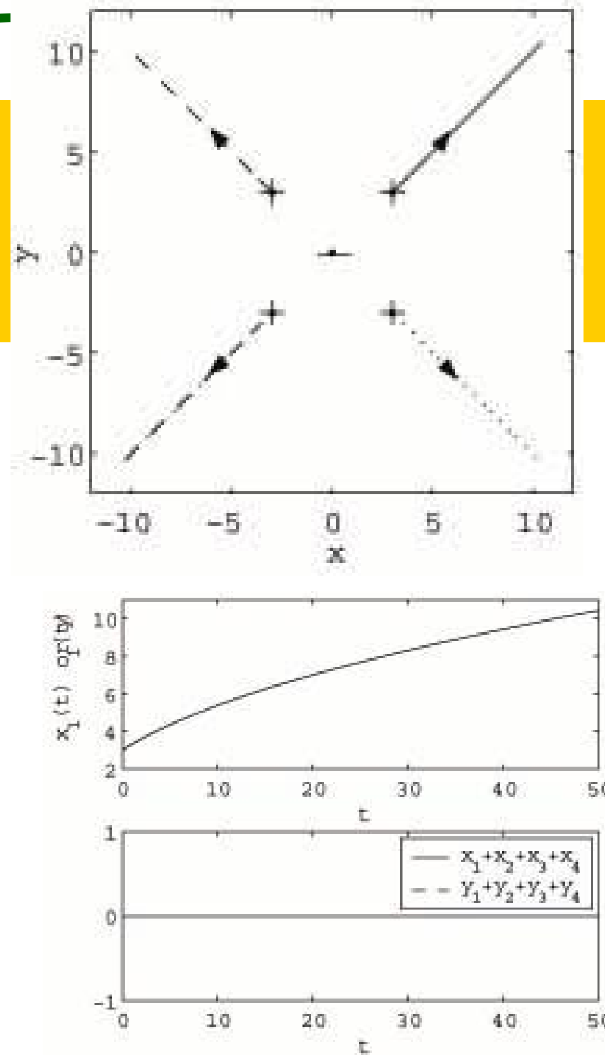
$$\vec{x}_N(t) = \vec{0}; \qquad \vec{x}_j(t) = a\ \left(\cos\left(\frac{2j\pi}{N-1} + \frac{N-4}{a^2}t\right), \ \sin\left(\frac{2j\pi}{N-1} + \frac{N-4}{a^2}t\right)\right)$$


y
x
$x_1(t)$ or $f(y)$
t
$x_1+x_2+x_3$
$y_1+y_2+y_3$
t

N=3

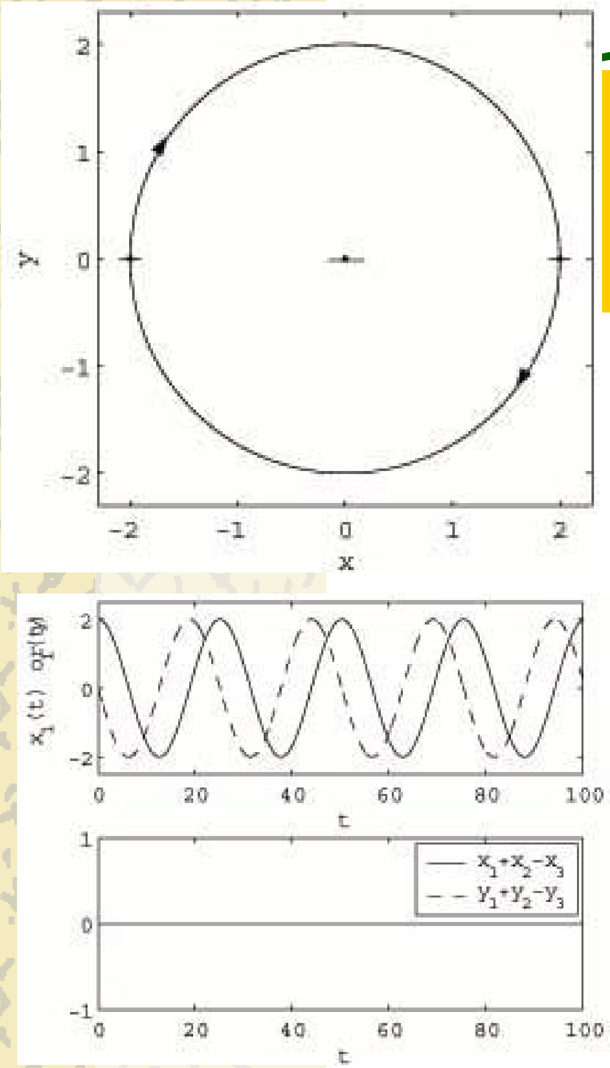

y
x
$x_1(t)$ or $f(y)$
t
$x_1+x_2+x_3+x_4$
$y_1+y_2+y_3+y_4$
t

N=4


y
x
$x_1(t)$ or $f(y)$
t
$x_1+x_2+x_3+x_4$
$y_1+y_2+y_3+y_4$
t

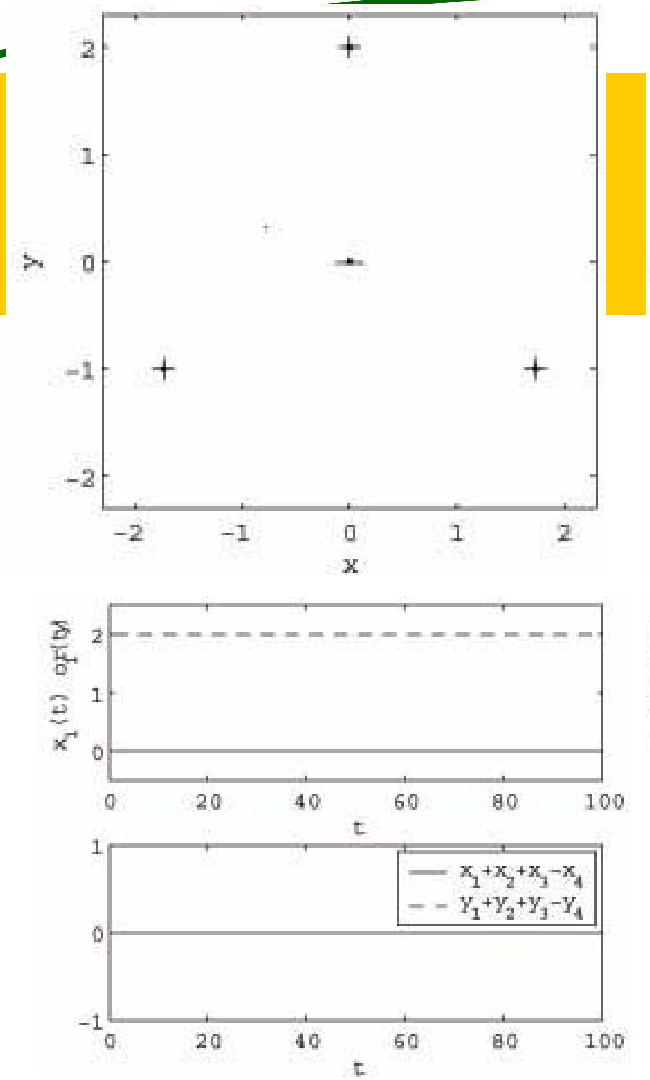
N=5

back



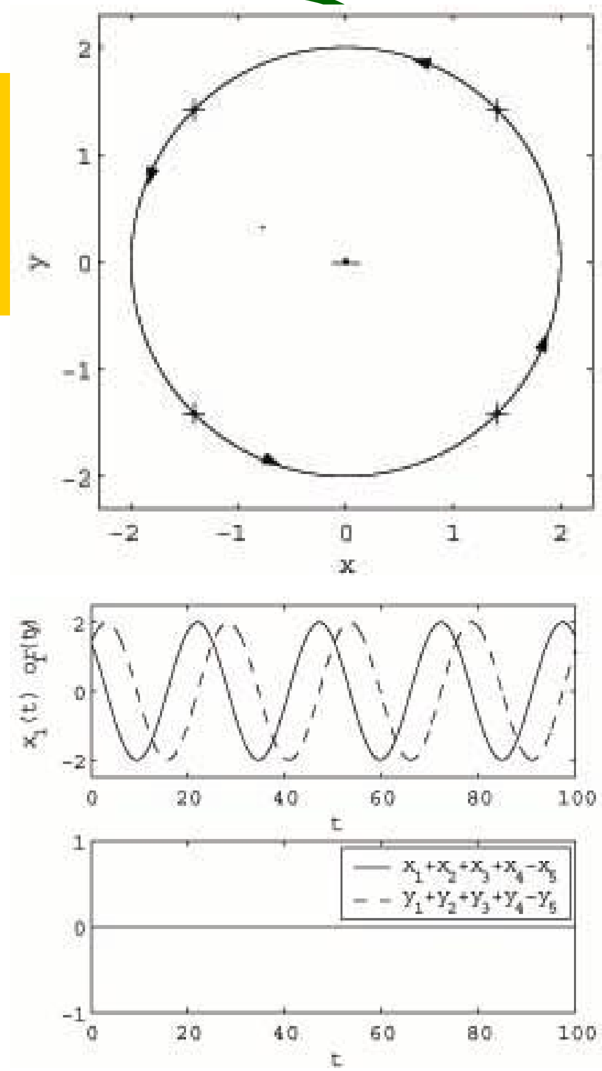
y
x
$x_1(t)$ or $y_1(t)$
t
$x_1+x_2-x_3$
$y_1+y_2-y_3$
t

