Cut-elimination for the logic of pragmatics

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Abstract

We present a sequent calculus for the Intuitionistic Logic of Pragmatics (**ILP**) with operators of *assertion* and *conjecture*, which is a conservative extension of ordinary intuitionistic logic and its dual, the logic of co-Heyting algebras [7]. Our sequent calculus is in the style of a **G3i** system [8], where the rules of weakening and contraction are *implicit* and where sequents have *privileged affine areas*. We give a deterministic cut-elimination procedure for this sequent calculus.

1 Introduction

The aim of this paper is to proof the cut-elimination for the sequent calculus of the Intuitionistic Logic of Pragmatics (**ILP**) with assertion and conjecture operators, which has been presented in [1, 3]. [1] The motivations for the logic for pragmatics, its informal interpretations and its models in terms of a topological and a Kripke-style semantics are given in [1], where it is also given a proof of the completeness of **ILP** with respect to Kripke's semantics over preordered frames, using an extension of Gödel, McKinsey and Tarski's translation of intuitionistic logic into the modal system of **S4**. The expressions of the language $\mathcal{L}^P$ are either *assertive* or *conjectural*. The elementary expressions $\vdash \alpha$ and $\mathcal{H}\,\alpha$ of $\mathcal{L}^P$ express the *assertion* and the *conjecture* that α is true, respectively; similarly, the usual intuitionistic implication $\delta_1 \supset \delta_2$ is assertive and its dual *subtraction* $\delta_1 \smallsetminus \delta_2$ is conjectural. Their translations in **S4** are:

$$\begin{array}{rclcrcl} (\vdash \alpha)^M & = & \Box\alpha & \qquad & (\mathcal{H}\,\alpha)^M & = & \Diamond\alpha \\ (\delta_1 \supset \delta_2)^M & = & \Box(\delta_1^M \to \delta_2^M) & \qquad & (\delta_1 \smallsetminus \delta_2)^M & = & \Box(\delta_1^M \wedge \neg\delta_2^M). \end{array}$$

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In a sequent calculus for standard intuitionistic logic there are only *assertive* formulas and Gentzen's restriction, which is characteristic of the sequent calculus **LJ**, requires that at most one formula should occur in the succedent of any sequent. In a sequent calculus for the dual intuitionistic system, formulas are *conjectural* and the analogue of Gentzen's restriction applies to the formulas in the antecedent. Of course, both constraints correspond to the restrictions on the $\Box$-R and $\Diamond$-L rules in the sequent calculus for **S4**. In [3] we have presented a sequent calculus for **ILP** in the style of **G3i** [8], where the rules of weakening and contraction are *implicit*. Here we have *assertive* and *conjectural* formulas, and we use sequents with privileged areas in the antecedent and in the succedent and also apply a generalization of Gentzen's restriction: *there can be no more than one formula in privileged position, i.e., no more than one conjectural formula in the antecedent or one assertive formula in the succedent.* As there are sequent calculi for intuitionistic logic **G3i**m where Gentzen's restriction on **LJ** sequents is relaxed for all propositional rules, except for the sequent-premise of a $\supset$-R, so there is a sequent calculus **FILP** for the logic of pragmatics where our restriction is relaxed [1, 3], making it easier to prove the completeness theorem. Indeed there are rules that are *invertible* in **FILP** while their counterparts in **ILP** may not be invertible. Moreover the system **FILP** is more concise; for instance, in **ILP** we need both rules ACC.6 and ACC.7 (which may be called *multiplicative* and *additive*) as it is shown by the following examples

$$\frac{;\upsilon \Rightarrow ; \upsilon, \vartheta \curlyvee \upsilon}{;\upsilon \Rightarrow ; \vartheta \curlyvee \upsilon}\ ACC.7 \qquad \frac{\vartheta ; \Rightarrow \vartheta ; \upsilon, \vartheta \curlyvee \upsilon, \Upsilon}{\vartheta ; \Rightarrow ; \vartheta \curlyvee \upsilon}\ ACC.6$$

while only (the multiplicative) one suffices in **FILP**. We shall not deal with **FILP** in this paper.

Since the formulas of our pragmatic language are naturally *polarized* as assertive and conjectural, the calculus **ILP** with its uniform specification of *affine areas*[2] in a sequent is the most appropriate formalism for the presentation of a deterministic and confluent notion of cut-elimination. Polarization allows us to avoid *non-determinism* of the cut-elimination process, which in conjunction with heavy structural transformations due to implicit weakening and contraction would make the cut-elimination process *non-confluent* (as in the classical sequent calculus **LK**). Since **ILP** is a polarized system, we can find a deterministic algorithm for cut-elimination: in particular, for commutative reductions the direction of the permutation is determined by

[2]The privileged areas in a sequent are affine, not linear, because we must allow implicit weakening in them to have completeness with respect to Kripke's semantics.

the global function of the cut-formulas in the derivation, and depends on the cut-formula which occurs in the privileged (affine) area of the sequent.

For the proof of the cut-elimination theorem we need to show that the rules of *weakening* and *contraction* are *admissible*. Some delicate decisions in the design of the system are required to guarantee this property. In the case of weakening formulas *outside* the privileged area this is easily achieved by introducing the weakening formula in all sequents of the proof-tree up the axioms; for a weakening formula which is introduced *inside* the privileged area, the inference rules where this area becomes empty must allow the introduction of a weakening formula ϵ [or ϵ'] in the antecedent [in the consequent]. Admissibility of contraction is harder to achieve: the proof proceeds by induction on the length of the proof, but it achieves a reduction of contraction to formulas of lower logical complexity. Notice that as in **LJ**, contraction is admissible only in the non-privileged areas. Where a rule of inference is *non-invertible* on a sequent-premise, the system has been designed so that an occurrence of the principal formula occurs in that premise and the induction hypothesis on the length of the proof can be applied. If an inference rule *can be inverted*, then we apply contraction to formulas of lower logical complexity. It should be noticed that this process, regarded as a transformation on proofs, may considerably modify the structure of the given proof, by pruning some branches of the proof-tree.

In conclusion, the present work is certainly related to the area of research inspired by J-Y.Girard which aims at the constructivization of classical proof theory through translations into linear logic. We expect that many facts and results observed in those computational studies of classical logic may be reproduced in the context of our intuitionistic logic of pragmatics with dual operators (see also [4]).

2 The logic for pragmatics

2.1 Language of the logic for pragmatics

Our language $\mathcal{L}^P$ is in two levels: there are *radical formulas* α, α_0, α_1 ... built from a set of atoms p, p_i, ... using the classical connectives $\neg$, $\wedge$ $\vee$ $\rightarrow$ and *elementary formulas* obtained from a radical formula α by prefixing it with a sign of *illocutionary force* $\vdash$ and $\mathcal{H}$. Here α is a proposition interpreted according to classical semantics, while the *elementary formulas* $\vdash \alpha$ and $\mathcal{H}\alpha$ express illocutionary acts of *assertion* and *conjecture*. There is an elementary constant for absurdity $\bigwedge$ and one for validity $\bigvee$. The calculus presented here is *intuitionistic* because the radical part of each elementary is regarded as constant throughout a derivation; a calculus where the inference rules act on the radical part is considered in [1].

Recall that illocutionary acts can be *justified* or *unjustified*: $\vdash \alpha$ is *justified* if and only if there is a proof that α is true; *unjustified* otherwise. $\mathcal{H}\,\alpha$ is *justified* if there is no proof that α is false; *unjustified* otherwise. The elementary constants $\bigwedge$ and $\bigvee$ express illocutionary acts that are never justified or always justified, respectively. Pragmatic expressions (or *sentential formulas*) are built using *pragmatic connectives* which are given Heyting's, not Tarski's semantics. Indeed pragmatic connectives apply to formulas that can be justified or unjustified, not to proposition having a truth value. Sentential formulas are built from the elementary formulas using *assertive* (or *strong*) pragmatic connectives, namely, *negation* $\sim$, *conjunction* $\cap$, *disjunction* $\cup$, *implication* $\supset$ and *subtraction* $\setminus\!\!\setminus$, and also *conjectural* (or *weak*) connectives, namely, *doubt* $\frown$, *disjunction* $\curlyvee$, *conjunction* $\curlywedge$, *implication* $\succ$ and *subtraction* $\smallsetminus$. The grammar is defined thus:

$$\alpha \ := \ p \mid \neg\alpha \mid \alpha \wedge \alpha \mid \alpha \vee \alpha \mid \alpha \rightarrow \alpha \mid$$

$$\delta \ := \ \vartheta \mid \upsilon \mid$$

$$\vartheta \ := \vdash \alpha \mid \bigwedge \mid \bigvee \mid \ \sim \delta \mid \delta \supset \delta \mid \delta \cap \delta \mid \delta \cup \delta \mid \delta \setminus\!\!\setminus \delta$$

$$\upsilon \ := \ \mathcal{H}\,\alpha \mid \bigwedge \mid \bigvee \mid \ \frown \delta \mid \delta \succ \delta \mid \delta \curlyvee \delta \mid \delta \curlywedge \delta \mid \delta \smallsetminus \delta$$

2.2 The sequent calculus for ILP

All the sequents S are of the form

$$\Theta \ ; \ \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon$$

where

- Θ is a sequence of assertive formulas $\vartheta_1, \ldots, \vartheta_m$;
- Υ is a sequence of conjectural formulas $\upsilon_1, \ldots, \upsilon_n$;
- ϵ is conjectural and ϵ' is assertive and at most one of ϵ, ϵ' occurs in S.

hence our sequents can have only one of the following three forms

$$\theta_1, .., \theta_n \ ; \ \upsilon \ \Rightarrow \ ; \ \upsilon_1, .., \upsilon_m$$
$$\theta_1, .., \theta_n \ ; \ \Rightarrow \ \theta \ ; \upsilon_1, .., \upsilon_m$$
$$\theta_1, .., \theta_n \ ; \ \Rightarrow \ ; \ \upsilon_1, .., \upsilon_m$$

The rules of **ILP** are given in the Appendix.

