Higher order basis functions for FEM

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We consider, for semplicity, the homogeneous Dirichlet problem.

1 One-dimensional case

In the one dimensional case Ω is an open interval and $X = H_0^1(\Omega)$. We just consider the space $X_h^2 = \{v_h \in X : v_h|_{T_h} \in \mathbb{P}_2(T_h)\}$. A polynomial of degree two on a interval is defined by three points, usually the two extreme points and the middle point. Therefore, given an original set of nodes $\{y_j\}_{j=1}^m \subset \Omega$, we have to consider the new set of nodes $\{x_i\}_{i=1}^{2m-1} \subset \Omega$ given by

$$\begin{cases} x_i = y_{(i+1)/2}, & i \text{ odd} \\ x_i = \dfrac{y_{i/2} + y_{i/2+1}}{2}, & i \text{ even} \end{cases}$$

and the set of basis functions

$$\varphi_i(x) \in X_h^2,\ \varphi_i(x_j) = \delta_{ij}, \quad 1 \le i, j \le 2m-1$$

On the element ℓ_j, with endpoints $\ell_{j,1}$ and $\ell_{j,3}$ and middle point $\ell_{j,2}$, the form of $\varphi_{\ell_{j,k}}$ is

$$\varphi_{\ell_{j,1}}(x) = \frac{\begin{vmatrix} 1 & 1 \\ x & x_{\ell_{j,2}} \end{vmatrix} \cdot \begin{vmatrix} 1 & 1 \\ x & x_{\ell_{j,3}} \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ x_{\ell_{j,1}} & x_{\ell_{j,2}} \end{vmatrix} \cdot \begin{vmatrix} 1 & 1 \\ x_{\ell_{j,1}} & x_{\ell_{j,3}} \end{vmatrix}}$$
$$\varphi_{\ell_{j,2}}(x) = \frac{\begin{vmatrix} 1 & 1 \\ x_{\ell_{j,1}} & x \end{vmatrix} \cdot \begin{vmatrix} 1 & 1 \\ x & x_{\ell_{j,3}} \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ x_{\ell_{j,1}} & x_{\ell_{j,2}} \end{vmatrix} \cdot \begin{vmatrix} 1 & 1 \\ x_{\ell_{j,2}} & x_{\ell_{j,3}} \end{vmatrix}}$$
$$\varphi_{\ell_{j,3}}(x) = \frac{\begin{vmatrix} 1 & 1 \\ x_{\ell_{j,1}} & x \end{vmatrix} \cdot \begin{vmatrix} 1 & 1 \\ x_{\ell_{j,2}} & x \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ x_{\ell_{j,1}} & x_{\ell_{j,3}} \end{vmatrix} \cdot \begin{vmatrix} 1 & 1 \\ x_{\ell_{j,2}} & x_{\ell_{j,3}} \end{vmatrix}}$$

Clearly now the basis function φ_i shares its support with $\varphi_{i-2}, \varphi_{i-1}, \varphi_{i+1}, \varphi_{i+2}$ and therefore the stiffness matrix, for instance, is a pentadiagonal matrix.

1.1 Error estimates

The weak formulation is

$$\text{find } u \in H^1(\Omega) \text{ such that } a(u, v) = \ell(v),\ \forall v \in H^1(\Omega)$$

with a SPD, bilinear, coercive, continuos and ℓ linear bounded. Therefore we assume that $u \in H^1(\Omega)$. Let us denote the generic triangle (edge) by K and its length by h_K. The maximum length of the triangles is h.

1.1.1 H^1 norm, X_h^r space

Let be $u_h \in X_h^r$. Then:

- if $u \in H^{p+1}(\Omega, \mathcal{T}_h)$ (u "piecewise regular") and $s = \min\{p, r\}$

$$\|u_h - u\|_{H^1(\Omega)} \leq C \sum_{K \in \mathcal{T}_h} \left(h_K^{2s} |u|^2_{H^{s+1}(K)} \right)^{1/2} \leq Ch^s |u|_{H^{s+1}(\Omega, \mathcal{T}_h)}$$

- if $u \in H^{p+1}(\Omega)$ (u "regular" and therefore "piecewise regular") and $s = \min\{p, r\}$

$$\|u_h - u\|_{H^1(\Omega)} \leq C \sum_{K \in \mathcal{T}_h} \left(h_K^{2s} |u|^2_{H^{s+1}(K)} \right)^{1/2} \leq Ch^s |u|_{H^{s+1}(\Omega)}$$

Of course, the seminorms on the right end sides can be overestimated by the corresponding norms.