N=3



y
x
$x_1(t)$ or $y_1(t)$
t
$x_1+x_2+x_3-x_4$
$y_1+y_2+y_3-y_4$
t

N=4



y
x
$x_1(t)$ or $y_1(t)$
t
$x_1+x_2+x_3+x_4-x_5$
$y_1+y_2+y_3+y_4-y_5$
t

N=5

back

Numerical difficulties

- Numerical difficulties:
  - Highly oscillatory nature in the transverse direction: resolution
  - Quadratic decay rate in radial direction: large domain
  - For NLSE, more difficulties:
    - Time reversible
    - Time transverse invariant
    - Dispersive equation
    - Keep conservation laws in discretized level
    - Radiation

Numerical methods

- For `bright-tail' vortex (Bao,Du & Zhang, Eur. J. Appl. Math., 07')
  - Apply a time-splitting technique: decouple nonlinearity
  - Adapt polar coordinate: resolve solution in transverse better
  - Use Fourier pesudo-spectral discretization in transverse
  - Use 2nd or 4th order FEM or FDM in radial direction
- For `dark-tail' vortex (Bao & Zhang, M3AS, 05')
  - Apply a time-splitting technique: decouple nonlinearity
  - Use Fourier pseudo-spectral discretization
  - Use generalized Laguerre-Hermite pseudo-spectral method

Vortex dynamics & interaction

- Data chosen (Bao, Du & Zhang, SIAM, J. Appl. Math., 07')

$$\varepsilon = 1, \qquad V(\vec{x},t) \equiv 1, \qquad \psi_0(\vec{x}) = \prod_{j=1}^{N} \phi_{n_j}(\vec{x}-\vec{x}_j), \quad m = \sum_{j=1}^{N} n_j$$

- Two vortices with like winding numbers

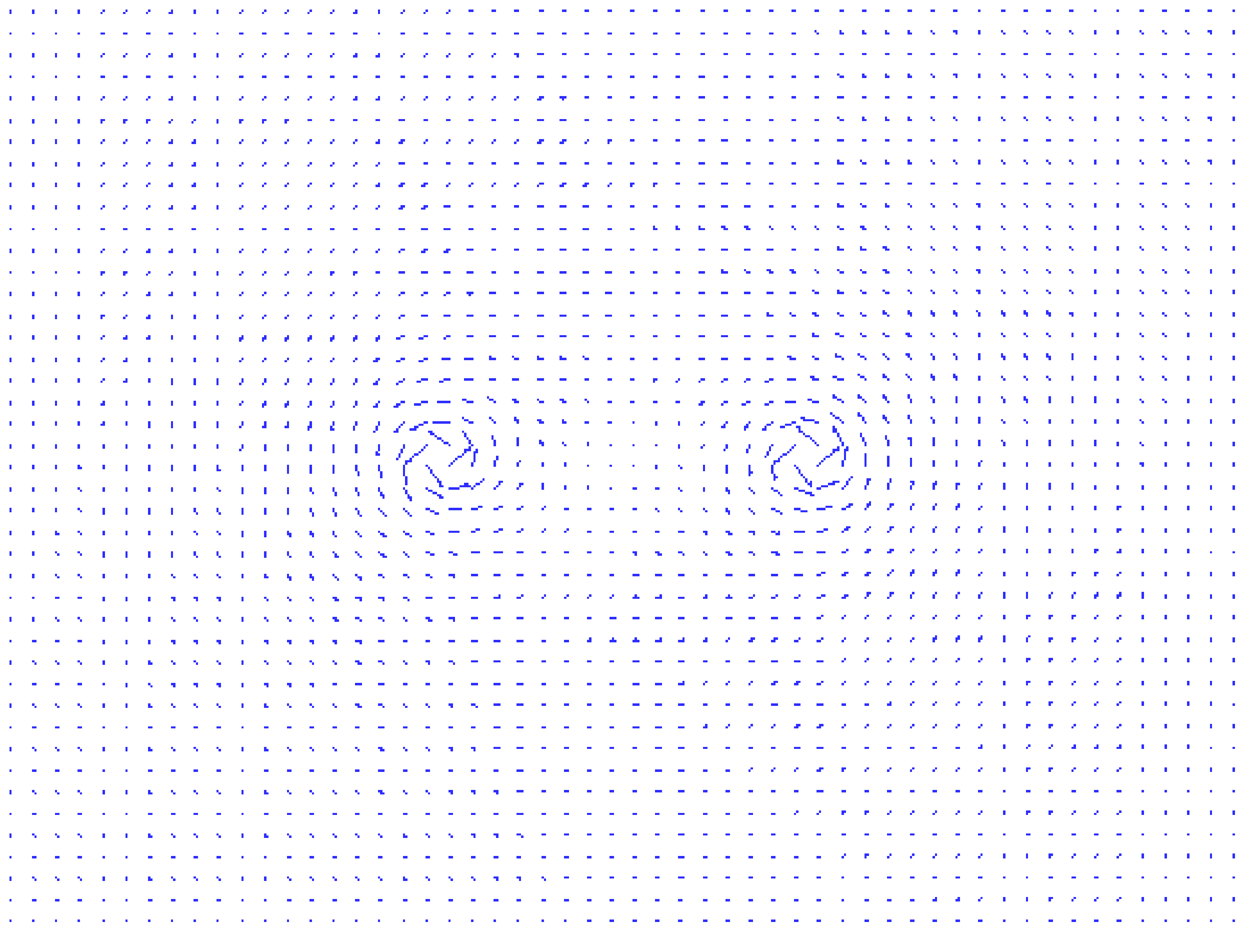
$$N = 2, \qquad n_1 = 1, \qquad n_2 = 1, \qquad \vec{x}_1 = (a,0), \qquad \vec{x}_2 = (-a,0)$$