In this paper we shall assume that *in the logical axioms*

S.1: logical axiom:	*S.2: logical axiom:*
$\Theta, \vartheta\ ;\ \Rightarrow\ \vartheta\ ; \Upsilon$	$\Theta;\ \upsilon\ \Rightarrow\ ;\ \upsilon, \Upsilon$

the principal formulas are elementary, i.e., $\vartheta = \ \vdash \alpha$ and $\upsilon = \ \nVdash \alpha$. It is easy to prove, by induction on the logical complexity of ϑ and υ, that the more general forms with ϑ and υ arbitrary are admissible.

3 Preliminaries for Cut-elimination in ILP

3.1 Inversion lemma

We write $\vdash_n \Theta; \epsilon \Rightarrow \epsilon'; \Upsilon$ to express the fact that there is a derivation of $\Theta; \epsilon \Rightarrow \epsilon'; \Upsilon$ in **ILP** where the derivation tree has depth at most n.

We say that a rule of inference **R** is *invertible on the left or right sequent-premise or in both preserving the depth of the derivation* if for every triple of sequents S_0, S_1 and S such that

$$\frac{S_0}{S}\mathbf{R} \qquad \text{or} \qquad \frac{S_0 \quad S_1}{S}\mathbf{R}$$

whenever $\vdash_n S$ then $\vdash_n S_0$ or $\vdash_n S_1$ or both, respectively.

We want to classify the rules **R** of **ILP** with respect to their behaviour with respect to depth-preserving invertibility. Invertibility fails because a loss of information occurs in passing from the sequent-conclusion S to a sequent-premise S_i, due to the restrictions on the privileged area: this happens because

(a) a one-premised rule has is *additive* (in the terminology of Linear Logic), like the rules **A.7, 8** $\cup$-R or

(b) there is an *implicit weakening*, i.e., a formula ϵ or ϵ' occurs in S but is not in S_i, like in the left premise of **A.4** $\supset$-L.

In both cases it is not difficult to find examples showing that the rule **R** is not invertible: for instance in the case of **A.4** S may be a logical axiom, and this property fails for S_1, because ϵ or ϵ' is missing. Notice also that the property of invertibility becomes *trivial* if all the formulas occurring in S occur also in S_i, like in the case of **C3.R**.

To facilitate the proof of the admissibility of contraction in the next section we distinguish invertibility of the rules that allow contraction in their principal formula and invertibility of the rules that don't.

Lemma. The rules of the sequent calculus for **ILP** can be classified as follows:

1. Invertible rules that don't allow contraction in their principal formula: **A.1**, **A.3**, **A.5**, **A.10**, **C.2**, **C.5**, **C.10**, **C.12**, **CA.1**, **ACA.1**, **ACA.3**, **ACA.9**, **CAA.4**, **CAA.10**, **CCA.1**, **CCA.3**, **CCA.9**, **AC.2**, **ACC.2**, **ACC.8**, **ACC.10**, **CAC.3**, **CAC.9**, **AAC.3**, **AAC.9**, **AAC.11**.

2. Non-invertible *additive* rules that don't allow contraction in their principal formula; **A.7,8**, **C.7,8**, **ACA.6,7**, **CAA.1,2**, **CAA.7,8**, **CCA.6,7**, **ACC.4,5**, **CAC.5,6**, **AAC.5,6**, **CAC.11,12**.

3. Rules that allow contraction in their principal formula and are *non-trivially* invertible. **A.4** right premise, **A.6**, **A.9**, **C.6**, **C.9**, **CA.2**,**ACA.8** left branch, **ACA.10**, **CAA.3**, **CAA.9** right branch, **CCA.2** left branch, **C.11** right branch, **AC.1**, **ACC.1**, **ACC.3** right branch, **CAC.4** left premise, **CAC.10**, **AAC.10** right branch.

4. Rules that allow contraction in their principal formula and are *trivially* invertible: **A.12**, **C.3**, **ACA.5**, **CAA.6**, **CCA.11**, **AAC.2**, **ACC.7**, **CAC.8**

5. Rules that allow contraction in their principal formula and are *non-invertible* because of *implicit weakening*: **A.2**, **A.4** left premise, **A.11**, **C.1**, **C.4**, **C.11** left branch, **ACA.2**, **ACA.4**, **ACA.8** right branch, **CAA.5**, **CAA.9** left branch, **CAA.11**,**CAA.12**, **CCA.10**, **CCA.2** right branch, **CCA.4,5**, **CCA.8**, **ACC.3** left branch, **ACC.6**, **CAC.1**, **CAC.2**, **CAC.4** right branch, **CAC.7**, **ACC.9**, **AAC.1**, **AAC.4**, **AAC.7**, **AAC.8**, **AAC.10** left branch.

For each rule in cases 1, 3 and 4, the inversion lemma can be stated more precisely as follows:

- Invertible rules that don't allow contraction in their principal formula:

  $$\text{if } \vdash_n \Theta; \Rightarrow \sim \vartheta; \Upsilon \text{ then } \vdash_n \Theta, \vartheta; \Rightarrow; \Upsilon \text{ (rule A.4)}$$

  and similary for all the other cases

- Rules that allow contraction in their principal formula and are *non-trivially* invertible

  if $\vdash_n \vartheta_1 \supset \vartheta_2, \Theta \, ; \, \epsilon \; \Rightarrow \; \epsilon' \, ; \, \Upsilon$ then $\vdash_n \vartheta_2, \Theta \, ; \, \epsilon \; \Rightarrow \; \epsilon' \, ; \, \Upsilon$ (A.4 right branch)

  and similary for all the other cases

- Rules that allow contraction in their principal formula and are *trivially* invertible:

  if $\vdash_n \vartheta_1 \setminus\!\!\setminus \vartheta_2, \Theta \, ; \, \epsilon \; \Rightarrow \; \epsilon' \, ; \, \Upsilon$ then $\vdash_n \vartheta_1 \setminus\!\!\setminus \vartheta_2, \vartheta_1, \Theta; \, \epsilon \Rightarrow \epsilon'; \, \Upsilon$ (A.12)

  and similary for all the other cases

Proof. By induction on n. As a typical example we prove the case correspondent to the inversion of the right premise of the rule **A.4**. Let $\vdash_n \vartheta_1 \supset \vartheta_2, \Theta \, ; \, \epsilon \; \Rightarrow \; \epsilon' \, ; \, \Upsilon$ by a deduction D. If D is an axiom then $\vartheta_1 \supset \vartheta_2$ is not principal and $\vartheta_2, \Theta \, ; \, \epsilon \; \Rightarrow \; \epsilon' \, ; \, \Upsilon$ is also an axiom. If D is not an axiom and $\vartheta_1 \supset \vartheta_2$ is not principal, we apply the induction to the premise(s) of the last rule used. If on the other hand $\vartheta_1 \supset \vartheta_2$ is principal, the deduction ends with

$$\frac{\vdash_{n-1} \vartheta_1 \supset \vartheta_2, \Theta; \; \Rightarrow \; \vartheta_1 \, ; \, \Upsilon \qquad \vdash_{n-1} \vartheta_2, \Theta \, ; \epsilon \; \Rightarrow \; \epsilon' \, ; \, \Upsilon}{\vdash_n \vartheta_1 \supset \vartheta_2, \Theta \, ; \, \epsilon \; \Rightarrow \; \epsilon' \, ; \, \Upsilon}$$

and the subeduction of the right premise leads us to the conclusion. Note that the inversion lemma doesn't hold for the left premise. Indeed if $\vartheta_1 \supset \vartheta_2, \Theta \, ; \, \epsilon \; \Rightarrow \; \epsilon' \, ; \, \Upsilon$ was an axiom with principal formula ϵ or ϵ' certainly $\vartheta_1 \supset \vartheta_2, \Theta; \; \Rightarrow \; \vartheta_1 \, ; \, \Upsilon$ would not be an axiom.

3.2 Admissibility of depth-preserving weakening

Lemma.

$$\text{If} \quad \vdash_n \Theta; \epsilon \Rightarrow \epsilon'; \Upsilon \quad \text{then} \quad \vdash_n \Theta', \Theta; \epsilon \Rightarrow \epsilon'; \Upsilon, \Upsilon'$$

$$\text{If} \quad \vdash_n \Theta; \Rightarrow; \Upsilon \quad \text{then} \quad \vdash_n \Theta; \epsilon \Rightarrow \epsilon'; \Upsilon$$

Proof. To prove the first proposition it suffices to add Θ' and Υ' to the antecedent and consequent, respectively, of all sequents in the given derivation of $\Theta; \epsilon \Rightarrow \epsilon'; \Upsilon$. Obviously, this transformation preserves the depth of the derivation.