1.1.2 L^2 norm, X_h^r space

Let be $u_h \in X_h^r$. If from $\ell(v) = \ell_f(v) = \int_\Omega fv$ (therefore $f \in L^2(\Omega)$) it follows that $u \in H^2(\Omega)$ (it is called *elliptic regularity*, for instance, Poisson problem), then

- if $u \in H^{p+1}(\Omega, \mathcal{T}_h)$ and $s = \min\{p, r\}$

$$\|u_h - u\|_{L^2(\Omega)} \leq Ch^{s+1}|u|_{H^{s+1}(\Omega,\mathcal{T}_h)}$$

- if $u \in H^{p+1}(\Omega)$ and $s = \min\{p, r\}$

$$\|u_h - u\|_{L^2(\Omega)} \leq Ch^{s+1}|u|_{H^{s+1}(\Omega)}$$

Of course, the seminorms on the right end sides can be overestimated by the corresponding norms.

2 Two-dimensional case

In the two-dimensional case Ω is a polygon and $X = H_0^1(\Omega)$. We just consider the space $X_h^2 = \{v_h \in X \cap \mathcal{C}^0(\bar{\Omega})\colon v_h|_{T_h} \in \mathbb{P}_2(T_h)\}$. A polynomial of degree two on a triangle is defined by six points in general position. Usually the three vertices and the three middle points of the edges are taken. We introduce the *barycentric coordinates:* any point x in a triangle ℓ_j with vertices $\{x_1, x_2, x_3\} \in \Omega$ can be written in a unique way as

$$x = \lambda_1(x)x_1 + \lambda_2(x)x_2 + \lambda_3(x)x_3, \quad \lambda_1(x) + \lambda_2(x) + \lambda_3(x) \equiv 1$$

We have that $\lambda_k(x)$ coincides, on the triangle, with the piecewise linear function $\varphi_{\ell_{j,k}}(x)$.

Proposition. *Given three non-collinear points $x_1, x_2, x_3 \in \Omega$ and the corresponding middle points x_{12}, x_{13}, x_{23}, a polynomial $p(x)$ of total degree two is well defined by the values of $p(x)$ at the six points.*

Proof. It is enough to prove that if $p(x_1) = p(x_2) = p(x_3) = p(x_{12}) = p(x_{13}) = p(x_{23}) = 0$, than $p \equiv 0$. Along the edge x_2x_3 p is a quadratic polynomial in one variable which is zero at three points. Therefore it is zero on the whole edge and we can write $p(x) = \lambda_1(x)w_1(x)$ with $w_1(x) \in$

$\mathbb{P}_1$. In the same way p is zero along the edge x_1x_3 and therefore $p(x) = \lambda_1(x)\lambda_2(x)w_0(x)$ with $w_0(x) = \gamma \in \mathbb{P}_0$. If we now take the point x_{12}, we have

$$0 = p(x_{12}) = \lambda_1(x_{12})\lambda_2(x_{12})\gamma = \frac{1}{2}\frac{1}{2}\gamma$$

and therefore $\gamma = 0$. $\square$

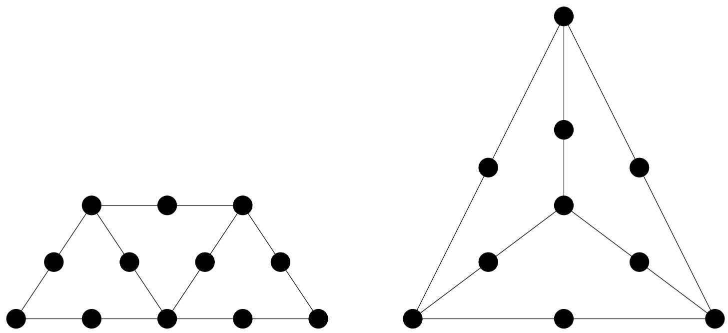

Figure 1: $m = 5$, $n = 3$ (right) and $m = 4$, $n = 3$ (left).