  - For GLE: velocity density
  - For GPE: velocity density
  - Trajectory

- Conclusion

next

Frame 001 | 06 May 2005 |



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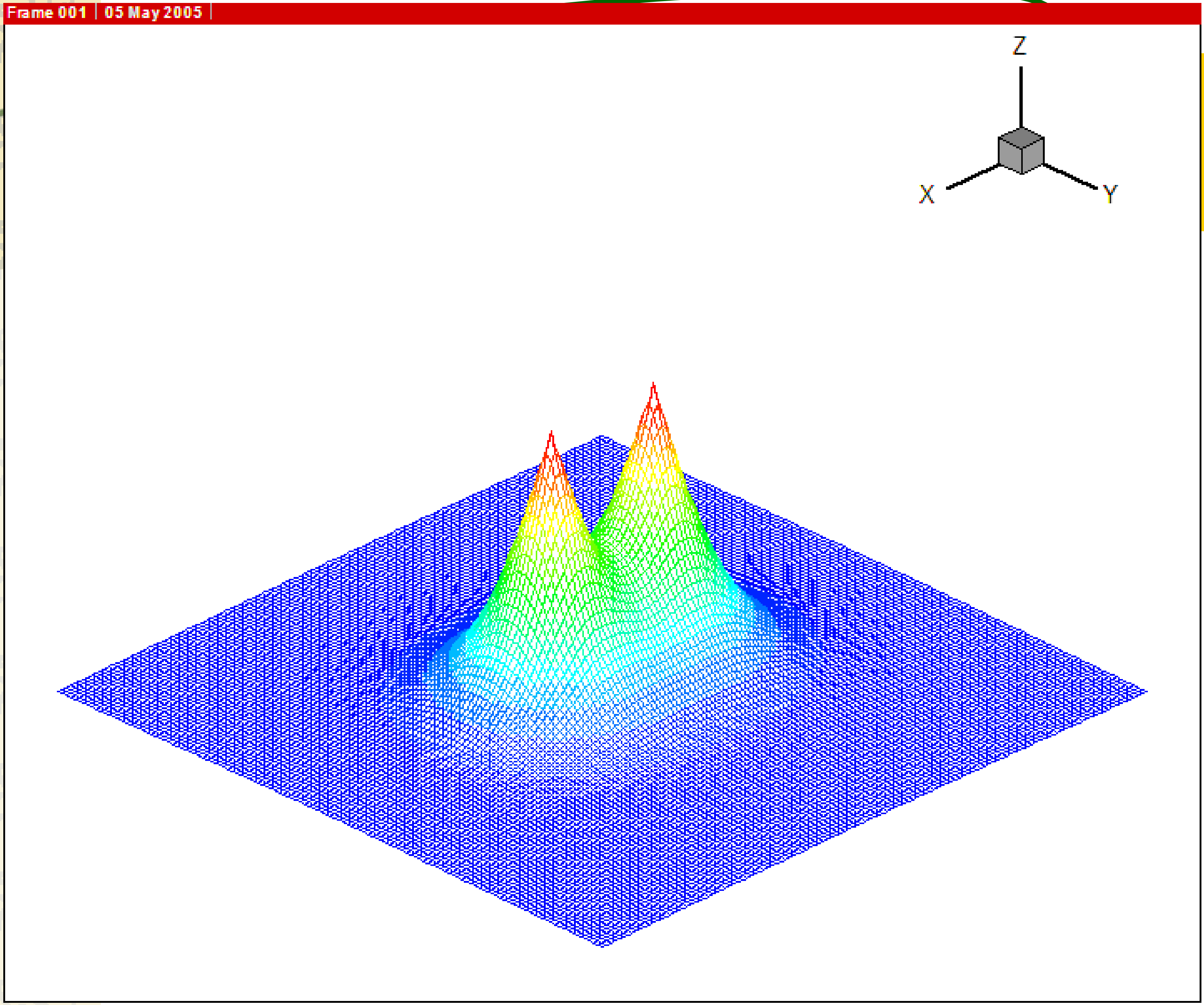

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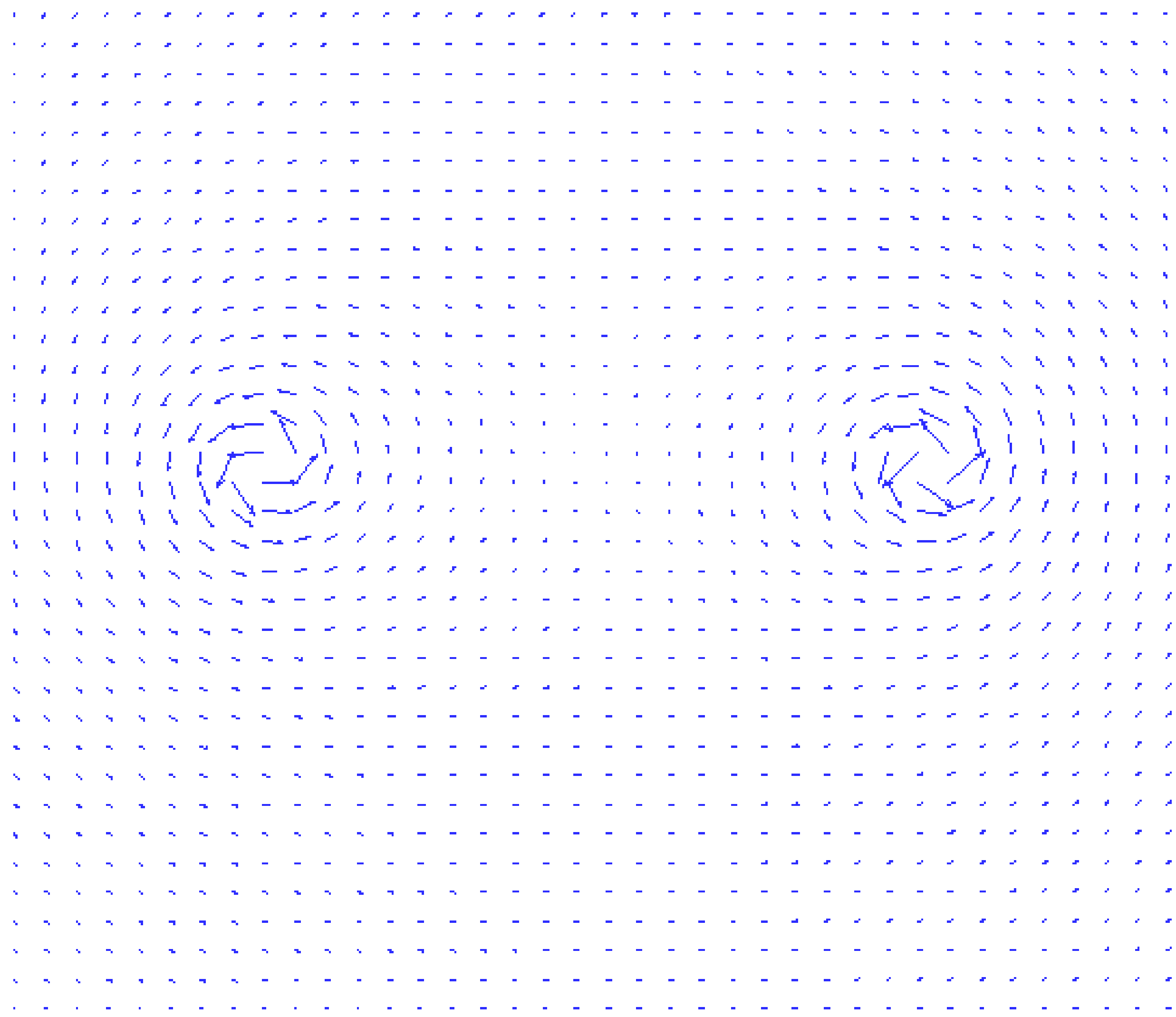