For the second proposition, we want to add ϵ or ϵ' in the endsequent. Working our way upwards, for each rule we rewrite the sequent-premises adding ϵ or ϵ', if possible, if ϵ or ϵ', respectively, occurs in the sequent-conclusion. Eventually, we either find a rule where ϵ or ϵ' occurs in the conclusion but cannot be added to one or both premises or we reach an *assertion-conjecture* axiom. In any cases the axioms and rules of inference remain valid and the depth of the given demonstration is preserved.

We shall write $d\{\Theta' \,;\Rightarrow;\; \Upsilon'\}$ for a derivation with conclusion $\Theta, \Theta' \,;\, \epsilon \;\Rightarrow\; \epsilon' \,;\, \Upsilon, \Upsilon'$ obtained by applying depth-preserving weakening (*dp-weakening*) to a deduction d with conclusion $\Theta \;;\; \epsilon \;\;\Rightarrow\;\; \epsilon' \;;\; \Upsilon$ Analogously we shall write $d\{;\epsilon \Rightarrow \epsilon';\}$ for a deduction with conclusion $\Theta;\epsilon \Rightarrow \epsilon';\Upsilon$ obtained by applying dp-weakening to a deduction d with conclusion $\Theta \;;\; \Rightarrow \;;\; \Upsilon$

3.3 Admissibility of depth-preserving contraction

The rule of contraction is inadmissible for formulas occurring as ϵ or ϵ' in the privileged part of a sequent: these are conjectural formulas in the antecedent and assertive formulas in the succedent. For formulas which do not occur in the privileged part we need to prove the admissibility of contraction preserving the depth of the derivation tree. The proof is by induction on the depth of the given derivation. If neither contraction formula is principal in the last inference, then we apply the induction hypothesis to the sequent-premise(s). Otherwise, the induction step depends on whether or not the last inference $\mathcal{I}$ is invertible preserving the depth of the derivation tree. If the last inference is a non-invertible or trivially invertible inference, then the principal formula of $\mathcal{I}$ occurs also in the sequent-premise(s) and then we can apply the induction hypothesis to the sequent-premise(s). Otherwise $\mathcal{I}$ is a non-trivially invertible inference, and we can invert the (ancestor of the) contraction-formula δ occurring in the sequent-premise(s), obtaining a derivation of lower depth than the given one; then we can applying contraction to the *immediate subformulas* of δ and the inductive step is concluded by applying the same rule of inference as $\mathcal{I}$.

Lemma.

1. if $\vdash_n \vartheta, \vartheta, \Theta; \epsilon \Rightarrow \epsilon'; \Upsilon$ then $\vdash_n \vartheta, \Theta; \epsilon \Rightarrow \epsilon'; \Upsilon$
2. if $\vdash_n \Theta; \epsilon \Rightarrow \epsilon'; \Upsilon, \upsilon, \upsilon$ then $\vdash_n \Theta; \epsilon \Rightarrow \epsilon'; \Upsilon, \upsilon$

Proof. By induction on n. We consider the first assertion; the second is treated symmetrically.
Let D a deduction of lenght $n+1$ of $\vartheta, \vartheta, \Theta; \epsilon \Rightarrow \epsilon'; \Upsilon$. If ϑ is not principal in the last rule applied in D, apply the induction to the premise. If ϑ is

Logic Rule	Invertibility
A.2	NO
A.4	only right premise
A.6	YES
A.9	YES
A.11	NO
A.12	YES(trivial)
C.1	NO
C.3	YES(trivial)
C.4	NO
C.6	YES
C.9	YES
C.11 AAC.10	only right premise
CA.2	YES
ACA.2	NO
ACA.4 CAA.5	NO
ACA.5 CAA.6	YES(trivial)
ACA.8	only left premise
ACA.10	YES
CAA.3	YES
CAA.9	only right premise
CAA.11 CAA.12	NO
CCA.2	only left premise
CCA.4 CCA.5	NO
CCA.8	NO
CCA.10	YES
CCA.11	YES(trivial)
AC.1	YES
ACC.1	YES
ACC.3	only right premise
ACC.9	NO
ACC.6 CAC.7	NO
ACC.7 CAC.8	YES(trivial)
CAC.1	NO
CAC.2	NO
CAC.4	only left premise
CAC.10	YES
AAC.1	NO
AAC.2	YES(trivial)
AAC.4	NO
AAC.7 AAC.8	NO

Table 1: Invertibility for rules that allow contraction on the principal formula

principal in the last rule applied, we distinguish cases (here we consider only three).

Last rule used is A.2

$$\frac{\vdash_n \sim \vartheta, \sim \vartheta, \Theta; \ \Rightarrow \ \vartheta \ ; \ \Upsilon}{\vdash_{n+1} \sim \vartheta, \sim \vartheta, \Theta \ ; \epsilon \Rightarrow \epsilon'; \Upsilon}$$

we apply the inductive hypothesis to the premise, obtaining

$$\vdash_n \sim \vartheta, \Theta; \ \Rightarrow \ \vartheta \ ; \ \Upsilon$$

from which, using **A.2**

$$\vdash_{n+1} \sim \vartheta, \Theta \ ; \epsilon \Rightarrow \epsilon'; \Upsilon$$

Last rule used is A.4

$$\frac{\vdash_n \vartheta_1 \supset \vartheta_2, \vartheta_1 \supset \vartheta_2, \Theta; \ \Rightarrow \ \vartheta_1 \ ; \ \Upsilon \qquad \vdash_n \vartheta_1 \supset \vartheta_2, \vartheta_2, \Theta \ ; \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon}{\vdash_{n+1} \vartheta_1 \supset \vartheta_2, \vartheta_1 \supset \vartheta_2, \Theta \ ; \ \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon}$$

By applying the inversion lemma to the right premise we obtain

$$\vdash_n \vartheta_2, \vartheta_2 \Theta \ ; \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon$$

by applying now the inductive hypthesis to both premises we have, for the right premise

$$\vdash_n \vartheta_2 \Theta \ ; \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon$$

and for the left premise

$$\vdash_n \vartheta_1 \supset \vartheta_2, \Theta; \ \Rightarrow \ \vartheta_1 \ ; \ \Upsilon$$

The rule **A.4** finally allows us to obtain the conlusion

$$\frac{\vdash_n \vartheta_1 \supset \vartheta_2, \Theta; \ \Rightarrow \ \vartheta_1 \ ; \ \Upsilon \qquad \vdash_n \vartheta_2, \Theta \ ; \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon}{\vdash_{n+1} \vartheta_1 \supset \vartheta_2, \Theta \ ; \ \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon}$$

Last rule used is A.9

$$\frac{\vdash_n \Theta, \vartheta_0 \cup \vartheta_1, \vartheta_0 \ ; \ \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon \qquad \vdash_n \Theta, \vartheta_0 \cup \vartheta_1, \vartheta_1 \ ; \ \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon}{\vdash_{n+1} \Theta, \vartheta_0 \cup \vartheta_1, \vartheta_0 \cup \vartheta_1 \ ; \ \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon}$$

By applying the inversion lemma to both premises we obtain for the left premise

$$\frac{\vdash_{n-1} \Theta, \vartheta_0, \vartheta_0 \ ; \ \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon \quad \vdash_{n-1} \Theta, \vartheta_1, \vartheta_0 \ ; \ \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon}{\vdash_n \Theta, \vartheta_0 \cup \vartheta_1, \vartheta_0; \ \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon}$$

from which, using the inductive hypothesis on the left premise we obtain

$$\vdash_{n-1} \Theta, \vartheta_0; \ \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon \qquad (1)$$

for the right premise we have

$$\frac{\vdash_{n-1} \Theta, \vartheta_0, \vartheta_1 \ ; \ \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon \quad \vdash_{n-1} \Theta, \vartheta_1, \vartheta_1 \ ; \ \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon}{\vdash_n \Theta, \vartheta_0 \cup \vartheta_1, \vartheta_1; \ \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon}$$

from which, using the inductive hypothesis on the right premise we obtain

$$\vdash_{n-1} \Theta, \vartheta_1; \ \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon \qquad (2)$$

using the rule A.9 with premises (1) and (2) allow us to obtain the conclusion

$$\frac{\vdash_{n-1} \Theta, \vartheta_0; \ \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon \quad \vdash_{n-1} \Theta, \vartheta_1 \ ; \ \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon}{\vdash_n \Theta, \vartheta_0 \cup \vartheta_1; \ \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon}$$

4 Cut-elimination for ILP

Given a derivation containing applications of the Cut rule, another derivation of the same sequent without applications of Cut (*Cut-free*) can be found. An immediate consequence of the cut-elimination theorem is the subformula property. Indeed every rule but Cut has the property that the sequent-premises consist of subformulas of the conclusion and as a consequence in a cut-free derivation d of $\Theta; \epsilon \Rightarrow \epsilon'; \Upsilon$ all the formulas occurring in d are subformulas of the formulas in $\Theta \cup (\epsilon \text{ or } \epsilon') \cup \Upsilon$. An important consequence of this theorem is the following: the full system **ILP** is a conservative extension of the ordinary intuitionistic fragment (based upon assertive connectives) and its dual (based on conjectural connectives).