Given the number m of original nodes and the number n of triangles, by Euler's formula we have that the number of edges is $m+(n+1)-2 = m+n-1$ (in Euler's formula it has to be counted also the unbounded region outside the triangularion). Therefore, the dimension of X_h^2 is $m+(m+n-1) = 2m+n-1$.

It is not possible, as well, to know a priori the structure of the stiffness matrix.

2.1 Bandwidth reduction

Even in the simplest case of piecewise linear basis function, an ordering of the nodes as in Figure 2 (left) would yield a sparsity pattern as in Figure 2 (right). The *degree* of a node is the number of adjacent to it. We can consider the following heuristic algorithm, called *Cuthill–McKee* reordering

- Select a node i and set the first element of the array R to i.
- Put the adjacent nodes of i in the increasing order of their degree in the array Q.
- DO UNTIL Q is empty

  Take the first node in Q: if it is already in R, delete it, otherwise add it to R, delete it from Q and add to Q the adjacent nodes of it which are not already in R, in the increasing order of their degree,

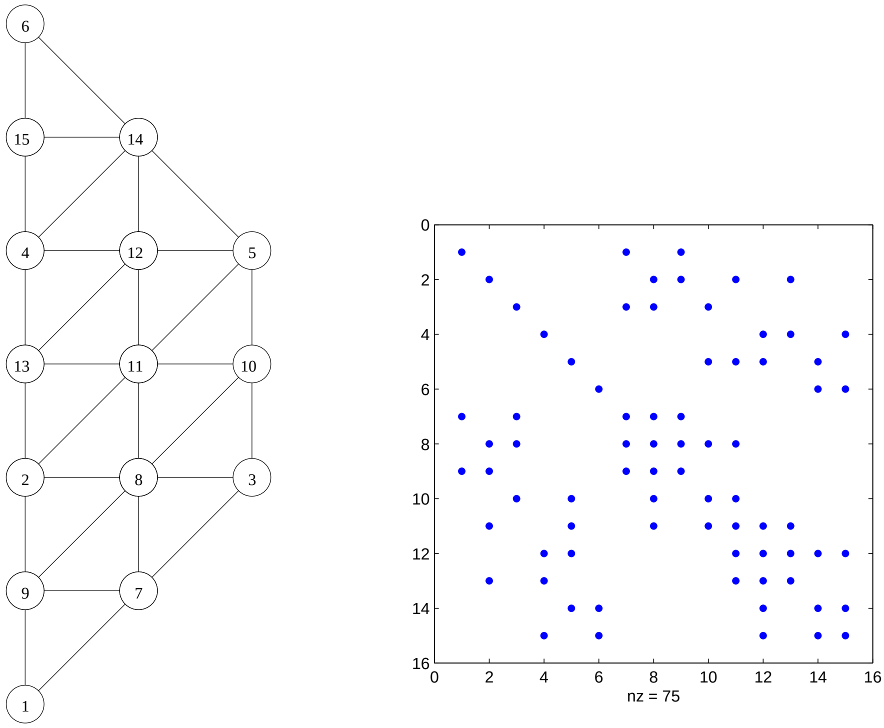

Figure 2: Unordered mesh and corresponding sparsity pattern.

The new label of node $R(j)$ is j. A variant is the so called *reverse Cuthill–McKee ordering*, in which the final ordering produced by the previous algorithm is reversed. The ordering produced by the reverse Cuthill–McKee algorithm with initial node 1 (a node with smallest degree) is shown in Figure 3.

2.2 Error estimates

The weak formulation is

$$\text{find } u \in H^1(\Omega) \text{ such that } a(u,v) = \ell(v),\, \forall v \in H^1(\Omega)$$

with a SPD, bilinear, coercive, continuos and ℓ linear bounded. Therefore we assume that $u \in H^1(\Omega)$. Let us denote the generic triangle by K and its diameter by h_K. The maximum diameter of the triangles is h.