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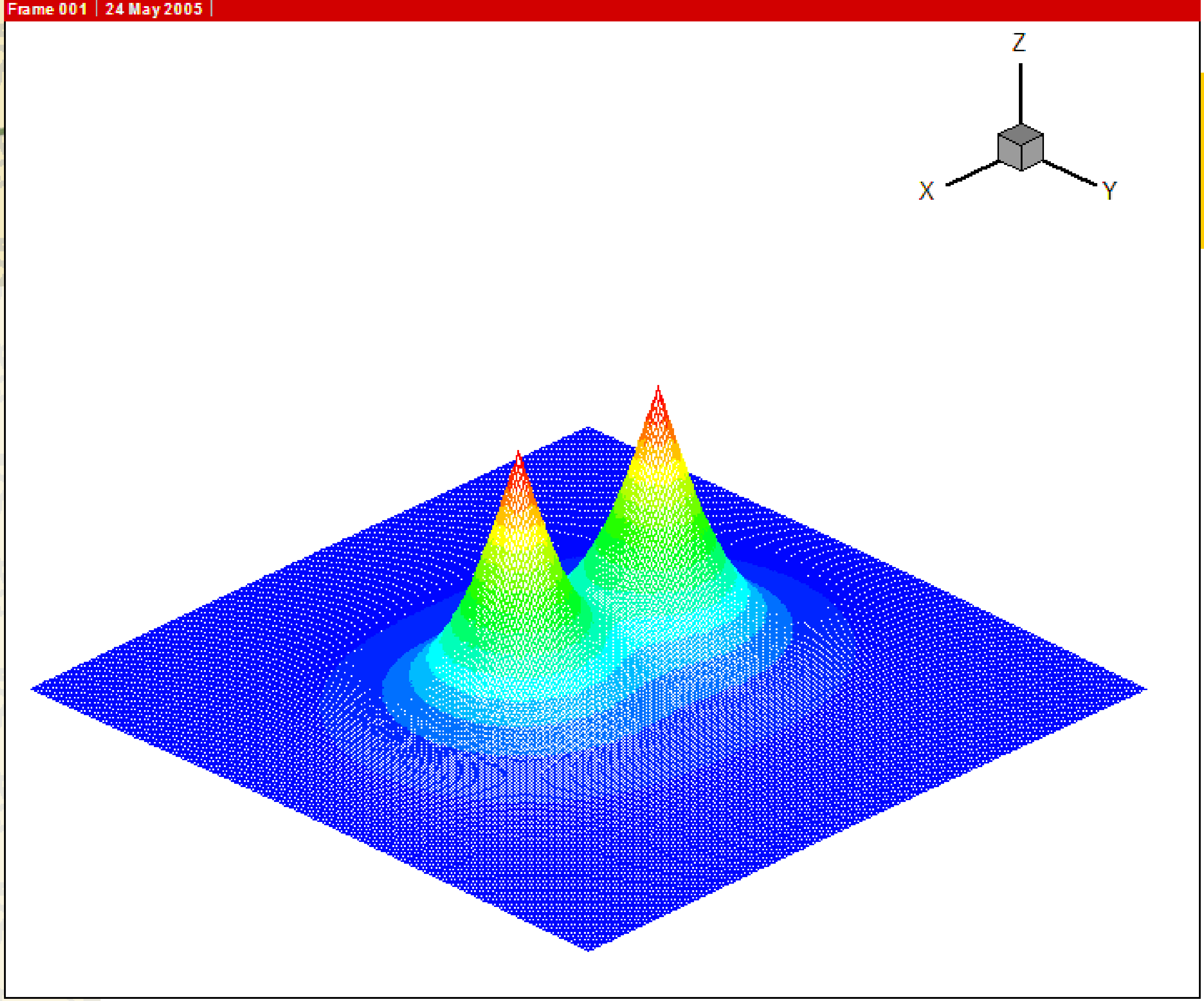

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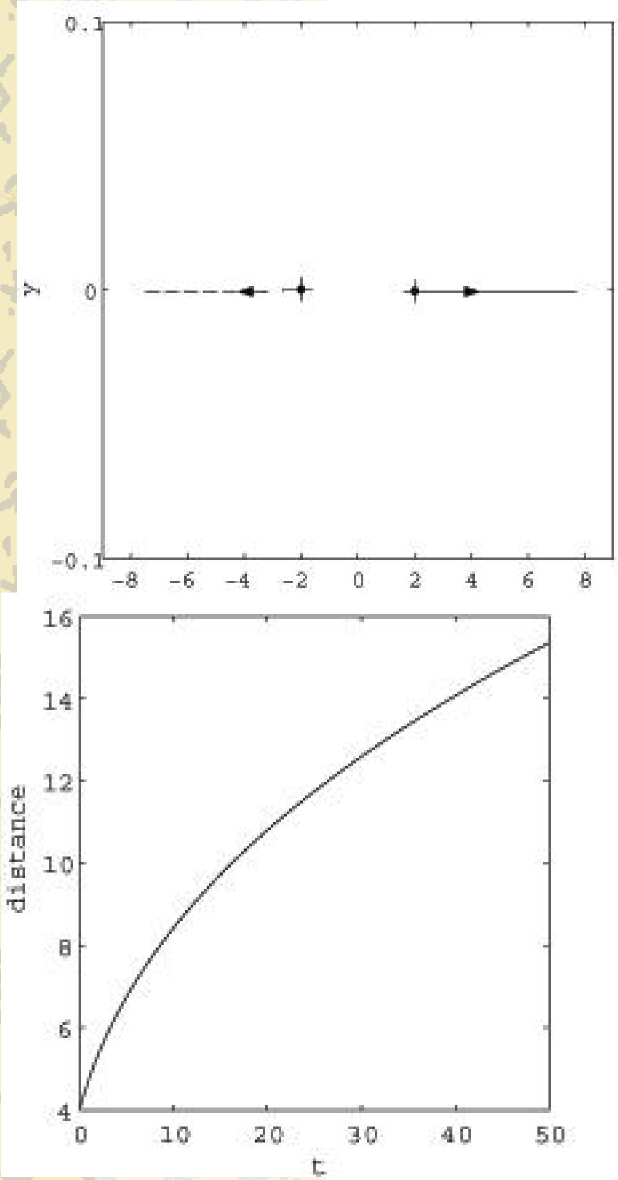
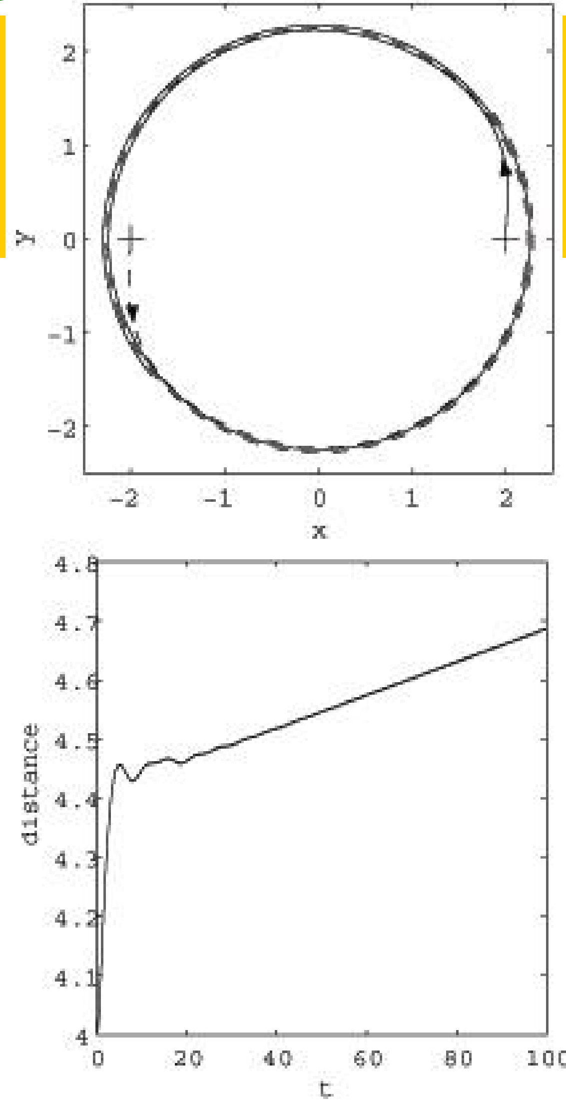
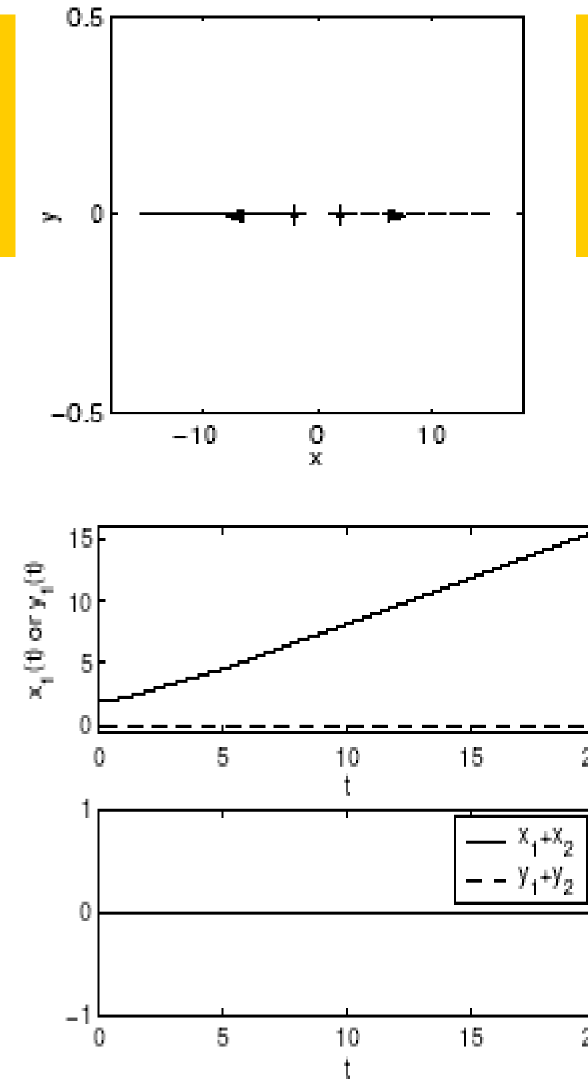