4.1 Admissibility of "context-sharing" Cut

A rule of Cut is *additive* or *context-sharing* if the same sequences of formulas occur in the two premises and in the conclusion with the exception only of the cut-formulas and of formulas occurring in the privileged areas. For instance, the context-sharing rule cut_1 has the form

$$\frac{\Theta \;;\; \Rightarrow \; \vartheta \;;\; \Upsilon \qquad \vartheta, \Theta \;;\; \epsilon \; \Rightarrow \; \epsilon' \;;\; \Upsilon}{\Theta \;;\; \epsilon \; \Rightarrow \; \epsilon' \;;\; \Upsilon}$$

Lemma For all systems close under *dp-weakenig*, any proof-tree with instances of Cut may be transformed into a proof-tree with instances of context-sharing Cut only. Hence eliminability of Cut is a consequence of eliminability of contex-sharing Cut .
Proof. It sufficies to take a topmost instance of Cut in the proof-tree, e.g.,

$$\frac{\Theta \;;\; \Rightarrow \; \vartheta \;;\; \Upsilon \qquad \vartheta, \Theta' \;;\; \epsilon \; \Rightarrow \; \epsilon' \;;\; \Upsilon'}{\Theta, \Theta' \;;\; \epsilon \; \Rightarrow \; \epsilon' \;;\; \Upsilon, \Upsilon'}$$

After replacing the subdeduction d_1 and d_2 of the premises with $d_1\{\Theta' \;; \Rightarrow \;; \Upsilon'\}$, $d_2\{\Theta \;; \Rightarrow \;; \Upsilon\}$, we apply now a context-sharing Cut to these two new deductions and obtain a new deduction with the same conclusion $\Theta, \Theta' \;;\; \epsilon \; \Rightarrow \; \epsilon' \;;\; \Upsilon, \Upsilon'$. Thus proceeding in this way we may successively replace all the istances of Cut with a context-sharing Cut throughout the derivation. In our Cut-elimination proof we will use cut_1 and cut_2 in the context-sharing version.

Notations:

- Given a deduction d let $last(d)$ be the last rule used in d.
- The *logical complexity* of a formula δ is the number of pragmatic connectives it contains; thus
  - $|\eta| = 0$ for $\eta = \vdash \alpha,\ \varkappa\, \alpha$ oppure $\bigwedge$;
  - $|\delta_0 \circ \delta_1| = |\delta_0| + |\delta_1| + 1$ if $\circ$ is a binary connective;
  - $|\circ\, \delta| = |\delta| + 1$ if $\circ$ is a unary connective.
- The *rank* of a application of a *Cut* rule with cut-formula δ is $|\delta| + 1$.
- The *cut-rank* of a deduction d is the maximal rank of the applications of Cut within it. Hence if d_1 [and d_2] are the subdeductions of the premises of $last(d)$ then

$$cr(d) = \begin{cases} 0 & \text{if last}(d) \text{ is the identity axiom} \\ cr(d_1) & \text{if last}(d) \text{ is a unary rule} \\ max[cr(d_1), cr(d_2)] & \text{if last}(d) \text{ is a binary rule} \\ max[|\delta| + 1, cr(d_1), cr(d_2)] & \text{if last}(d) \text{ is a cut withc ut-formula } \delta \end{cases}$$

- The *depth* of a deduction d is its depth as a tree. More precisely, we have the following inductive definition, where d_1 [and d_2] are the derivation(s) of the sequent-premises of $last(d)$ (*immediate subderivations*):

$$|d| \begin{cases} 0 & \text{if last}(d)\text{=axiom} \\ |d_1| + 1 & \text{if last}(d)\text{=unary rule} \\ max(|d_1|, |d_2|) + 1 & \text{if last}(d)\text{=binary rule} \end{cases}$$

- The *level* of a Cut having as immediate subdeductions d_1 and d_2 is defined as the sum of the depths d_1 and d_2.

4.2 Cut Elimination Theorem

Theorem. All application of the Cut rules, except for those with elementary cut-formulas occurring in an axiom **S.5**, are eliminable in a **ILP** deduction.

Proof. Our strategy is to consider the Cuts whose rank is equal to the rank of the whole deduction, and among these to remove the Cuts which are highest in the proof-tree, hence the ones of a lowest level. To implement this strategy it sufficies to show how to replace a subdeduction d of the form

$$\frac{\Theta\ ;\ \Rightarrow\ \vartheta\ ;\ \Upsilon \qquad \vartheta, \Theta\ ;\ \epsilon\ \Rightarrow\ \epsilon'\ ;\ \Upsilon}{\Theta\ ;\ \epsilon\ \Rightarrow\ \epsilon'\ ;\ \Upsilon}$$

having $cr(d) = |\vartheta|+1$ with a new deduction d' having the same conclusion but cut-rank strictly lower: $cr(d') < cr(d)$. Note that the inequality is strict only if we proceed by successively deleting Cuts with maximal rank and minimum level. The proof uses a main induction on the cut-rank and a subinduction on the level of the Cut at the bottom of d. We have four different reductions.

4.2.1 Axiom reductions

The reduction steps of this form apply when one premise of a Cut is an axiom. These reductions lower the cut-rank by removing the Cut itself. We consider the cases of our axioms, with the exception of the **Axiom S.5** which is a *non-logical* axiom and it is not reduced.

Axiom S.1

$$\frac{\vartheta, \Theta; \Rightarrow \vartheta; \Upsilon \qquad \dfrac{\vdots}{\vartheta, \vartheta, \Theta; \epsilon \Rightarrow \epsilon'; \Upsilon}}{\vartheta, \Theta; \epsilon \Rightarrow \epsilon'; \Upsilon}\ cut_1$$

using dp-contraction on the right premise we have that $\vdash_n \vartheta, \vartheta, \Theta; \epsilon \Rightarrow \epsilon'; \Upsilon$ implies $\vdash_n \vartheta, \Theta; \epsilon \Rightarrow \epsilon'; \Upsilon$. We can take the latter as the new deduction d'. The cut-rank of d' is now strictly lower than $|\vartheta| + 1$ and the Cut has been deleted. Analogue considerations for the axiom **S.2**.

Axiom S.4

$$\dfrac{\begin{matrix} d_1 \\ \vdots \\ \Theta; \epsilon \Rightarrow \epsilon'; \Upsilon, \bigwedge \end{matrix} \qquad \Theta; \bigwedge \Rightarrow ; \Upsilon}{\Theta; \epsilon \Rightarrow \epsilon'; \Upsilon} \; cut_2$$

We have three subcases:

(i) the cut-formula is principal in the left sequent-premise, thus d_1 is a logical axiom **S.1** where ϵ is $\bigwedge$: we are back to the case of an axiom **S.1**, and the reduction yields the right sequent-premise (which is the same as the sequent-conclusion);

(ii) the cut-formula is not principal in the left sequent-premise, and d_1 is a logical axiom **S.1**, **S.2** or an absurdity axiom **S.3**, **S.4**, or a validity axiom **S.6**, **S.7**: the reduction yields the sequent-conclusion regarded as an axiom of the same nature as the left sequent-premise.

(iii) otherwise, d_1 ends with an inference whose principal formula is different from the cut-fomula $\bigwedge$: we apply a commutative reduction described below.

Notice that in cases (i) and (ii) the cut-rank is lowered, because a Cut has been deleted with cut-rank $|\bigwedge| + 1 = 1$.

The case where the right sequent-premise is an absurdity axiom **S.3** and the cut-formula $\bigwedge$ is eliminated using cut_1 is similar, but in subcase (ii) d_1 can only be an axiom **S.3** or **S.7**.

The treatment of validity axioms is dual to that of absurdity axioms and is omitted.

4.2.2 Commutative reductions

Commutative reductions permute a Cut inference above another inference; they apply when a sequent-premise of a Cut is the conclusion of a rule whose principal formula is different from the cut-formula and (an ancestor of) the cut-formula occurs in the sequent-premise. These reductions don't change the cut-rank of the proof, but reduce the level of the Cut; by iterating them, eventually we come to a situation where the cut-rank can be lowered either by a symmetric reduction or an axiom reductions or a weakening reduction.

Sometimes there are different ways of lowering the level of a Cut, by permuting it above an inference either in the left or in the right immediate subderivation; since the cut-elimination process requires structural transformations such as *dp-contraction* and weakening reductions, the non-determinism of commutative reductions may make the cut-elimination process *non-confluent* (as in the classical sequent calculus **LK**). However, since **ILP** is a polarized system, we can find a deterministic algorithm for commutative reductions. For $i = 0$ or 1, let d_i be the subderivation which end with the sequent-premise where the cut-formula occurs in the *privileged* area: (a) *we always permute first the Cut above the inferences in* d_i *whenever possible, and then* (b) *we permute the Cut above the inferences in* d_{1-i} *if possible.* Here we consider the case of commutation of Cuts cut_1 and cut_2 with an application of the rule **A.4** which illustrate the two cases.

