Topics in Model-Based Reasoning

Towards Integration of Proving and Solving

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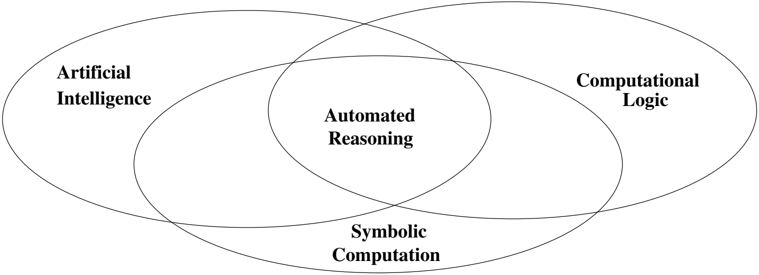
Introduction: Towards model-based reasoning

I part: A classic from the literature: DPLL-CDCL

II part: Solver + prover in DPLL($\Gamma+\mathcal{T}$)

III part: Discussion of current trends in the field

Automated reasoning



- In AI we work with symbols: automated reasoning is symbolic reasoning
- Symbolic reasoning: Logico-deductive, Probabilistic ...

The gist of this lecture I

- Automated reasoning from proofs to models
- Models are relevant to applications, e.g.:
  - Program verification: a program state is a model
  - Program testing: model as "mole" in automated test generation
  - Program synthesis: model as example in example-driven synthesis

The gist of this lecture II

- Proofs mean proving
- Models mean solving
- Towards integrations of proving and solving

Symbolic reasoning: Proving

- Validity: $\mathcal{T} \models \varphi$
- Refutationally: $\mathcal{T} \cup \{\neg\varphi\}$ unsatisfiable
- Inference: $\mathcal{T} \cup \{\neg\varphi\} \vdash \bot$ (Success!)
- If not: $\mathcal{T}$-model of $\neg\varphi$, counter-example for φ

Symbolic reasoning: Solving

- Satisfiability: is there a $\mathcal{T}$-model of φ?
- Search: solution found (Success!)
- If not: $\mathcal{T} \cup \{\varphi\}$ unsatisfiable, $\mathcal{T} \models \neg\varphi$

Theorem proving strategies (Semi-decision procedures)

- First-order logic with equality
- Unsatisfiability is semi-decidable, satisfiability is not
- Search for proof (refutation)
- Models for semantic guidance,e.g.:
  - Hyper-resolution
  - Set of support
  - Semantic resolution

Algorithmic reasoning (Decision procedures)

- Satisfiability decidable: Symmetry restored
- Propositional logic
- Decidable (fragments of) first-order theories, e.g.:
  - QFF: equality, recursive data structures (e.g., lists), arrays
  - Linear arithmetic (integers, rationals), arithmetic (algebraic reals)

Towards integration I

- Integrating solvers (e.g., arithmetic) into first-order reasoners

Towards integration II

Integration in the reasoner's operations:

- Deduction guides search for model
- Candidate partial model guides deduction
- How?

I part: A classic from the literature: DPLL-CDCL

- DPLL: The Davis-Putnam-Logemann-Loveland procedure with
- Conflict driven clause learning (DPLL-CDCL), or
- How the integration of search and inference in propositional logic brought Boolean satisfiability from theoretical hardness to practical success

Propositional logic (SAT)

- Davis-Putnam-Logemann-Loveland (DPLL) procedure
- Decision procedure:
  model found: return sat;
  failure: return unsat
- Backtracking search for model

DPLL

- Build candidate model M
- State of derivation: $M \parallel F$
  M: sequence of truth assignments
  F: clauses to satisfy
- Depth-first search with backtracking

DPLL-CDCL I

State of derivation: $M \parallel F$

- Decide: guess L is true, add it to M (decided literal)
- UnitPropagate: propagate consequences of assignment (implied literals)
- Conflict: detect $L_1 \vee \ldots \vee L_n$ all false
- Unsat: conflict clause is $\Box$ (nothing else to try)
- Sat: all variables assigned