GLE



GPE



NLWE

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Conclusion of interaction

- For GLE (Bao, Du & Zhang, SIAP, 07')
  - Undergo a repulsive interaction
  - Core size of the two vortices will well separated
  - Energy decreasing: $E(\psi(\vec{x},t)) \to 2E(\psi_1(\vec{x})), \quad t \to \infty$
- For NLSE (Bao, Du & Zhang, SIAP, 07')
  - Behave like point vortices in ideal fluid
  - The two vortex centers move almost along a circle
  - Energy conservation & radiation back
- For NLWE (Bao & Zhang, 07')
  - Similar as GLE but with different speed

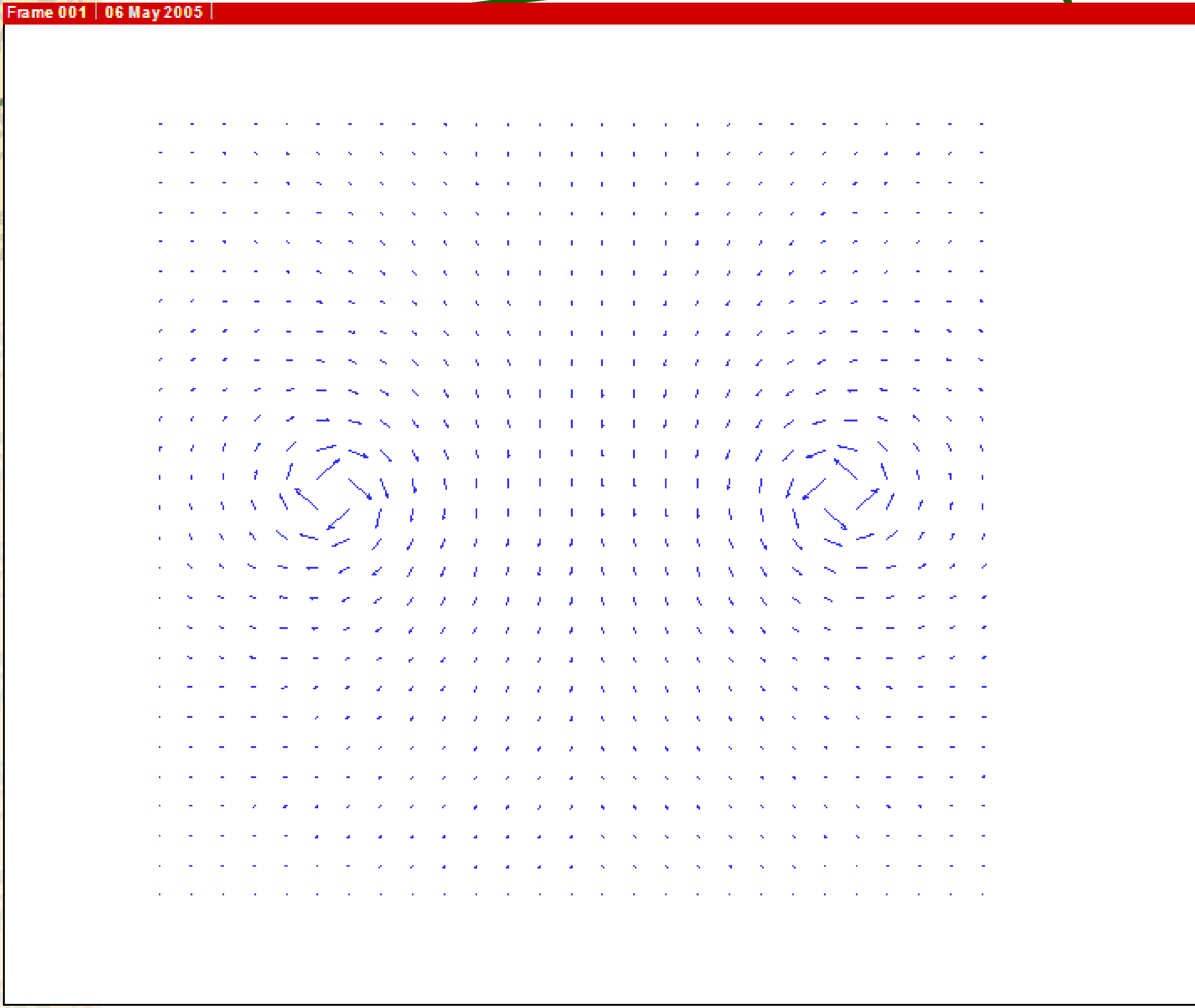
Vortex dynamics & interaction

- Two vortices with opposite winding numbers

$$N=2, \qquad n_1=1, \qquad n_2=-1, \qquad \vec{x}_1=(a,0), \qquad \vec{x}_2=(-a,0)$$

  - For GLE: velocity density
  - For NLSE:
    - Case 1: velocity density
    - Case 2: velocity density
  - Trajectory
- Conclusion

next



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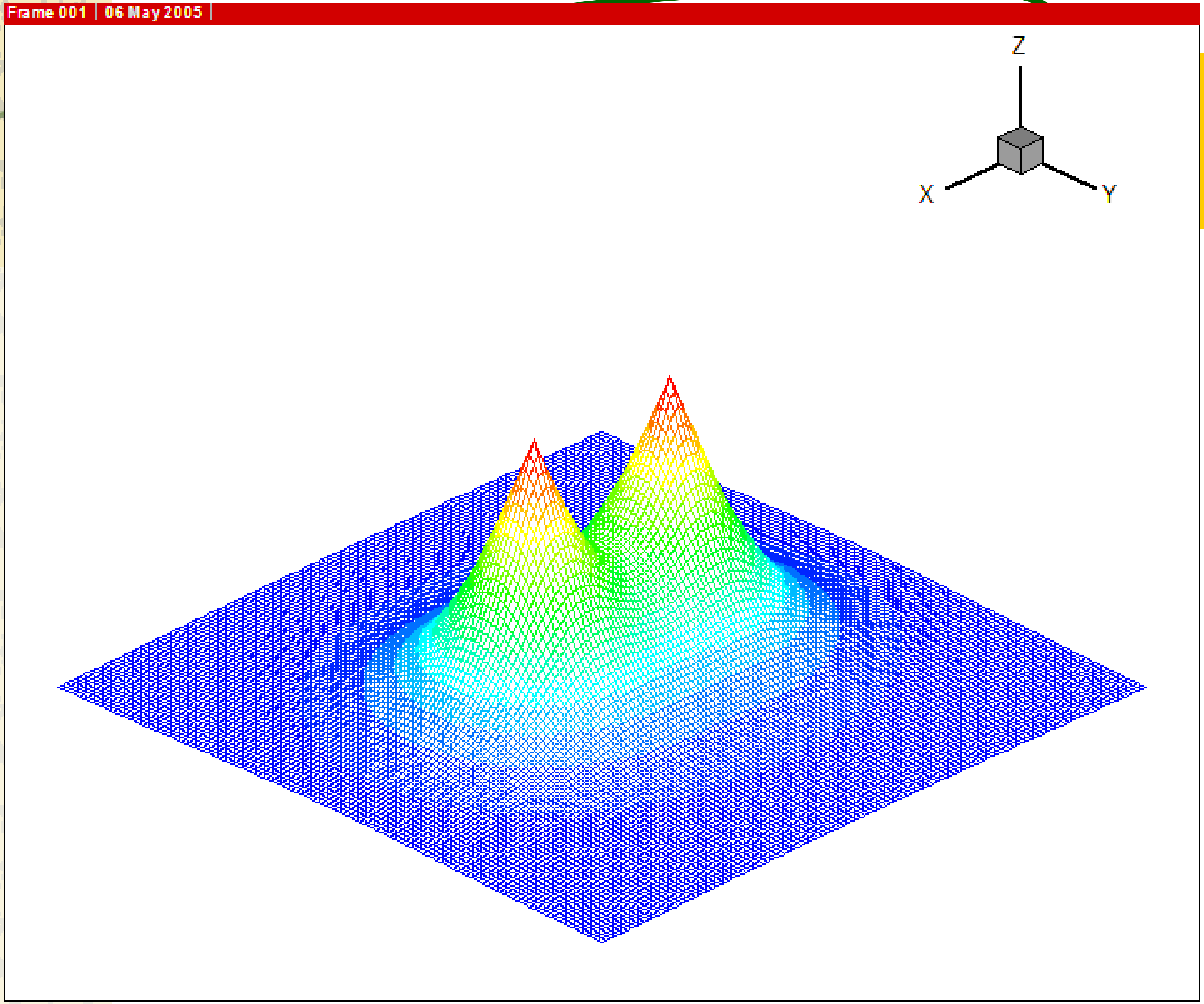

