On Model-Based Reasoning

Recent Trends and Current Developments

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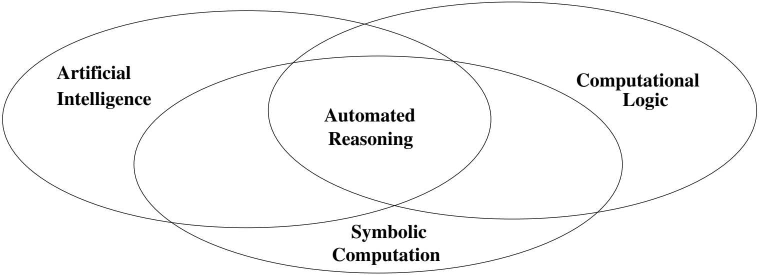
The gist of this talk

- Automated reasoning from proofs to models
- Models are relevant to applications
  (e.g., program testing, program synthesis)
- Theorem provers that terminate on satisfiable inputs
  (Decision procedures)
- Trade-off between decidability and expressivity

Automated reasoning



- Logico-deductive reasoning
- Other kinds: Probabilistic ...

Logico-deductive reasoning

- Proofs and Models
- Theorem Proving
  - Validity: $\mathcal{T} \models \varphi$
  - Refutationally: $\mathcal{T} \cup \{\neg\varphi\}$ unsatisfiable
  - If not: $\mathcal{T}$-model of $\neg\varphi$, counter-example for φ
- Model Building
  - Satisfiability: is there a $\mathcal{T}$-model of φ?
  - If not: $\mathcal{T} \cup \{\varphi\}$ unsatisfiable, $\mathcal{T} \models \neg\varphi$

Theorem proving strategies (Semi-decision procedures)

- First-order logic with equality
- Unsatisfiability is semi-decidable, satisfiability is not
- Search for proof (refutation)
- Models for semantic guidance:
  - Hyper-resolution [Alan Robinson 1965]
  - Set of support [Larry Wos et al. 1965]
  - Semantic resolution [James Slagle 1967]
  - ...

Algorithmic reasoning (Decision procedures)

- Satisfiability decidable: Symmetry restored
- Propositional logic
- Decidable (fragments of) first-order theories
  - QFF: equality, recursive data structures, arrays
  - Linear arithmetic (integers, rationals), arithmetic (reals)

Symmetry in the reasoner's operations

- Deduction guides search for model
- Candidate partial model guides deduction
- How?

Propositional logic (SAT)

- Davis-Putnam-Logemann-Loveland (DPLL) procedure

  [Martin Davis and Hilary Putnam 1960]

  [Martin Davis and George Logemann and Donald Loveland 1962]

- Backtracking search for model
- State of derivation: $M \parallel F$
  M: sequence of truth assignments
  F: clauses to satisfy

Conflict-Driven Clause Learning (CDCL)

- Conflict: M falsifies clause $L_1 \vee \ldots \vee L_n$: conflict clause
- Explain: resolve and get another conflict clause
  $L_1 \vee \ldots \vee L_n$
  $\neg L_1 \vee Q_2 \ldots \vee Q_k$
- Learn: may add resolvent(s)
- Backjump: undoes at least an assignment, jumps back as far as possible to state where learnt resolvent can be satisfied

[João P. Marques-Silva and Karem A. Sakallah 1997]

[Matthew W. Moskewicz, Conor F. Madigan, Ying Zhao, Lintao Zhang and Sharad Malik 2001]

Example of CDCL

$F = \{\neg a \vee b,\ \neg c \vee d,\ \neg e \vee \neg f,\ f \vee \neg e \vee \neg b\}$
$M = a\ b\ c\ d\ e\ \neg f$
blue: assignments; violet: propagations

Conflict: $f \vee \neg e \vee \neg b$
Explain by resolving $f \vee \neg e \vee \neg b$ and $\neg e \vee \neg f$: $\neg e \vee \neg b$
Learn $\neg e \vee \neg b$: no model with e and b true
Jump back to earliest state with $\neg b$ false and $\neg e$ unassigned:
$M = a\ b\ \neg e$

Chronological backtracking: $M = a\ b\ c\ d\ \neg e$

Satisfiability modulo theories (SMT)

- DPLL($\mathcal{T}$) procedure
- Integrate $\mathcal{T}$-satisfiability procedure in DPLL
- Ground first-order literals abstracted to propositional variables
- CDCL: same

[Robert Nieuwenhuis, Albert Oliveras and Cesare Tinelli 2006]