A.4 – cut₁ *Case (a):*

$$\dfrac{\dfrac{\overset{\vdots\, d_{01}}{\vartheta_1 \supset \vartheta_2, \Theta;\ \Rightarrow \vartheta_1; \Upsilon} \qquad \overset{\vdots\, d_{02}}{\vartheta_2, \Theta;\ \Rightarrow \vartheta; \Upsilon}}{\vartheta_1 \supset \vartheta_2, \Theta;\ \Rightarrow \vartheta; \Upsilon}\,A.4 \qquad \overset{\vdots\, d_1}{\vartheta, \vartheta_1 \supset \vartheta_2, \Theta; \epsilon \Rightarrow \epsilon'; \Upsilon}}{\vartheta_1 \supset \vartheta_2, \Theta; \epsilon \Rightarrow \epsilon'; \Upsilon}\,cut_1$$

reduces to

$$\dfrac{\overset{\vdots\, d_{01}\{\vartheta_1 \supset \vartheta_2; \Rightarrow;\}}{\vartheta_1 \supset \vartheta_2, \vartheta_1 \supset \vartheta_2, \Theta; \Rightarrow \vartheta_1; \Upsilon} \qquad \dfrac{\overset{\vdots\, d_{02}\{\vartheta_1 \supset \vartheta_2;\}}{\vartheta_2, \vartheta_1 \supset \vartheta_2, \Theta;\ \Rightarrow \vartheta; \Upsilon} \qquad \overset{\vdots\, d_1\{\vartheta_2; \Rightarrow;\}}{\vartheta, \vartheta_2, \vartheta_1 \supset \vartheta_2, \Theta; \epsilon \Rightarrow \epsilon'; \Upsilon}}{\vartheta_2, \vartheta_1 \supset \vartheta_2, \Theta; \epsilon \Rightarrow \epsilon'; \Upsilon}\,cut_1}{\vartheta_1 \supset \vartheta_2, \vartheta_1 \supset \vartheta_2, \Theta; \epsilon \Rightarrow \epsilon'; \Upsilon}\,A.4$$

using closure under *dp-weakening* to obtain the weakened versions of d_{01}, d_{02} and d_1. To obtain the desired conclusion we apply *dp-contraction:*

$$\text{if} \quad \vdash_n \vartheta_1 \supset \vartheta_2, \vartheta_1 \supset \vartheta_2, \Theta; \epsilon \Rightarrow \epsilon'; \Upsilon \quad \text{then} \quad \vdash_n \vartheta_1 \supset \vartheta_2, \Theta; \epsilon \Rightarrow \epsilon'; \Upsilon$$

Case (b): If ϑ does not occur in the sequent premise of the last inference of d_0, then

$$\dfrac{\overset{\vdots\, d_0}{\vartheta_1 \supset \vartheta_2, \Theta;\ \Rightarrow \vartheta; \Upsilon} \qquad \dfrac{\overset{\vdots\, d_{11}}{\vartheta, \vartheta_1 \supset \vartheta_2, \Theta; \Rightarrow \vartheta_1; \Upsilon} \qquad \overset{\vdots\, d_{12}}{\vartheta, \vartheta_2, \Theta; \epsilon \Rightarrow \epsilon'; \Upsilon}}{\vartheta, \vartheta_1 \supset \vartheta_2, \Theta; \epsilon \Rightarrow \epsilon'; \Upsilon}\,A.4}{\vartheta_1 \supset \vartheta_2, \Theta; \epsilon \Rightarrow \epsilon'; \Upsilon}\,cut_1$$

reduces to

$$
\dfrac{
\dfrac{
\begin{array}{c}\vdots\, d_0\{\supset;\Rightarrow:\}\\ \supset^2,\Theta;\Rightarrow\vartheta;\Upsilon\end{array}
\quad
\begin{array}{c}\vdots\, d_{11}\{\supset;\Rightarrow:\}\\ \vartheta,\supset^2,\Theta;\Rightarrow\vartheta_1;\Upsilon\end{array}
}{\supset^2,\Theta;\Rightarrow\vartheta_1;\Upsilon}\,cut_1
\quad
\dfrac{
\begin{array}{c}\vdots\, d_0\{\vartheta_2;\Rightarrow;\}\\ \vartheta_2,\supset,\Theta;\Rightarrow\vartheta;\Upsilon\end{array}
\quad
\begin{array}{c}\vdots\, d_{12}\{\supset;\Rightarrow:\}\\ \vartheta,\vartheta_2,\supset,\Theta;\epsilon\Rightarrow\epsilon';\Upsilon\end{array}
}{\vartheta_2,\vartheta_1\supset\vartheta_2,\Theta;\epsilon\Rightarrow\epsilon';\Upsilon}\,cut_1
}{\vartheta_1\supset\vartheta_2,\vartheta_1\supset\vartheta_2,\Theta;\epsilon\Rightarrow\epsilon';\Upsilon}\,A.4
$$

where we write "$\supset$" for $\vartheta_1\supset\vartheta_2$ and "$\supset^2$" for $\vartheta_1\supset\vartheta_2,\vartheta_1\supset\vartheta_2$.

Finally, using *dp-contraction*, from $\vdash_n \vartheta_1\supset\vartheta_2,\vartheta_1\supset\vartheta_2,\Theta;\epsilon\Rightarrow\epsilon';\Upsilon$ we obtain $\vdash_n \vartheta_1\supset\vartheta_2,\Theta;\epsilon\Rightarrow\epsilon';\Upsilon$.

A.4 – cut$_2$ *Case (a):*

$$
\dfrac{
\begin{array}{c}\vdots\, d_0\\ \vartheta_1\supset\vartheta_2,\Theta;\epsilon\Rightarrow\epsilon';\Upsilon,\upsilon\end{array}
\quad
\dfrac{
\begin{array}{c}\vdots\, d_{11}\\ \vartheta_1\supset\vartheta_2,\Theta;\Rightarrow\vartheta_1;\Upsilon\end{array}
\quad
\begin{array}{c}\vdots\, d_{12}\\ \vartheta_2,\Theta;\upsilon\Rightarrow\ \Upsilon\end{array}
}{\vartheta_1\supset\vartheta_2,\Theta;\upsilon\Rightarrow;\Upsilon}\,A.4
}{\vartheta_1\supset\vartheta_2,\Theta;\epsilon\Rightarrow\epsilon';\Upsilon}\,cut_2
$$

reduces to

$$
\dfrac{
\begin{array}{c}\vdots\, d_{11}\{\vartheta_1\supset\vartheta_2;\Rightarrow;\}\\ \vartheta_1\supset\vartheta_2,\vartheta_1\supset\vartheta_2,\Theta;\Rightarrow\vartheta_1;\Upsilon\end{array}
\quad
\dfrac{
\begin{array}{c}\vdots\, d_0\{\vartheta_2;\Rightarrow;\}\\ \vartheta_2,\supset,\Theta;\epsilon\Rightarrow\epsilon';\Upsilon,\upsilon\end{array}
\quad
\begin{array}{c}\vdots\, d_{12}\{\vartheta_1\supset\vartheta_2;\Rightarrow;\}\\ \vartheta_2,\supset,\Theta;\upsilon\Rightarrow;\Upsilon\end{array}
}{\vartheta_2,\vartheta_1\supset\vartheta_2,\Theta;\epsilon\Rightarrow\epsilon';\Upsilon}\,cut_2
}{\vartheta_1\supset\vartheta_2,\vartheta_1\supset\vartheta_2,\Theta;\epsilon\Rightarrow\epsilon';\Upsilon}\,A.4
$$

Case (b): If υ does not occur in the sequent premise of the last inference of d_1, then

$$
\dfrac{
\dfrac{
\begin{array}{c}\vdots\, d_{01}\\ \vartheta_1\supset\vartheta_2,\Theta;\Rightarrow\vartheta_1;\Upsilon\upsilon\end{array}
\quad
\begin{array}{c}\vdots\, d_{02}\\ \vartheta_2,\Theta;\epsilon\Rightarrow\epsilon';\Upsilon,\upsilon\end{array}
}{\vartheta_1\supset\vartheta_2,\Theta;\epsilon\Rightarrow\epsilon';\Upsilon,\upsilon}\,A.4
\quad
\begin{array}{c}\vdots\, d_1\\ \vartheta_1\supset\vartheta_2,\Theta;\upsilon\Rightarrow;\Upsilon\end{array}
}{\vartheta_1\supset\vartheta_2,\Theta;\epsilon\Rightarrow\epsilon';\Upsilon}\,cut_2
$$

reduces to

$$
\dfrac{
\dfrac{
\begin{array}{c}\vdots\, d'_{01}\\ \supset^2,\Theta;\Rightarrow\vartheta_1;\Upsilon,\upsilon\end{array}
\quad
\begin{array}{c}\vdots\, d'_1\\ \supset^2,\Theta;\upsilon\Rightarrow;\Upsilon\end{array}
}{\vartheta_1\supset\vartheta_2,\vartheta_1\supset\vartheta_2,\Theta;\Rightarrow\vartheta_1;\Upsilon}\,cut_2
\quad
\dfrac{
\begin{array}{c}\vdots\, d'_{02}\\ \vartheta_2,\supset,\Theta;\epsilon\Rightarrow\epsilon';\Upsilon,\upsilon\end{array}
\quad
\begin{array}{c}\vdots\, d''_1\\ \vartheta_2,\supset,\Theta;\upsilon\Rightarrow;\Upsilon\end{array}
}{\vartheta_2,\vartheta_1\supset\vartheta_2,\Theta;\epsilon\Rightarrow\epsilon';\Upsilon}\,cut_2
}{\vartheta_1\supset\vartheta_2,\vartheta_1\supset\vartheta_2,\Theta;\epsilon\Rightarrow\epsilon';\Upsilon}\,A.4
$$

where $d'_{01} = d_{01}\{\vartheta_1 \supset \vartheta_2; \Rightarrow;\}$, $d'_{02} = d_{02}\{\vartheta_1 \supset \vartheta_2; \Rightarrow;\}$
$d''_1 = d_1\{\vartheta_2; \Rightarrow;\}$, $d'_1 = d_1\{\vartheta_1 \supset \vartheta_2; \Rightarrow;\}$

4.2.3 Symmetric reduction

These reductions apply when both cut-formulas are the principal formula of the last inference in both immediate subderivations. In order to achieve determinism of the cut-elimination procedure, when the given Cut is replaced with new Cuts, including more than one Cut of lower rank, we must *fix the order* in which these new Cuts occur: on one hand, a Cut with the same rank must obviously occur highest (in this way its level is reduced); on the other hand, Cuts of lower rank may be applied (from top down) in the typographical order of the occurrences of the new cut-formulas a subformulas of the given cut-formula, if the latter is assertive, and in the reverse order if the given cut-formula is conjectural, as it is shown in the examples below.