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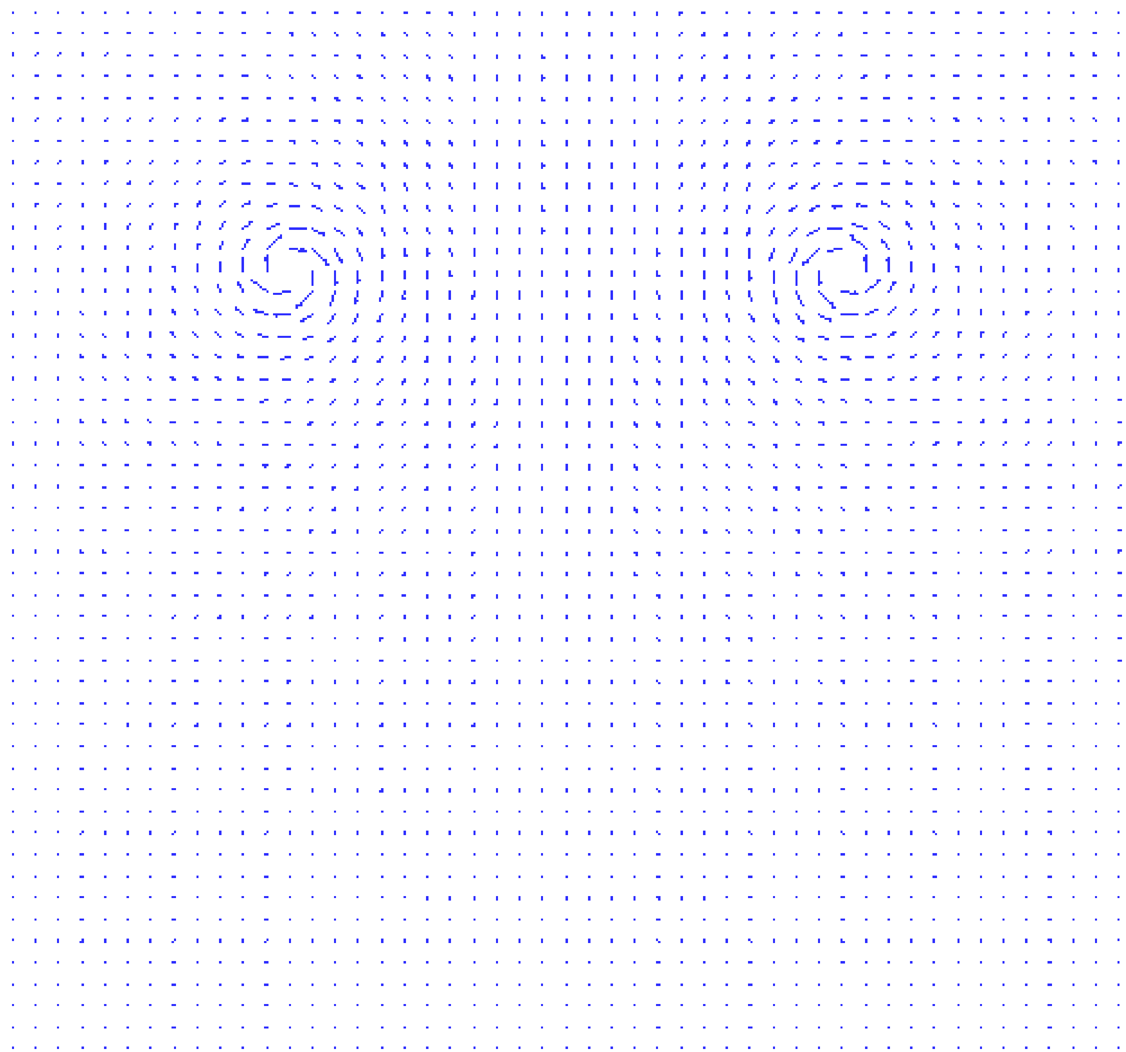

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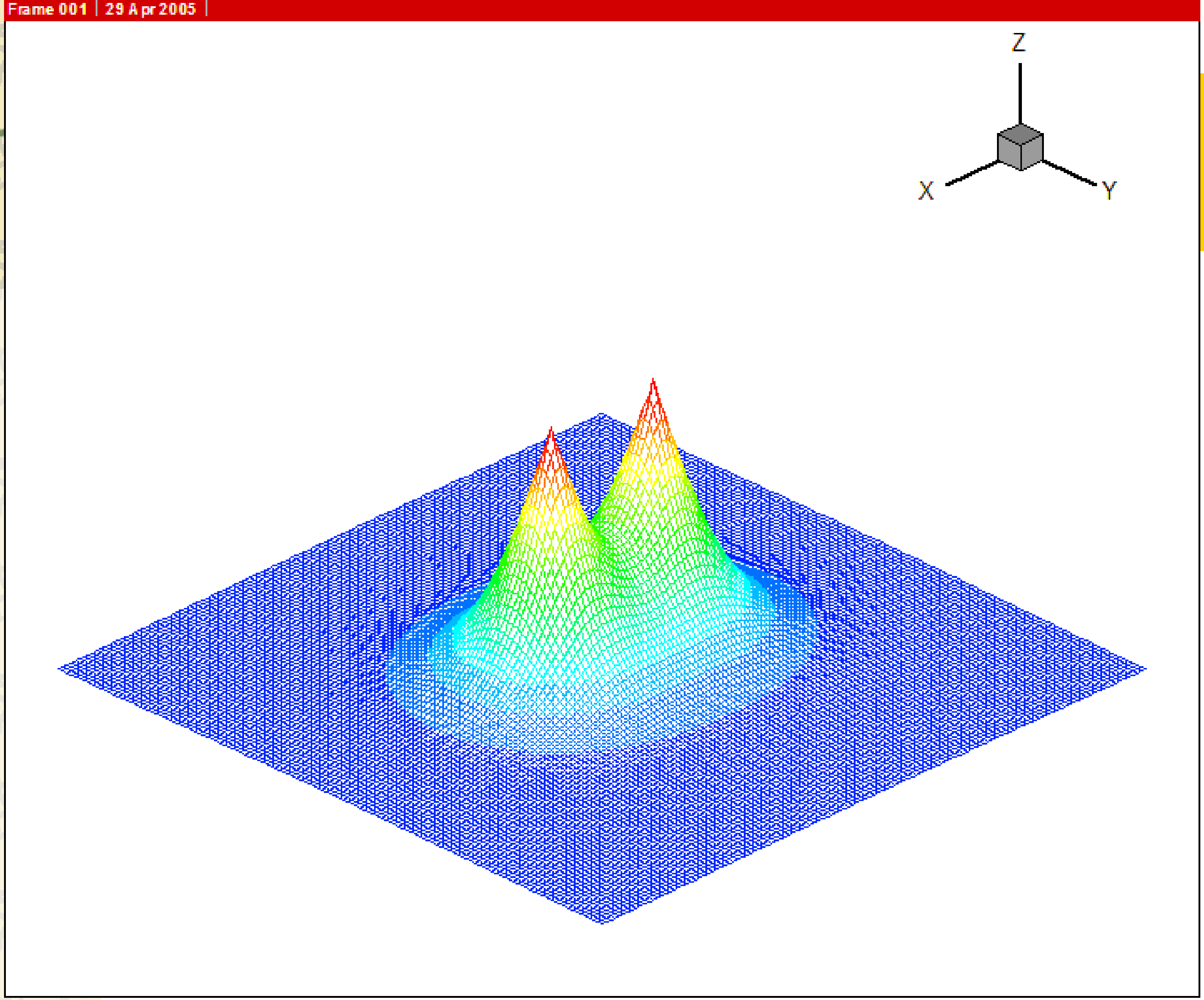