Theory combination by equality sharing

- Theories $\mathcal{T}_1, \ldots, \mathcal{T}_n$
- $\mathcal{T} = \bigcup_{i=1}^{n} \mathcal{T}_i$
- $\mathcal{T}_i$-satisfiability procedures
- Disjoint: share only $\simeq$ and uninterpreted constants
- Need to compute arrangement: which shared constants are equal and which are not
- Conservative approach: propagate all entailed (disjunctions of) equalities between shared constants

[Greg Nelson and Derek C. Oppen 1979]

Model-based theory combination (MBTC)

- Every $\mathcal{T}_i$-satisfiability procedure builds a $\mathcal{T}_i$-model
- Optimistic approach: propagate equalities true in $\mathcal{T}_i$-model
- If not entailed: conflict + backjumping with CDCL + update $\mathcal{T}_i$-model
- Rationale: few equalities matter in practice

[Leonardo de Moura and Nikolaj Bjørner 2007]

CDCL for ∃-fragments of arithmetic

- Linear arithmetic (rationals)

  [Ken McMillan, A. Kuehlmann and Mooly Sagiv 2009]

  [Konstantin Korovin, Nestan Tsiskaridze and Andrei Voronkov 2009] [Scott Cotton 2010]

- Linear arithmetic (integers)

  [Dejan Jovanović and Leonardo de Moura 2011]

- Non-linear arithmetic (reals)

  [Dejan Jovanović and Leonardo de Moura 2012]

- Floating-point binary arithmetic

  [Leopold Haller, Alberto Griggio, Martin Brain and Daniel Kroening 2012]

Model-constructing satisfiability procedures (MCsat)

- Satisfiability modulo assignment (SMA)
- M: both L (means $L \leftarrow true$) and $x \leftarrow 3$
- CDCL + MBTC
- Theory CDCL: explain theory conflicts and theory propagations
- Beyond input literals: finite bag for termination
- Equality, lists, arrays, linear arithmetic (rationals)

[Leonardo de Moura and Dejan Jovanović 2013]

[Dejan Jovanović, Clark Barrett and Leonardo de Moura 2013]

Example of theory explanation (equality)

$$F = \{\ldots,\ v \simeq f(a),\ w \simeq f(b),\ \ldots\}$$

$$M = \ldots\ a \leftarrow \alpha \quad b \leftarrow \alpha \quad w \leftarrow \beta_1 \quad v \leftarrow \beta_2\ \ldots$$

Conflict!

Explain by $a \simeq b \supset f(a) \simeq f(b)$
(instance of substitutivity)

Summary: Recent trends in model-based reasoning

- Deduction guides search for model
- Candidate model guides deduction

- Propositional CDCL (both DPLL and DPLL($\mathcal{T}$))
- Model-based theory combination (MBTC)
- CDCL for arithmetic (aka Natural domain SMT)
- Model-constructing satisfiability procedures (MCsat)

Motivation

- Decision procedures are most desirable, but ...
- Formulæ from SW verification tools (verifying compiler, static analyzer, test generator, synthesizer, model checker) use quantifiers to write
  - invariants
  - axioms of theories without decision procedure
- Need for generic first-order inferences

Shape of problem

- Background theory $\mathcal{T}$
  - $\mathcal{T} = \bigcup_{i=1}^{n} \mathcal{T}_i$ (linear arithmetic, data structures)
- Set of formulæ: $\mathcal{R} \cup P$
  - $\mathcal{R}$: set of non-ground clauses without $\mathcal{T}$-symbols
  - P: large ground formula (set of ground clauses) typically with $\mathcal{T}$-symbols
- Determine whether $\mathcal{R} \cup P$ is satisfiable modulo $\mathcal{T}$

DPLL($\Gamma+\mathcal{T}$): integrate Γ in DPLL($\mathcal{T}$)

- Superposition-based inference system Γ:
  - FOL+= clauses with universally quantified variables
  - Expansion: generate clauses (resolution, superposition)
  - Contraction: delete redundant clauses (subsumption, simplification)
  - Well-founded ordering and literal selection
  - Decision procedure for several theories of data structures (e.g., lists, arrays, records)
- Model-based deduction:
  literals in M as premises of Γ-inferences!