DPLL-CDCL II

State of derivation: $M \parallel F$

- Explain: unfold implied literals in conflict clause by resolution
- Learn conflict clause $C \vee L$
- Backjump: when only L assigned at current decision level, jump back to least recent level where C false and L unassigned, undo at least one decision, make L true (implied by $C \vee L$)

Conflict-Driven Clause Learning (CDCL)

- Conflict: M falsifies clause $L_1 \vee \ldots \vee L_n$: conflict clause
- Explain: resolve and get another conflict clause
  $L_1 \vee \ldots \vee L_n$
  $\neg L_1 \vee Q_2 \ldots \vee Q_k$
- Learn: may add resolvent(s)
- Backjump: undoes at least an assignment, jumps back as far as possible to state where learnt resolvent can be satisfied

Example of CDCL

$F = \{\neg a \vee b,\ \neg c \vee d,\ \neg e \vee \neg f,\ f \vee \neg e \vee \neg b\}$
$M = a\ b\ c\ d\ e\ \neg f$
blue: assignments; violet: propagations

Conflict: $f \vee \neg e \vee \neg b$
Explain by resolving $f \vee \neg e \vee \neg b$ and $\neg e \vee \neg f$: $\neg e \vee \neg b$
Learn $\neg e \vee \neg b$: no model with e and b true
Jump back to earliest state with $\neg b$ false and $\neg e$ unassigned:
$M = a\ b\ \neg e$

Chronological backtracking: $M = a\ b\ c\ d\ \neg e$

Decision procedures

- Davis-Putnam-Logemann-Loveland (DPLL) procedure for SAT
- $\mathcal{T}$-solver: Decision procedure for $\mathcal{T}$
  Equality: congruence closure (CC)
- DPLL($\mathcal{T}$)-based SMT-solver: Decision procedure for $\mathcal{T} = \bigcup_{i=1}^{n} \mathcal{T}_i$ with
- Combination of $\mathcal{T}_i$-sat procedures by a method called equality sharing

Satisfiability modulo theories (SMT)

- DPLL($\mathcal{T}$) procedure
- Integrate $\mathcal{T}$-satisfiability procedure in DPLL
- Ground first-order literals abstracted to propositional variables
- CDCL: same

DPLL($\mathcal{T}$)

State of derivation: $M \parallel F$

- $\mathcal{T}$-Propagate: add to M an L that is $\mathcal{T}$-consequence of M
- $\mathcal{T}$-Conflict: detect that $L_1, \ldots, L_n$ in M are $\mathcal{T}$-inconsistent

Theory combination by equality sharing I

- Theories $\mathcal{T}_1, \ldots, \mathcal{T}_n$
- $\mathcal{T} = \bigcup_{i=1}^{n} \mathcal{T}_i$
- $\mathcal{T}_i$-satisfiability procedures
- Disjoint: share only $\simeq$ and uninterpreted constants
- Mixed terms separated by introducing new constants
- Need to agree on:
  - Shared constants
  - Cardinalities of shared sorts

Theory combination by equality sharing II

- Compute arrangement: which shared constants are equal and which are not
- $\mathcal{T}_i$-solvers generate and propagate all entailed (disjunctions of) equalities between shared constants
- For cardinalities: assume stably infinite: every $\mathcal{T}_i$-sat ground formula has $\mathcal{T}_i$-model with infinite cardinality

Model-based theory combination (MBTC)

- Assume $\mathcal{T}_i$-satisfiability procedure that builds a $\mathcal{T}_i$-model (e.g., linear arithmetic)
- Optimistic approach: propagate equalities true in $\mathcal{T}_i$-model
- If not entailed: conflict + backjumping with CDCL + update $\mathcal{T}_i$-model
- Rationale: few equalities matter in practice

II part: Solver + prover in DPLL($\Gamma+\mathcal{T}$)

- Γ: first-order inference system
- DPLL($\mathcal{T}$): SMT-solver with DPLL-CDCL and equality sharing
- A tight integration: the DPLL($\Gamma+\mathcal{T}$) method

Motivation

- Decision procedures are most desirable, but ...
- Formulæ from SW verification tools (verifying compiler, static analyzer, test generator, synthesizer, model checker) use quantifiers to write
  - invariants
  - axioms of theories without decision procedure
- Need for generic first-order inferences