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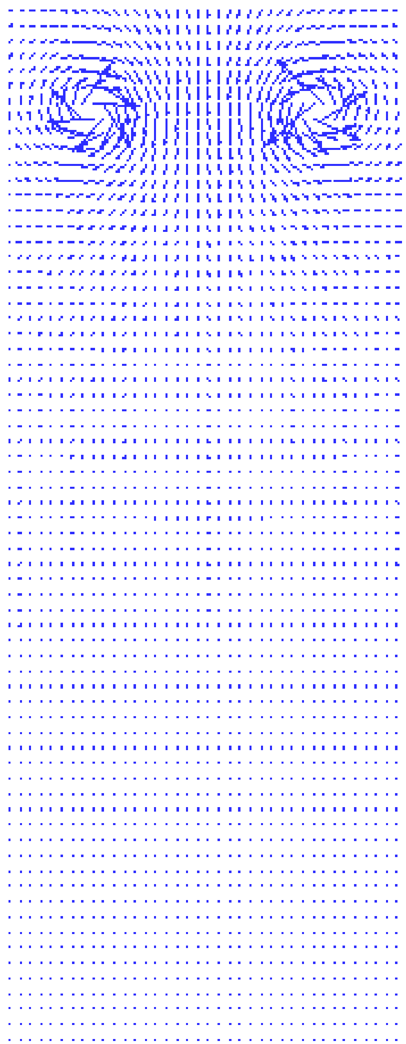

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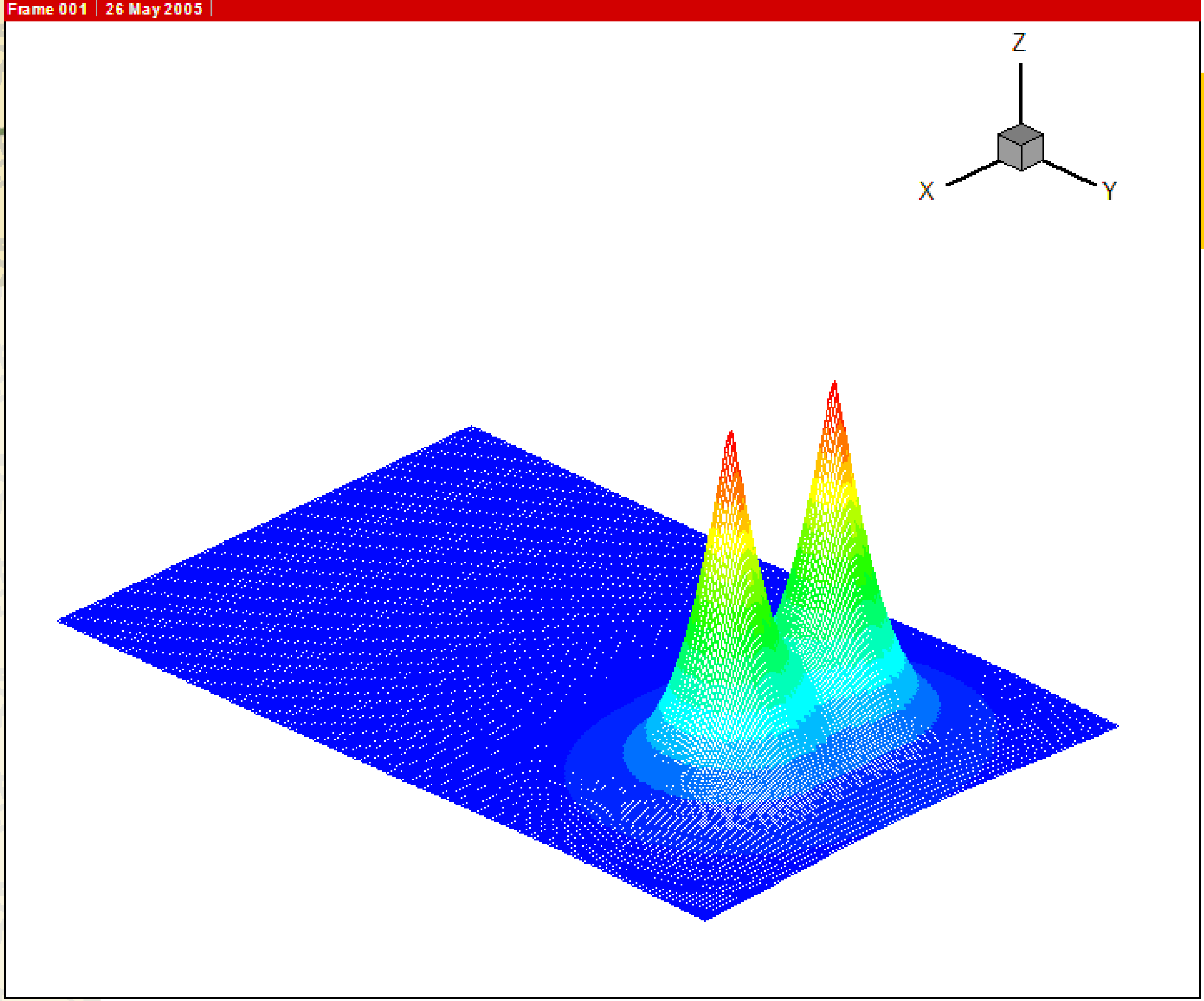

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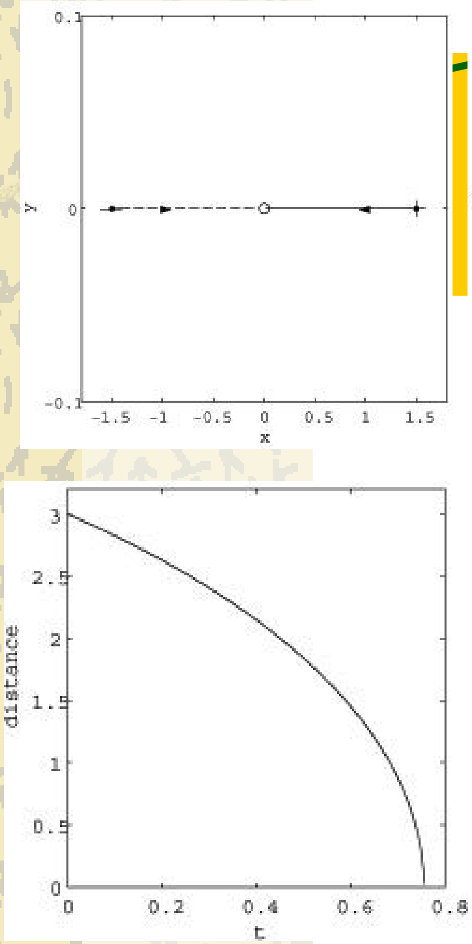

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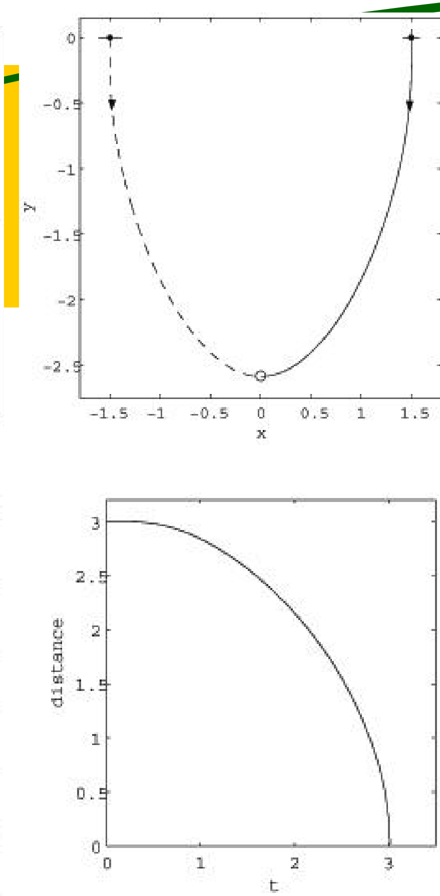
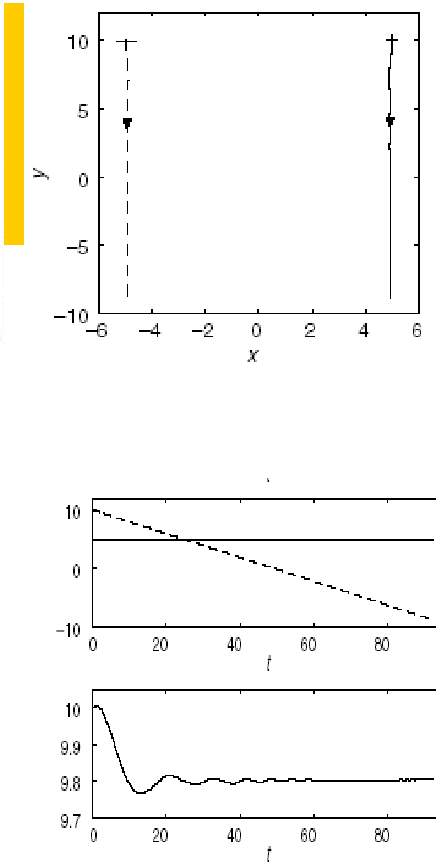
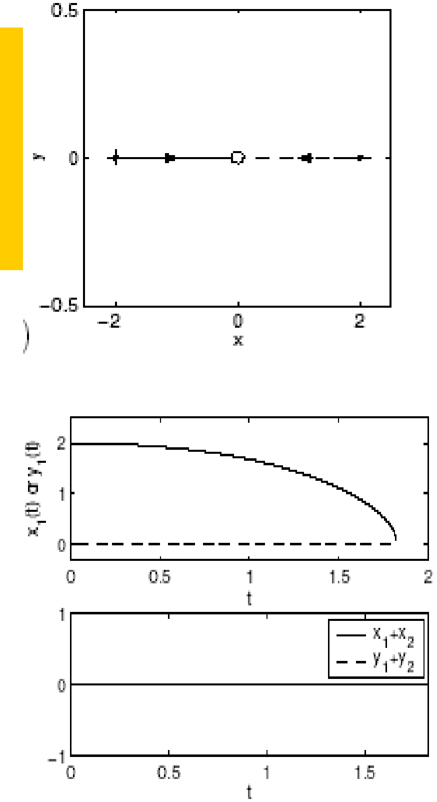