[Alessandro Armando, Maria Paola Bonacina, Silvio Ranise and Stephan Schulz 2009]

[Leonardo de Moura and Nikolaj Bjørner 2008]

Hypothetical clauses

- Literals from M used as premises of Γ-inferences stored as hypotheses in inferred clause:
  $(L_1 \wedge \ldots \wedge L_n) \triangleright (L'_1 \vee \ldots L'_m)$
  interpreted as
  $\neg L_1 \vee \ldots \vee \neg L_n \vee L'_1 \vee \ldots \vee L'_m$
- Inferred clauses inherit hypotheses from premises
- Backjump: remove hypothetical clauses depending on undone assignments

DPLL($\Gamma+\mathcal{T}$): expansion inferences

- If non-ground clauses $C_1, \ldots, C_m$ and ground $\mathcal{R}$-literals $L_{m+1}, \ldots, L_n$ generate C :
  $H_1 \triangleright C_1, \ldots, H_m \triangleright C_m$ and $L_{m+1}, \ldots, L_n$ in M generate
  $H_1 \cup \ldots \cup H_m \cup \{L_{m+1}, \ldots, L_n\} \triangleright C$
- Only $\mathcal{R}$-literals: Γ-inferences ignore $\mathcal{T}$-literals
- Take ground unit $\mathcal{R}$-clauses from M as MBTC puts them there

DPLL($\Gamma+\mathcal{T}$): contraction inferences

- Don't delete clause if clauses that make it redundant gone by backjumping
  - Level of a literal in M: its decision level
  - Level of a set of literals: the maximum
- If non-ground clauses $C_1, \ldots, C_m$ and ground $\mathcal{R}$-literals $L_{m+1}, \ldots, L_n$ simplify C to C' :
  $H_1 \triangleright C_1, \ldots, H_m \triangleright C_m$ and $L_{m+1}, \ldots, L_n$ in M simplify $H \triangleright C$ to $H \cup H_1 \cup \ldots \cup H_m \cup \{L_{m+1}, \ldots, L_n\} \triangleright C'$
  - If $\mathit{level}(H) \geq \mathit{level}(H')$: delete
  - If $\mathit{level}(H) < \mathit{level}(H')$: disable
    (re-enable when backjumping $\mathit{level}(H')$)

Completeness of DPLL($\Gamma+\mathcal{T}$)

- Refutational completeness of the inference system:
  - From that of Γ, DPLL($\mathcal{T}$) and equality sharing
  - Combines both built-in and axiomatized theories
- Fairness of the search plan:
  - Depth-first search fair only for ground SMT problems;
  - Add iterative deepening on inference depth:
    k-bounded DPLL($\Gamma+\mathcal{T}$)

DPLL($\Gamma+\mathcal{T}$): Summary

Use each engine for what is best at:

- DPLL($\mathcal{T}$) works on ground clauses and built-in theory
- Γ works on non-ground clauses and ground unit clauses taken from M: Γ works on $\mathcal{R}$-satisfiability problem
- Γ-inferences guided by current partial model

Can DPLL($\Gamma+\mathcal{T}$) still be a decision procedure?

Problematic axioms do occur in relevant inputs:

1. $\neg(x \sqsubseteq y) \vee f(x) \sqsubseteq f(y)$ (Monotonicity)
2. $a \sqsubseteq b$ generates by resolution
3. $\{f^i(a) \sqsubseteq f^i(b)\}_{i \geq 0}$

When $f(a) \sqsubseteq f(b)$ or $f^2(a) \sqsubseteq f^2(b)$ often suffice to show satisfiability

Idea: Allow speculative inferences

1. $\neg(x \sqsubseteq y) \vee f(x) \sqsubseteq f(y)$
2. $a \sqsubseteq b$
3. $a \sqsubseteq f(c)$
4. $\neg(a \sqsubseteq c)$

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1. Add $f(x) \simeq x$
2. Rewrite $a \sqsubseteq f(c)$ into $a \sqsubseteq c$ and get $\Box$: backtrack!

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2. Rewrite $a \sqsubseteq f(c)$ into $a \sqsubseteq c$ and get $\Box$: backtrack!
3. Add $f(f(x)) \simeq x$
4. $a \sqsubseteq b$ yields only $f(a) \sqsubseteq f(b)$
5. $a \sqsubseteq f(c)$ yields only $f(a) \sqsubseteq c$
6. Terminate and detect satisfiability

Speculative inferences in DPLL($\Gamma+\mathcal{T}$)

- Speculative inference: add arbitrary clause C
- To induce termination on satisfiable input
- What if it makes problem unsatisfiable?!
- Detect conflict and backjump:
  - $\lceil C \rceil$: new propositional variable (a "name" for C)
  - Add $\lceil C \rceil \triangleright C$ to clauses and $\lceil C \rceil$ to M
  - Speculative inferences are reversible

Example as done by system

1. $\neg(x \sqsubseteq y) \vee f(x) \sqsubseteq f(y)$
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1. Add $\lceil f(x) \simeq x \rceil \triangleright f(x) \simeq x$
2. Rewrite $a \sqsubseteq f(c)$ into $\lceil f(x) \simeq x \rceil \triangleright a \sqsubseteq c$