Shape of problem

- Background theory $\mathcal{T}$
  - $\mathcal{T} = \bigcup_{i=1}^{n} \mathcal{T}_i$ (linear arithmetic, data structures)
- Set of formulæ: $\mathcal{R} \cup P$
  - $\mathcal{R}$: set of non-ground clauses without $\mathcal{T}$-symbols
  - P: large ground formula (set of ground clauses) typically with $\mathcal{T}$-symbols
- Determine whether $\mathcal{R} \cup P$ is satisfiable modulo $\mathcal{T}$

Superposition-based inference system Γ

- FOL+= clauses with universally quantified variables
- Axiomatized theories
- Deduce clauses from clauses (expansion)
- Remove redundant clauses (contraction)
- Well-founded ordering $\succ$ on terms and literals to restrict expansion and define contraction
- Semi-decision procedure
- No backtracking

Inference system Γ

State of derivation: set of clauses F

- Resolution
- Superposition/Paramodulation: resolution with equality built-in
- Simplification: by well-founded rewriting
- Subsumption: eliminate less general clauses
- Other rules: e.g., Factoring rules, Deletion of trivial clauses

Theorem-proving strategy as decision procedure

- Termination results by analysis of inferences: Γ is $\mathcal{T}$-satisfiability procedure
- Covered theories include: lists, arrays and records with or without extensionality, recursive data structures

Also for combination of theories

- If Γ terminates on $\mathcal{R}_i$-sat problems, it terminates also on $\mathcal{R}$-sat problems for $\mathcal{R} = \bigcup_{i=1}^{n} \mathcal{R}_i$, if the $\mathcal{R}_i$'s are disjoint and variable-inactive
- Variable-inactivity: no maximal literals of the form $t \simeq x$ where $x \notin Var(t)$ (no paramodulation from variables)
- The only inferences across theories are paramodulations from shared constants (correspond to equalities between shared constants in equality sharing)

Variable inactivity implies stable infiniteness

- If $\mathcal{R}$ is variable-inactive, then it is stably infinite
- Γ reveals lack of stable infiniteness by generating a cardinality constraint (e.g., $y \simeq x \vee y \simeq z$) which is not variable-inactive

Recap on first-order inference systems

- Resolution/superposition-based engines good for reasoning on formulæ with quantified variables: automated instantiation
- Not for large non-Horn clauses
- Not for theories such as linear arithmetic or bit-vectors
- Unexpected: they are satisfiability-procedures for theories such as lists, arrays, records and their combinations

DPLL($\Gamma+\mathcal{T}$): integrate Γ in DPLL($\mathcal{T}$) I

- Model-based deduction:
  literals in M can be premises of Γ-inferences
- Stored as hypotheses in inferred clause
- Hypothetical clause: $(L_1 \wedge \ldots \wedge L_n) \triangleright (L'_1 \vee \ldots L'_m)$
  interpreted as $\neg L_1 \vee \ldots \vee \neg L_n \vee L'_1 \vee \ldots \vee L'_m$

DPLL($\Gamma+\mathcal{T}$): integrate Γ in DPLL($\mathcal{T}$) II

- Inferred clauses inherit hypotheses from premises
- Backjump: remove hypothetical clauses depending on undone assignments

DPLL($\Gamma+\mathcal{T}$): expansion inferences

- If non-ground clauses $C_1, \ldots, C_m$ and ground $\mathcal{R}$-literals $L_{m+1}, \ldots, L_n$ generate C :
  $H_1 \triangleright C_1, \ldots, H_m \triangleright C_m$ and $L_{m+1}, \ldots, L_n$ in M generate
  $H_1 \cup \ldots \cup H_m \cup \{L_{m+1}, \ldots, L_n\} \triangleright C$
- Only $\mathcal{R}$-literals: Γ-inferences ignore $\mathcal{T}$-literals
- Take ground unit $\mathcal{R}$-clauses from M as MBTC puts them there

DPLL($\Gamma+\mathcal{T}$): contraction inferences

- Don't delete clause if clauses that make it redundant gone by backjumping
  - Level of a literal in M: its decision level
  - Level of a set of literals: the maximum