A.4-A.3

$$\dfrac{\dfrac{\overset{\vdots\ d_1}{\vartheta_1, \Theta; \Rightarrow \vartheta_2; \Upsilon}}{\Theta; \Rightarrow \vartheta_1 \supset \vartheta_2; \Upsilon}\,A.3 \qquad \dfrac{\overset{\vdots\ d_2}{\vartheta_1 \supset \vartheta_2, \Theta; \Rightarrow \vartheta_1; \Upsilon} \quad \overset{\vdots\ d_3}{\vartheta_2, \Theta; \epsilon \Rightarrow \epsilon'; \Upsilon}}{\vartheta_1 \supset \vartheta_2, \Theta; \epsilon \Rightarrow \epsilon'; \Upsilon}\,A.4}{\Theta; \epsilon \Rightarrow \epsilon'; \Upsilon}\,cut_1$$

reduces in

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{\vdots\ d_1}{\Theta, \vartheta_1; \Rightarrow \vartheta_2; \Upsilon}}{\Theta; \Rightarrow \vartheta_1 \supset \vartheta_2; \Upsilon}\,A.4 \quad \overset{\vdots\ d_2}{\vartheta_1 \supset \vartheta_2, \Theta; \Rightarrow \vartheta_1; \Upsilon}}{\Theta; \Rightarrow \vartheta_1; \Upsilon}\,cut_1 \quad \overset{\vdots\ d_1}{\Theta, \vartheta_1; \Rightarrow \vartheta_2; \Upsilon}}{\Theta; \Rightarrow \vartheta_2; \Upsilon}\,cut_1 \quad \overset{\vdots\ d_3}{\vartheta_2, \Theta; \epsilon \Rightarrow \epsilon'; \Upsilon}}{\Theta; \epsilon \Rightarrow \epsilon'; \Upsilon}\,cut_1$$

the cut-level on $\vartheta_1 \supset \vartheta_2$ has been lowered from $|d_1| + max(|d_2|, |d_3|) + 2$ to $|d_1| + |d_2| + 1$. The two new Cuts introduced have cut-rank $|\vartheta_2| + 1$ and $|\vartheta_1| + 1$ both lower than the removed Cut with cut-rank $|\vartheta_1 \supset \vartheta_2| + 1$.

ACC.1-ACC.2

$$\dfrac{\dfrac{\overset{\vdots\ d_0}{\vartheta, \Theta; \epsilon \Rightarrow \epsilon'; \Upsilon, \upsilon}}{\Theta; \epsilon \Rightarrow \epsilon'; \Upsilon, \vartheta \succ \upsilon}\,ACC.1 \qquad \dfrac{\overset{\vdots\ d_1}{\Theta; \Rightarrow \vartheta; \Upsilon} \quad \overset{\vdots\ d_2}{\Theta; \upsilon \Rightarrow; \Upsilon}}{\Theta; \vartheta \succ \upsilon \Rightarrow; \Upsilon}\,ACC.2}{\Theta; \epsilon \Rightarrow \epsilon'; \Upsilon}\,cut_2$$

in this case we do not introduce a new Cut of the same rank:

$$\dfrac{\begin{matrix}\vdots\, d_1\\ \Theta; \Rightarrow \vartheta; \Upsilon\end{matrix} \quad \dfrac{\begin{matrix}\vdots\, d_0\\ \vartheta, \Theta; \epsilon \Rightarrow \epsilon'; \Upsilon, \upsilon\end{matrix} \quad \begin{matrix}\vdots\, d_2\{\vartheta; \Rightarrow;\}\\ \vartheta, \Theta; \upsilon \Rightarrow; \Upsilon\end{matrix}}{\vartheta, \Theta; \epsilon \Rightarrow \epsilon'; \Upsilon}\, cut_2}{\Theta; \epsilon \Rightarrow \epsilon'; \Upsilon}\, cut_1$$

4.2.4 Weakening reductions

These reductions directly delete the Cut and also part of the derivation; they apply when a cut-formula is introduced by an implicit weakening in the privileged affine area of a sequent-premise of the Cut. The significant new cases are those in which the other sequent premise is either an axiom where the cut-formula is not principal (*case ii*) or is a consequence of a rule of inference where the cut-formula is the principal formula (*case iv*).

R − cut₁

$$\dfrac{\dfrac{\begin{matrix}\vdots\, d_0\\ \Theta\ ; \epsilon_1 \Rightarrow\ ; \Upsilon_1\end{matrix}}{\Theta\ ;\ \Rightarrow\ \vartheta; \Upsilon}\, R \qquad \begin{matrix}\vdots\, d_1\\ \vartheta, \Theta\ ;\ \epsilon\ \Rightarrow\ \epsilon'\ ; \Upsilon\end{matrix}}{\Theta\ ;\ \epsilon\ \Rightarrow\ \epsilon'\ ; \Upsilon}\, cut_1$$

There are four cases:
Case (i): d_1 is a logical axiom **S.1** with principal formula ϑ: then we are back to an axiom reduction and the given derivation reduces to

$$\dfrac{\begin{matrix}\vdots\, d_0\\ \Theta\ ; \epsilon_1 \Rightarrow\ ; \Upsilon_1\end{matrix}}{\Theta\ ; \epsilon \Rightarrow\ \epsilon'; \Upsilon}\, R$$

Case (ii): d_1 is an axiom **S.2**, **S.3**, **S.4**, **S.6** or **S.7**: then d_0 and the Cut inference are deleted and the conclusion is an axiom of the same nature as d_1.
Case (iii): otherwise, if the last inference of d_1 does not have ϑ as principal formula, then we are back to a commutative reduction.
Case (iv): finally, if ϑ is the principal formula of the last inference of d_1, say an application of **ACA.4** with $\vartheta = \vartheta' \cap \upsilon'$ and with sequent-premise

$$\dfrac{\begin{matrix}\vdots\, d_1'\\ \vartheta, \vartheta', \Theta\ ;\ \upsilon'\ \Rightarrow\ ; \Upsilon\end{matrix}}{\vartheta, \Theta\ ;\ \epsilon\ \Rightarrow\ \epsilon'\ ; \Upsilon}\, ACA.4$$

then the derivation reduces to

$$
\dfrac{
\dfrac{\overset{\vdots\; d_0}{\Theta\, ; \epsilon_1 \Rightarrow ; \Upsilon_1}}{\Theta\, ; \;\Rightarrow ; \Upsilon, \upsilon'}\,R
\qquad
\dfrac{
\dfrac{\overset{\vdots\; d_0}{\Theta\, ; \epsilon_1 \Rightarrow ; \Upsilon_1}}{\Theta\, ; \;\Rightarrow\; \vartheta' ; \Upsilon}\,R
\qquad
\dfrac{
\dfrac{\overset{\vdots\; d_0\{\vartheta'; \Rightarrow ; \}}{\vartheta', \Theta\, ; \epsilon_1 \Rightarrow ; \Upsilon_1}}{\vartheta', \Theta\, ; \;\Rightarrow\; \vartheta ; \Upsilon}\,R
\qquad
\overset{\vdots\; d'_1}{\vartheta, \vartheta', \Theta\, ;\; \upsilon' \;\Rightarrow ; \Upsilon}
}{\vartheta', \Theta\, ;\; \upsilon' \;\Rightarrow ;\; \Upsilon}\,cut''_1
}{\Theta\, ;\; \upsilon' \;\Rightarrow ;\; \Upsilon}\,cut'_1
}{\Theta\, ;\; \epsilon \;\Rightarrow\; \epsilon' ; \Upsilon}\,cut_2
$$

The inference cut''_1 has lower level that the given cut_1, while cut'_1 and cut_2 have lower cut-rank.

References

[1] G. Bellin, Towards a formal pragmatics: An intuitionistic theory of assertive and conjectural operators, with an extension of Gödel, McKinsey and Tarski's S4 translation, in Gorè R., Mendler M., de Paiva V. (Hrsg.) *Proceedings of the International Workshop on Intuitionistic Modal Logic and Applications (IMLA'02)*, Copenhagen, July 2002, Bamberger Beiträge zur Wirtschaftsinformatik und Angewandten Informatik, 61 (2002), Otto-Friedrich-Universität Bamberg

[2] G. Bellin and C. Dalla Pozza, A pragmatic interpretation of substructural logics, *Feferman Festschrift*, 2001.

[3] Corrado Biasi *Verso una logica degli operatori prammatici: asserzioni e congetture*, tesi di laurea in the Computer Science department of the Science university of Verona 2001-2002.