NLSE



NLSE

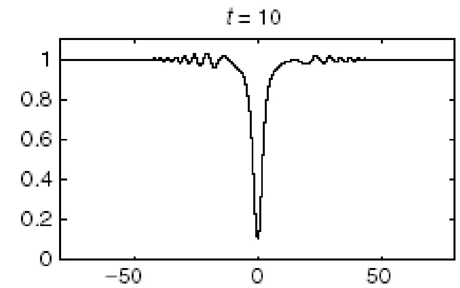


NLWE

back

Conclusion of interaction

- For GLE (Bao, Du & Zhang, SIAP, 07')
  - Undergo an attractive interaction & merge at finite time
  - Energy decreasing: $E(\psi(\vec{x},t)) \to 0, \quad t \to \infty$



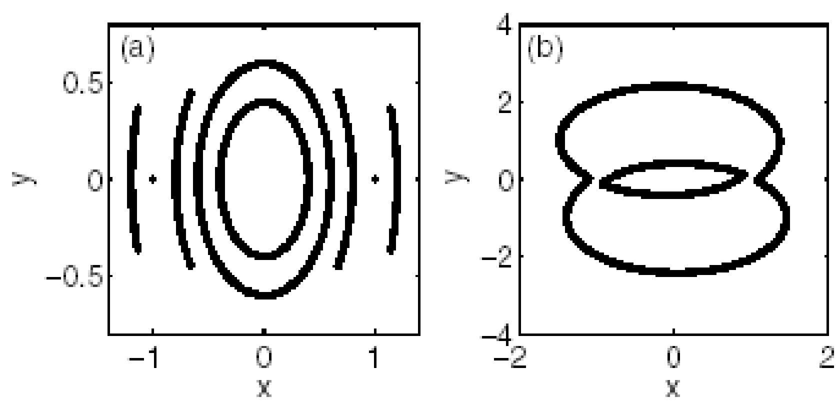
- For GPE (Bao, Du & Zhang, SIAP, 07')
  - depends on initial distance of the two centers
    - Small: merge at finite time & generate shock wave
    - Large: move almost paralleling & don't merge, solitary wave
  - Energy conservation & radiation
  - Sound wave generation

back

- For NLWE (Bao & Zhang, 07')
  - Undergo an attractive interaction & merge at finite time

Reduced dynamic laws

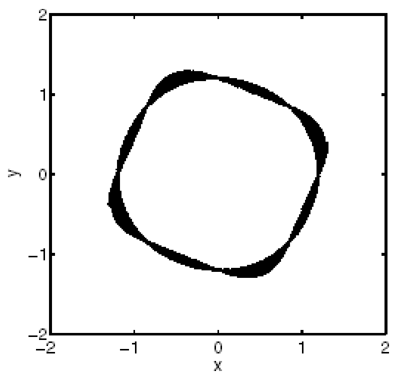
- For `dark-tail' vortex in BEC (Klein, Jaksch, Zhang & Bao, PRA, 07')
  - For linear with harmonic potential
    - N=1: $x_v(t)=x_1\cos(t)$ $\quad y_v(t)=x_1\sin(t).$
    - Vortex pair: $x_{vp,j}(t)=x_j\cos(t)-y_j\sin(t)$ $\quad y_{vp,j}(t)=x_j\sin(t)+y_j\cos(t),$
    - Vortex dipole: $y_{vd}(t)=\sin(t)(x_0^2-1)/x_0$ $\quad x_{vd}(t)=\pm\sqrt{x_0^2-y_{vd}^2(t)}.$

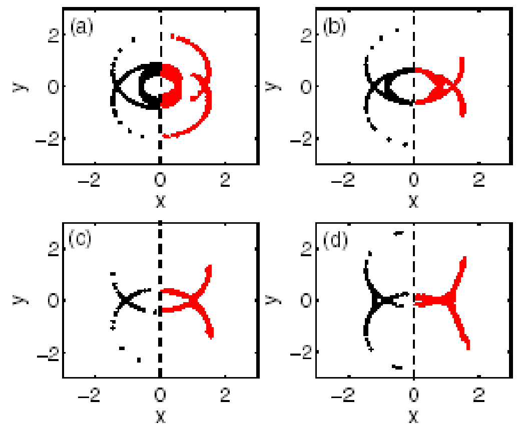


Reduced dynamic laws

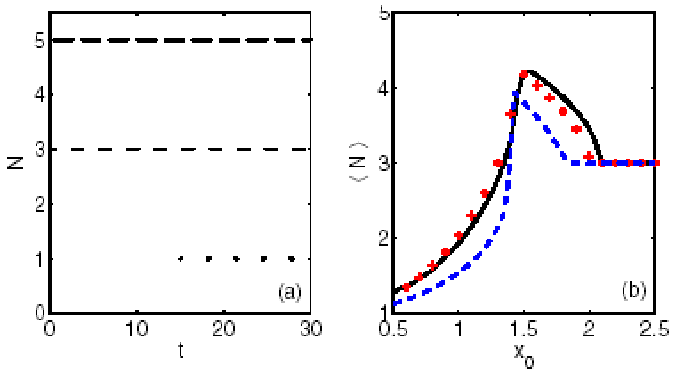
– For weak nonlinear case

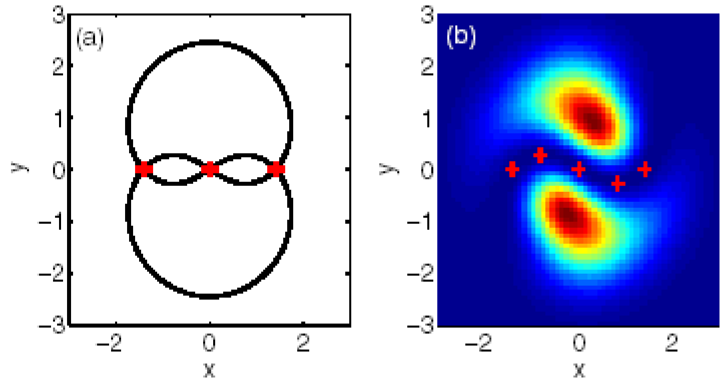
- Vortex pair





- Vortex dipole





– For strong interaction: ???

Conclusions

- Analytical results for reduced dynamical laws
  - Mass center is conserved in GLE
  - Signed mass center is conserved in GPE
  - Solve analytically for a few types initial data
- Numerical results
  - Study numerically vortex dynamics in GLE, GPE, NLWE & NLSE
    - Stability of a vortex with different winding number
    - Interaction of vortex pair, vortex dipole, vortex tripole, ….