DPLL($\Gamma+\mathcal{T}$): contraction inferences

- If non-ground clauses $C_1, \ldots, C_m$ and ground $\mathcal{R}$-literals $L_{m+1}, \ldots, L_n$ simplify C to C' :
  $H_1 \triangleright C_1, \ldots, H_m \triangleright C_m$ and $L_{m+1}, \ldots, L_n$ in M simplify $H \triangleright C$ to $H \cup H_1 \cup \ldots \cup H_m \cup \{L_{m+1}, \ldots, L_n\} \triangleright C'$
  - If $\mathit{level}(H) \geq \mathit{level}(H')$: delete
  - If $\mathit{level}(H) < \mathit{level}(H')$: disable
    (re-enable when backjumping $\mathit{level}(H')$)

Completeness of DPLL($\Gamma+\mathcal{T}$)

- Refutational completeness of the inference system:
  - From that of Γ, DPLL($\mathcal{T}$) and equality sharing
  - Combines both built-in and axiomatized theories
- Fairness of the search plan:
  - Depth-first search fair only for ground SMT problems;
  - Add iterative deepening on inference depth: k-bounded DPLL($\Gamma+\mathcal{T}$)

DPLL($\Gamma+\mathcal{T}$): Summary

Use each engine for what is best at:

- DPLL($\mathcal{T}$) works on ground clauses and built-in theory
- Γ works on non-ground clauses and ground unit clauses taken from M: Γ works on $\mathcal{R}$-satisfiability problem
- Γ-inferences guided by current partial model

Can DPLL($\Gamma+\mathcal{T}$) still be a decision procedure?

Problematic axioms do occur in relevant inputs:

1. $\neg(x \sqsubseteq y) \vee f(x) \sqsubseteq f(y)$ (Monotonicity)
2. $a \sqsubseteq b$ generates by resolution
3. $\{f^i(a) \sqsubseteq f^i(b)\}_{i \geq 0}$

When $f(a) \sqsubseteq f(b)$ or $f^2(a) \sqsubseteq f^2(b)$ often suffice to show satisfiability

Idea: Allow speculative inferences

1. $\neg(x \sqsubseteq y) \vee f(x) \sqsubseteq f(y)$
2. $a \sqsubseteq b$
3. $a \sqsubseteq f(c)$
4. $\neg(a \sqsubseteq c)$

1. Add $f(x) \simeq x$
2. Rewrite $a \sqsubseteq f(c)$ into $a \sqsubseteq c$ and get $\Box$: backtrack!
3. Add $f(f(x)) \simeq x$
4. $a \sqsubseteq b$ yields only $f(a) \sqsubseteq f(b)$
5. $a \sqsubseteq f(c)$ yields only $f(a) \sqsubseteq c$
6. Terminate and detect satisfiability

Speculative inferences in DPLL($\Gamma+\mathcal{T}$)

- Speculative inference: add arbitrary clause C
- To induce termination on satisfiable input
- What if it makes problem unsatisfiable?!
- Detect conflict and backjump:
  - $\lceil C \rceil$: new propositional variable (a "name" for C)
  - Add $\lceil C \rceil \triangleright C$ to clauses and $\lceil C \rceil$ to M
  - Speculative inferences are reversible

Example as done by system

1. $\neg(x \sqsubseteq y) \vee f(x) \sqsubseteq f(y)$
2. $a \sqsubseteq b$
3. $a \sqsubseteq f(c)$
4. $\neg(a \sqsubseteq c)$

1. Add $\lceil f(x) \simeq x \rceil \triangleright f(x) \simeq x$
2. Rewrite $a \sqsubseteq f(c)$ into $\lceil f(x) \simeq x \rceil \triangleright a \sqsubseteq c$
3. Generate $\lceil f(x) \simeq x \rceil \triangleright \square$; Backtrack, learn $\neg\lceil f(x) \simeq x \rceil$
4. Add $\lceil f(f(x)) \simeq x \rceil \triangleright f(f(x)) \simeq x$
5. $a \sqsubseteq b$ yields only $f(a) \sqsubseteq f(b)$
6. $a \sqsubseteq f(c)$ yields only $\lceil f(f(x)) = x \rceil \triangleright f(a) \sqsubseteq c$
7. Terminate and detect satisfiability