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1. Add $\lceil f(x) \simeq x \rceil \triangleright f(x) \simeq x$
2. Rewrite $a \sqsubseteq f(c)$ into $\lceil f(x) \simeq x \rceil \triangleright a \sqsubseteq c$
3. Generate $\lceil f(x) \simeq x \rceil \triangleright \Box$; Backtrack, learn $\neg\lceil f(x) \simeq x \rceil$

Example as done by system

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2. Rewrite $a \sqsubseteq f(c)$ into $\lceil f(x) \simeq x \rceil \triangleright a \sqsubseteq c$
3. Generate $\lceil f(x) \simeq x \rceil \triangleright \square$; Backtrack, learn $\neg\lceil f(x) \simeq x \rceil$
4. Add $\lceil f(f(x)) \simeq x \rceil \triangleright f(f(x)) \simeq x$
5. $a \sqsubseteq b$ yields only $f(a) \sqsubseteq f(b)$
6. $a \sqsubseteq f(c)$ yields only $\lceil f(f(x)) = x \rceil \triangleright f(a) \sqsubseteq c$
7. Terminate and detect satisfiability

Decision procedures with speculative inferences

To decide satisfiability modulo $\mathcal{T}$ of $\mathcal{R} \cup P$:

- Find sequence of speculative axioms U
- Show that there exists k s.t. k-bounded DPLL($\Gamma+\mathcal{T}$) is guaranteed to terminate
  - returning Unsat if $\mathcal{R} \cup P$ is $\mathcal{T}$-unsatisfiable
  - in a state which is not stuck at k otherwise

Decision procedures

- $\mathcal{R}$ has single monadic function symbol f
- Essentially finite: if $\mathcal{R} \cup P$ is satisfiable, has model where range of f is finite
- Such a model satisfies $f^j(x) \simeq f^k(x)$ for some $j \neq k$

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- Use $f^j(x) \simeq f^k(x)$ as rewrite rule to limit term depth
- Clause length limited by properties of Γ and $\mathcal{R}$
- Only finitely many clauses generated: termination

Situations where clause length is limited

Γ: Superposition, Resolution + negative selection, Simplification

Negative selection: only positive literals in positive clauses resolve or superpose

- $\mathcal{R}$ is Horn: number of literals in each clause is bounded
- $\mathcal{R}$ is ground-preserving: all variables appear also in negative literals
  the only positive clauses are ground
  only finitely many clauses generated

Axiomatizations of type systems

$$
\begin{array}{rll}
\text{Reflexivity} & x \sqsubseteq x & (1) \\
\text{Transitivity} & \neg(x \sqsubseteq y) \vee \neg(y \sqsubseteq z) \vee x \sqsubseteq z & (2) \\
\text{Anti-Symmetry} & \neg(x \sqsubseteq y) \vee \neg(y \sqsubseteq x) \vee x \simeq y & (3) \\
\text{Monotonicity} & \neg(x \sqsubseteq y) \vee f(x) \sqsubseteq f(y) & (4) \\
\text{Tree-Property} & \neg(z \sqsubseteq x) \vee \neg(z \sqsubseteq y) \vee x \sqsubseteq y \vee y \sqsubseteq x & (5)
\end{array}
$$

Multiple inheritance: $\mathsf{MI} = \{(1), (2), (3), (4)\}$
Single inheritance: $\mathsf{SI} = \mathsf{MI} \cup \{(5)\}$

Concrete examples of decision procedures

DPLL($\Gamma+\mathcal{T}$) with addition of $f^j(x) \simeq f^k(x)$ for $j > k$ decides the satisfiability modulo $\mathcal{T}$ of problems

- $\mathrm{MI} \cup P$
- $\mathrm{SI} \cup P$
- $\mathrm{MI} \cup \mathrm{TR} \cup P$ and $\mathrm{SI} \cup \mathrm{TR} \cup P$

where $\mathrm{TR} = \{\neg(g(x) \simeq \mathit{null}),\ h(g(x)) \simeq x\}$ has only infinite models!
(because g is injective, since it has left inverse, but not surjective, since there is no pre-image for *null*)

[Maria Paola Bonacina, Chris Lynch and Leonardo de Moura 2011]

Current and future work

- MCsat procedures for more first-order theories
  e.g., Boolean algebra with Presburger arithmetic (BAPA)
- Many-sorted DPLL($\Gamma+\mathcal{T}$)
- Weakening conditions for completeness
- More decision procedures by speculative inferences
- MCsat + Γ

[Joint work with Serdar Erbatur]