[4] T. Crolard. *Extension de l'isomorphisme de Curry-Howard au traitement des exceptions (application d'une ètude de la dualité en logique intuitionniste)*. Thèse de Doctorat, Université de Paris 7, 1996.

[5] J-Y. Girard. On the unity of logic. *Annals of Pure and Applied Logic* 59, 1993, pp.201-217.

[6] S. A. Kripke, Semantical analysis of modal logic I: Normal modal propostional calculi, *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, 9, 1963.

[7] G. Reyes and H. Zolfaghari, Bi-Heyting algebras, Toposes and Modalities, in *Journal of Philosophical Logic*, 25, 1996, pp. 25-43.

[8] A. S. Troelstra and H. Schwichtenberg, *Basic Proof Theory*, Cambridge Tracts in Theoretical Computer Science 43, Cambridge University Press, Second Edition, 2000.

[9] Steven Vickers *Topology Via Logic*, Cambridge Tracts in Theoretical Computer Science 5, Cambridge University Press, 1989.

5 Appendix

S.1: logical axiom:
$$\Theta, \vartheta \; ; \; \Rightarrow \; \vartheta \; ; \Upsilon$$

S.2: logical axiom:
$$\Theta; \; \upsilon \; \Rightarrow \; ; \; \upsilon, \Upsilon$$

S.3: absurdity axiom:
$$\Theta, \bigwedge \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon$$

S.4: absurdity axiom:
$$\Theta \; ; \; \bigwedge \; \Rightarrow \; ; \; \Upsilon$$

S.5: assertion-conjecture:
$$\Theta, \; \vdash \alpha \; ; \; \Rightarrow \; ; \; \mathcal{H}\,\alpha, \Upsilon$$

S.6: validity axiom:
$$\Theta \; ; \; \Rightarrow \; \bigvee \; ; \; \Upsilon$$

S.7: validity axiom:
$$\Theta \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon, \bigvee$$

S.8: cut$_1$:
$$\frac{\Theta \; ; \; \Rightarrow \; \vartheta \; ; \; \Upsilon \qquad \vartheta, \Theta' \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon'}{\Theta, \Theta' \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon, \Upsilon'}$$

S.9: cut$_2$:
$$\frac{\Theta \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon, \upsilon \qquad \Theta' \; ; \; \upsilon \; \Rightarrow \; \Upsilon'}{\Theta, \Theta' \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon, \Upsilon'}$$

S.10: exchange left:
$$\frac{\Theta, \vartheta_1, \vartheta_2, \Theta' \; ; \; \epsilon \Rightarrow \; \epsilon' \; ; \; \Upsilon}{\Theta, \vartheta_2, \vartheta_1, \Theta' \; ; \; \epsilon \; \Rightarrow \; \epsilon' \; ; \; \Upsilon'}$$

S.11: exchange right:
$$\frac{\Theta \; ; \; \epsilon \Rightarrow \; \epsilon' \; ; \; \Upsilon, \upsilon_1, \upsilon_2, \Upsilon'}{\Theta \; ; \; \epsilon \Rightarrow \; \epsilon' \; ; \; \Upsilon, \upsilon_2, \upsilon_1, \Upsilon'}$$

Table 2: **ILP**, identity and structural rules

logical rules, connectives of type $\vartheta \to \vartheta$

A.1: ∼R:

$$\frac{\Theta, \vartheta \ ; \ \Rightarrow \ ; \ \Upsilon}{\Theta \ ; \ \Rightarrow \sim \vartheta \ ; \ \Upsilon}$$

A.2: ∼L:

$$\frac{\sim \vartheta, \Theta; \ \Rightarrow \ \vartheta \ ; \ \Upsilon}{\sim \vartheta, \Theta \ ; \epsilon \Rightarrow \epsilon'; \Upsilon}$$

logical rules, connectives of type $\vartheta \times \vartheta \to \vartheta$

A.3: ⊃R:

$$\frac{\Theta, \vartheta_1 \ ; \ \Rightarrow \ \vartheta_2 \ ; \ \Upsilon}{\Theta \ ; \ \Rightarrow \ \vartheta_1 \supset \vartheta_2 \ ; \ \Upsilon}$$

A.4: ⊃L:

$$\frac{\vartheta_1 \supset \vartheta_2, \Theta; \ \Rightarrow \ \vartheta_1 \ ; \ \Upsilon \qquad \vartheta_2, \Theta \ ; \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon}{\vartheta_1 \supset \vartheta_2, \Theta \ ; \ \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon}$$

A.5: ∩R:

$$\frac{\Theta \ ; \ \Rightarrow \ \vartheta_1 \ ; \ \Upsilon \qquad \Theta \ ; \ \Rightarrow \ \vartheta_2 \ ; \ \Upsilon}{\Theta \ ; \ \Rightarrow \ \vartheta_1 \cap \vartheta_2 \ ; \ \Upsilon}$$

A.6: ∩L:

$$\frac{\Theta, \vartheta_0, \vartheta_1 \ ; \ \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon}{\Theta, \vartheta_0 \cap \vartheta_1 \ ; \ \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon}$$

A.7,8: ∪R:

$$\frac{\Theta \ ; \ \Rightarrow \ \vartheta_i \ ; \ \Upsilon}{\Theta \ ; \ \Rightarrow \ \vartheta_0 \cup \vartheta_1 \ ; \ \Upsilon}$$

for $i = 0, 1$.

A.9: ∪L:

$$\frac{\Theta, \vartheta_0 \ ; \ \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon \quad \Theta, \vartheta_1 \ ; \ \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon}{\Theta, \vartheta_0 \cup \vartheta_1 \ ; \ \epsilon \ \Rightarrow \ \epsilon' \ ; \ \Upsilon}$$

Table 3: **ILP**, the assertive fragment.

logical rules, connectives of type $\upsilon \to \upsilon$

C.1: $\frown R$:

$$\frac{\Theta \; ; \; \upsilon \;\Rightarrow ; \; \Upsilon, \frown \upsilon}{\Theta \; ; \epsilon \Rightarrow \;\; \epsilon'; \Upsilon, \frown \upsilon}$$

C.2: $\frown L$

$$\frac{\Theta; \;\Rightarrow \; ; \; \Upsilon, \upsilon}{\Theta \; ; \frown \upsilon \;\; \Rightarrow ; \; \Upsilon}$$

logical rules, connectives of type $\upsilon \times \upsilon \to \upsilon$

C.3: $\succ R$:

$$\frac{\Theta \; ; \; \epsilon \;\Rightarrow\; \epsilon' \; ; \; \upsilon_2, \Upsilon, \upsilon_1 \succ \upsilon_2}{\Theta \; ; \; \epsilon \;\Rightarrow\; \epsilon' \; ; \; \Upsilon, \upsilon_1 \succ \upsilon_2}$$

C.4: $\succ R$:

$$\frac{\Theta \; ; \; \upsilon_1 \;\Rightarrow \; ; \; \Upsilon, \upsilon_2, \upsilon_1 \succ \upsilon_2}{\Theta \; ; \epsilon \Rightarrow \epsilon'; \; \Upsilon, \upsilon_1 \succ \upsilon_2}$$

C.5: $\succ L$:

$$\frac{\Theta \; ; \;\Rightarrow\; ; \; \Upsilon, \upsilon_1 \qquad \Theta \; ; \upsilon_2 \;\Rightarrow \; ; \; \Upsilon}{\Theta \; ; \; \upsilon_1 \succ \upsilon_2 \;\Rightarrow \; ; \; \Upsilon}$$

C.6: $\curlywedge R$:

$$\frac{\Theta \; ; \; \epsilon \;\Rightarrow\; \epsilon' \; ; \; \Upsilon, \upsilon_0 \qquad \Theta \; ; \; \epsilon \;\Rightarrow\; \epsilon' \; ; \; \Upsilon, \upsilon_1}{\Theta \; ; \; \epsilon \;\Rightarrow\; \epsilon' \; ; \; \Upsilon, \upsilon_0 \curlywedge \upsilon_1}$$

C.7,8: $\curlywedge L$:

$$\frac{\Theta \; ; \; \upsilon_i \;\Rightarrow \; ; \; \Upsilon}{\Theta \; ; \; \upsilon_0 \curlywedge \upsilon_1 \;\Rightarrow \; ; \; \Upsilon}$$

for $i = 0, 1$.