Decision procedures with speculative inferences

To decide satisfiability modulo $\mathcal{T}$ of $\mathcal{R} \cup P$:

- Find sequence of speculative axioms U
- Show that there exists k s.t. k-bounded DPLL($\Gamma+\mathcal{T}$) is guaranteed to terminate
  - returning Unsat if $\mathcal{R} \cup P$ is $\mathcal{T}$-unsatisfiable
  - in a state which is not stuck at k otherwise

Decision procedures

- $\mathcal{R}$ has single monadic function symbol f
- Essentially finite: if $\mathcal{R} \cup P$ is satisfiable, has model where range of f is finite
- Such a model satisfies $f^j(x) \simeq f^k(x)$ for some $j \neq k$
- Add pseudo-axioms $f^j(x) \simeq f^k(x)$, $j > k$
- Use $f^j(x) \simeq f^k(x)$ as rewrite rule to limit term depth
- Clause length limited by properties of Γ and $\mathcal{R}$
- Only finitely many clauses generated: termination

Situations where clause length is limited

Γ: Superposition, Resolution + negative selection, Simplification

Negative selection: only positive literals in positive clauses resolve or superpose

- $\mathcal{R}$ is Horn: number of literals in each clause is bounded
- $\mathcal{R}$ is ground-preserving: all variables appear also in negative literals
  the only positive clauses are ground
  only finitely many clauses generated

Axiomatizations of type systems

$$
\begin{aligned}
&\text{Reflexivity} && x \sqsubseteq x && (1)\\
&\text{Transitivity} && \neg(x \sqsubseteq y) \vee \neg(y \sqsubseteq z) \vee x \sqsubseteq z && (2)\\
&\text{Anti-Symmetry} && \neg(x \sqsubseteq y) \vee \neg(y \sqsubseteq x) \vee x \simeq y && (3)\\
&\text{Monotonicity} && \neg(x \sqsubseteq y) \vee f(x) \sqsubseteq f(y) && (4)\\
&\text{Tree-Property} && \neg(z \sqsubseteq x) \vee \neg(z \sqsubseteq y) \vee x \sqsubseteq y \vee y \sqsubseteq x && (5)
\end{aligned}
$$

Multiple inheritance: $\mathrm{MI} = \{(1), (2), (3), (4)\}$
Single inheritance: $\mathrm{SI} = \mathrm{MI} \cup \{(5)\}$

Concrete examples of decision procedures

DPLL(Γ+$\mathcal{T}$) with addition of $f^j(x) \simeq f^k(x)$ for $j > k$ decides the satisfiability modulo $\mathcal{T}$ of problems

- $\mathsf{MI} \cup P$
- $\mathsf{SI} \cup P$
- $\mathsf{MI} \cup \mathsf{TR} \cup P$ and $\mathsf{SI} \cup \mathsf{TR} \cup P$

where $\mathsf{TR} = \{\neg(g(x) \simeq \mathit{null}),\ h(g(x)) \simeq x\}$ has only infinite models!
(because g is injective, since it has left inverse, but not surjective, since there is no pre-image for null)

III part: Discussion of current trends in the field

- Integration of search and inference in first-order theories
- CDCL beyond propositional logic?
- MBTC beyond linear integer arithmetic?

Model-constructing satisfiability procedures (MCsat)

- Satisfiability modulo assignment (SMA)
- M: both L (means $L \leftarrow true$) and $x \leftarrow 3$
- CDCL + MBTC
- Theory CDCL: explain theory conflicts and theory propagations
- Beyond input literals: finite bag for termination
- Equality, lists, arrays, linear arithmetic (rationals)

Example of theory explanation (equality)

$$F = \{\ldots,\ v \simeq f(a),\ w \simeq f(b),\ \ldots\}$$

$$M = \ldots\ \ a \leftarrow \alpha \quad b \leftarrow \alpha \quad w \leftarrow \beta_1 \quad v \leftarrow \beta_2\ \ \ldots$$

Conflict!