2.2.1 H^1 norm, X_h^r space

Let be $\{\mathcal{T}_h\}_h$ a family of regular triangulations and $u_h \in X_h^r$. Then Let be $u_h \in X_h^r$. Then:

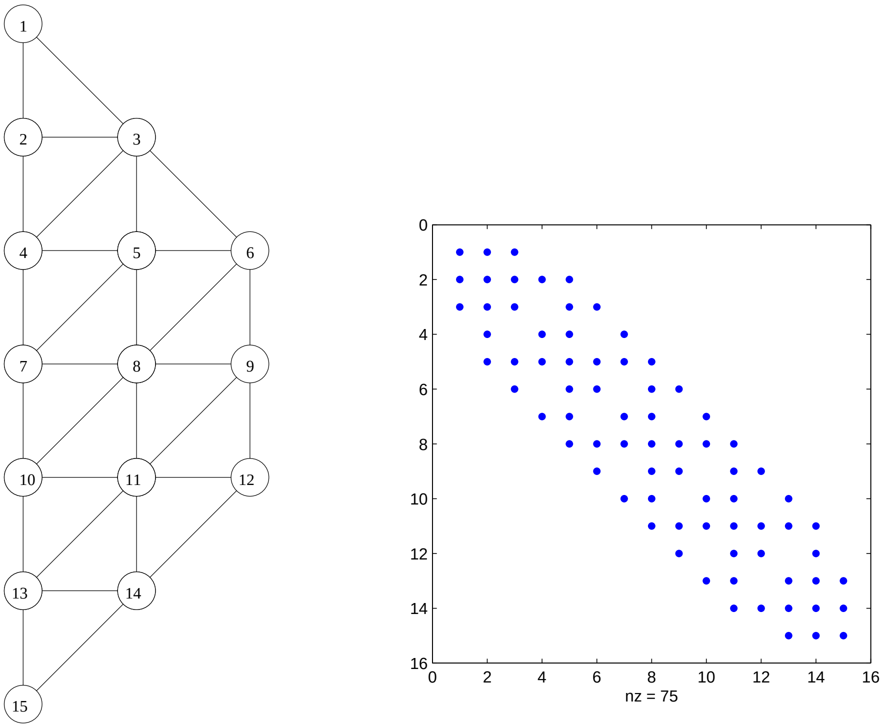

Figure 3: Reverse Cuthill–McKee ordered mesh and corresponding sparsity pattern.

- if $u \in H^{p+1}(\Omega, \mathcal{T}_h)$ (u "piecewise regular") and $s = \min\{p, r\}$

$$\|u_h - u\|_{H^1(\Omega)} \leq C \sum_{K \in \mathcal{T}_h} \left(h_K^{2s} |u|^2_{H^{s+1}(K)} \right)^{1/2} \leq C h^s |u|_{H^{s+1}(\Omega, \mathcal{T}_h)}$$

- if $u \in H^{p+1}(\Omega)$ (u "regular" and therefore "piecewise regular") and $s = \min\{p, r\}$

$$\|u_h - u\|_{H^1(\Omega)} \leq C \sum_{K \in \mathcal{T}_h} \left(h_K^{2s} |u|^2_{H^{s+1}(K)} \right)^{1/2} \leq C h^s |u|_{H^{s+1}(\Omega)}$$

Of course, the seminorms on the right end sides can be overestimated by the corresponding norms.

2.2.2 L^2 norm, X_h^r space

Let be $\{\mathcal{T}_h\}_h$ a family of regular triangulations and $u_h \in X_h^r$. If from $\ell(v) = \ell_f(v) = \int_\Omega f v$ (therefore $f \in L^2(\Omega)$) and Ω convex it follows that $u \in H^2(\Omega)$ (it is called *elliptic regularity*, for instance, Poisson problem), then

- if $u \in H^{p+1}(\Omega, \mathcal{T}_h)$ and $s = \min\{p, r\}$

$$\|u_h - u\|_{L^2(\Omega)} \leq Ch^{s+1}|u|_{H^{s+1}(\Omega,\mathcal{T}_h)}$$

- if $u \in H^{p+1}(\Omega)$ and $s = \min\{p, r\}$

$$\|u_h - u\|_{L^2(\Omega)} \leq Ch^{s+1}|u|_{H^{s+1}(\Omega)}$$

Of course, the seminorms on the right end sides can be overestimated by the corresponding norms.