Explain by $a \simeq b \supset f(a) \simeq f(b)$
(instance of substitutivity)

Example of theory explanation (arithmetic) I

$F = \{x \geq 2,\ \neg(x \geq 1) \vee y \geq 1,\ x^2 + y^2 \leq 1 \vee xy > 1\}$

- $M = \emptyset$
- Propagation: $M = x \geq 2$
- Theory Propagation: $M = x \geq 2,\ x \geq 1$
- Boolean Propagation: $M = x \geq 2,\ x \geq 1,\ y \geq 1$
- Boolean Decision: $M = x \geq 2,\ x \geq 1,\ y \geq 1,\ x^2 + y^2 \leq 1$
- Semantic Decision:
  $M = x \geq 2,\ x \geq 1,\ y \geq 1,\ x^2 + y^2 \leq 1,\ x \leftarrow 2$
- Conflict!: no value for y such that $4 + y^2 \leq 1$

Example of theory explanation (arithmetic) II

$$F = \{x \geq 2,\ \neg(x \geq 1) \vee y \geq 1,\ x^2 + y^2 \leq 1 \vee xy > 1\}$$

- Assume we'd learn $\neg(x = 2)$:
  $M = x \geq 2,\ x \geq 1,\ y \geq 1,\ x^2 + y^2 \leq 1,\ \neg(x = 2)$
- Semantic Decision:
  $M = x \geq 2,\ x \geq 1,\ y \geq 1,\ x^2 + y^2 \leq 1,\ \neg(x = 2),\ x \leftarrow 3$
- Another conflict!
- We don't want to learn $\neg(x = 2),\ \neg(x = 3),\ \neg(x = 4) \ldots$!

Example of theory explanation (arithmetic) III

$$F = \{x \geq 2,\ \neg(x \geq 1) \vee y \geq 1,\ x^2 + y^2 \leq 1 \vee xy > 1\}$$

- Solution: theory explanation by interpolation
- $x^2 + y^2 \leq 1$ implies $-1 \leq x \wedge x \leq 1$ which is inconsistent with $x = 2$
- Learn $\neg(x^2 + y^2 \leq 1) \vee x \leq 1$
- $M = x \geq 2,\ x \geq 1,\ y \geq 1,\ x^2 + y^2 \leq 1,\ x \leq 1$

Example of theory explanation (arithmetic) IV

$$F = \{x \geq 2,\ \neg(x \geq 1) \vee y \geq 1,\ x^2 + y^2 \leq 1 \vee xy > 1\}$$

- $M = x \geq 2,\ x \geq 1,\ y \geq 1,\ x^2 + y^2 \leq 1,\ x \leq 1$
- Theory conflict: $x \geq 2$ and $x \leq 1$
- Learn lemma: $\neg(x \geq 2) \vee \neg(x \leq 1)$
- Boolean Explanation (by resolution): $\neg(x^2 + y^2 \leq 1) \vee x \leq 1$ and $\neg(x \geq 2) \vee \neg(x \leq 1)$ yield $\neg(x^2 + y^2 \leq 1) \vee \neg(x \geq 2)$
- Boolean Explanation (by resolution): $\neg(x^2 + y^2 \leq 1) \vee \neg(x \geq 2)$ and $x \geq 2$ yield $\neg(x^2 + y^2 \leq 1)$
- $M = x \geq 2,\ x \geq 1,\ y \geq 1,\ \neg(x^2 + y^2 \leq 1)$

Recent trends in model-based reasoning

- Deduction guides search for model
- Candidate model guides deduction

- Propositional CDCL (both DPLL and DPLL($\mathcal{T}$))
- Model-based theory combination (MBTC)
- DPLL($\Gamma+\mathcal{T}$)
- CDCL for arithmetic (aka Natural domain SMT)
- Model-constructing satisfiability procedures (MCsat)

Ideas for future work

- MCsat procedures for more first-order theories
  e.g., Boolean algebra with Presburger arithmetic (BAPA)
- More decision procedures by speculative inferences
- MCsat + Γ