C.9: $\curlyvee R$:

$$\frac{\Theta \; ; \; \epsilon \;\Rightarrow\; \epsilon' \; ; \; \Upsilon, \upsilon_1, \upsilon_2}{\Theta \; ; \; \epsilon \;\Rightarrow\; \epsilon' \; ; \; \Upsilon, \upsilon_1 \curlyvee \upsilon_2}$$

C.10: $\curlyvee L$:

$$\frac{\Theta \; ; \; \upsilon_1 \;\Rightarrow \; ; \; \Upsilon \qquad \Theta \; ; \; \upsilon_2 \;\Rightarrow \; ; \; \Upsilon}{\Theta \; ; \; \upsilon_1 \curlyvee \upsilon_2 \;\Rightarrow \; ; \; \Upsilon}$$

Table 4: **ILP**, the conjectural fragment.

logical rules, connectives of type $\upsilon \to \vartheta$

CA.1: right negation:
$$\frac{\Theta\ ;\ \upsilon\ \Rightarrow\ ;\ \Upsilon}{\Theta\ ;\ \Rightarrow \sim\upsilon\ ;\ \Upsilon}$$

CA.2: left negation
$$\frac{\Theta;\ \epsilon\ \Rightarrow\ \epsilon'\ ;\ \Upsilon, \upsilon}{\sim\upsilon, \Theta\ ;\ \epsilon\ \Rightarrow\ \epsilon'\ ;\ \Upsilon}$$

logical rules, connectives of type $\vartheta \times \upsilon \to \vartheta$

ACA.1: $\supset$R:
$$\frac{\Theta, \vartheta\ ;\ \Rightarrow\ ;\ \Upsilon, \upsilon}{\Theta\ ;\ \Rightarrow\ \vartheta \supset \upsilon\ ;\ \Upsilon}$$

ACA.2: $\supset$L:
$$\frac{\vartheta \supset \upsilon, \Theta;\ \Rightarrow \vartheta\ ;\ \Upsilon \qquad \vartheta \supset \upsilon, \Theta\ ;\upsilon\ \Rightarrow\ ;\ \Upsilon}{\vartheta \supset \upsilon, \Theta\ ;\epsilon \Rightarrow \epsilon';\ \Upsilon}$$

ACA.3: $\cap$R:
$$\frac{\Theta\ ;\ \Rightarrow\ \vartheta\ ;\ \Upsilon \qquad \Theta\ ;\ \Rightarrow\ ;\ \upsilon, \Upsilon}{\Theta\ ;\ \Rightarrow\ \vartheta \cap \upsilon\ ;\ \Upsilon}$$

ACA.4: $\cap$L:
$$\frac{\Theta, \vartheta \cap \upsilon, \vartheta\ ;\ \upsilon\ \Rightarrow\ ;\ \Upsilon}{\Theta, \vartheta \cap \upsilon\ ;\epsilon \Rightarrow \epsilon';\ \Upsilon}$$

ACA.5: $\cap$L:
$$\frac{\Theta, \vartheta \cap \upsilon, \vartheta\ ;\ \epsilon\ \Rightarrow\ \epsilon'\ ;\ \Upsilon}{\Theta, \vartheta \cap \upsilon\ ;\ \epsilon\ \Rightarrow\ \epsilon'\ ;\ \Upsilon}$$

ACA.6: $\cup$R:
$$\frac{\Theta\ ;\ \Rightarrow\ \vartheta\ ;\ \Upsilon}{\Theta\ ;\ \Rightarrow\ \vartheta \cup \upsilon\ ;\ \Upsilon}$$

ACA.7: $\cup$R:
$$\frac{\Theta\ ;\ \Rightarrow\ ;\ \upsilon, \Upsilon}{\Theta\ ;\ \Rightarrow\ \vartheta \cup \upsilon\ ;\ \Upsilon}$$

ACA.8: $\cup$L:
$$\frac{\Theta, \vartheta\ ;\ \epsilon\ \Rightarrow\ \epsilon'\ ;\ \Upsilon \quad \Theta, \vartheta \cup \upsilon;\ \upsilon\ \Rightarrow\ ;\ \Upsilon}{\Theta, \vartheta \cup \upsilon\ ;\ \epsilon\ \Rightarrow\ \epsilon'\ ;\ \Upsilon}$$

logical rules, connectives of type $\upsilon \times \vartheta \to \vartheta$

CAA.1: $\supset$R:
$$\frac{\Theta\ ;\ \Rightarrow\ \vartheta\ ;\ \Upsilon}{\Theta\ ;\ \Rightarrow\ \upsilon \supset \vartheta\ ;\ \Upsilon}$$

CAA.2: $\supset$R:
$$\frac{\Theta\ ;\ \upsilon\ \Rightarrow\ ;\ \Upsilon}{\Theta\ ;\ \Rightarrow\ \upsilon \supset \vartheta\ ;\ \Upsilon}$$

CAA.3: $\supset$L:
$$\frac{\Theta;\ \epsilon\ \Rightarrow\ \epsilon'\ ;\ \Upsilon, \upsilon \quad \vartheta, \Theta\ ;\epsilon\ \Rightarrow\ \epsilon'\ ;\ \Upsilon}{\upsilon \supset \vartheta, \Theta\ ;\ \epsilon\ \Rightarrow\ \epsilon'\ ;\ \Upsilon}$$

CAA.4: $\cap$R:
$$\frac{\Theta\ ;\ \Rightarrow\ ;\ \upsilon, \Upsilon \qquad \Theta\ ;\ \Rightarrow\ \vartheta\ ;\ \Upsilon}{\Theta\ ;\ \Rightarrow\ \upsilon \cap \vartheta\ ;\ \Upsilon}$$

CAA.5: $\cap$L:
$$\frac{\Theta, \upsilon \cap \vartheta, \vartheta\ ;\ \upsilon\ \Rightarrow\ ;\ \Upsilon}{\Theta, \upsilon \cap \vartheta\ ;\epsilon \Rightarrow\ ;\epsilon'\Upsilon}$$

CAA.6: $\cap$L:
$$\frac{\Theta, \upsilon \cap \vartheta, \vartheta\ ;\ \epsilon\ \Rightarrow\ \epsilon'\ ;\ \Upsilon}{\Theta, \upsilon \cap \vartheta\ ;\ \epsilon\ \Rightarrow\ \epsilon'\ ;\ \Upsilon}$$

CAA.7: $\cup$R:
$$\frac{\Theta\ ;\ \Rightarrow\ ;\ \upsilon, \Upsilon}{\Theta\ ;\ \Rightarrow\ \upsilon \cup \vartheta\ ;\ \Upsilon}$$

CAA.8: $\cup$R:
$$\frac{\Theta\ ;\ \Rightarrow\ \vartheta\ ;\ \Upsilon}{\Theta\ ;\ \Rightarrow\ \upsilon \cup \vartheta\ ;\ \Upsilon}$$

CAA.9: $\cup$L:
$$\frac{\Theta, \upsilon \cup \vartheta;\ \upsilon\ \Rightarrow\ ;\ \Upsilon \quad \Theta, \vartheta\ ;\ \epsilon\ \Rightarrow\ \epsilon'\ ;\ \Upsilon}{\Theta, \upsilon \cup \vartheta\ ;\ \epsilon\ \Rightarrow\ \epsilon'\ ;\ \Upsilon}$$

logical rules, connectives of type $\upsilon \times \upsilon \to \vartheta$

CCA.1: $\supset$R:
$$\frac{\Theta\ ;\ \upsilon_1\ \Rightarrow\ ;\ \Upsilon, \upsilon_2}{\Theta\ ;\ \Rightarrow\ \upsilon_1 \supset \upsilon_2\ ;\ \Upsilon}$$

CCA.2: $\supset$L:
$$\frac{\Theta;\ \epsilon\ \Rightarrow\ \epsilon'\ ;\ \Upsilon, \upsilon_1 \qquad \upsilon_1 \supset \upsilon_2, \Theta\ ;\upsilon_2\ \Rightarrow\ ;\ \Upsilon}{\upsilon_1 \supset \upsilon_2, \Theta\ ;\ \epsilon\ \Rightarrow\ \epsilon'\ ;\ \Upsilon}$$

CCA.3: $\cap$R:
$$\frac{\Theta\ ;\ \Rightarrow\ ;\ \upsilon_0, \Upsilon \qquad \Theta\ ;\ \Rightarrow\ ;\ \upsilon_1, \Upsilon}{\Theta\ ;\ \Rightarrow\ \upsilon_0 \cap \upsilon_1\ ;\ \Upsilon}$$

CCA.4,5: $\cap$L:
$$\frac{\Theta, \upsilon_0 \cap \upsilon_1\ ;\ \upsilon_i\ \Rightarrow\ ;\ \Upsilon}{\Theta, \upsilon_0 \cap \upsilon_1\ ;\epsilon \Rightarrow\ ;\epsilon'\Upsilon}$$

CCA.6,7: $\cup$R:
$$\frac{\Theta\ ;\ \Rightarrow\ ;\ \upsilon_i, \Upsilon}{\Theta\ ;\ \Rightarrow\ \upsilon_0 \cup \upsilon_1\ ;\ \Upsilon}$$

CCA.8: $\cup$L:
$$\frac{\Theta, \upsilon_1 \cup \upsilon_2;\ \upsilon_1\ \Rightarrow\ ;\ \Upsilon \quad \Theta, \upsilon_1 \cup \upsilon_2;\ \upsilon_2\ \Rightarrow\ ;\ \Upsilon}{\Theta, \upsilon_1 \cup \upsilon_2\ ;\epsilon \Rightarrow \epsilon';\ \Upsilon}$$

Table 5: **ILP**, assertive mixed rules

logical rules, connectives of type $\vartheta \to \upsilon$

AC1: right doubt:
$$\frac{\Theta, \vartheta \,;\, \epsilon \;\Rightarrow\; \epsilon' \,;\, \Upsilon}{\Theta \,; \epsilon \Rightarrow\; \epsilon';\; \Upsilon, \frown \vartheta}$$

AC.2: left doubt
$$\frac{\Theta; \;\Rightarrow\; \vartheta \,;\, \Upsilon}{\Theta \,;\, \frown \vartheta \;\Rightarrow\,;\, \Upsilon}$$

logical rules, connectives of type $\vartheta \times \upsilon \to \upsilon$

ACC.1: $\succ$R:
$$\frac{\Theta, \vartheta \,;\, \epsilon \;\Rightarrow\; \epsilon' \,;\, \Upsilon, \upsilon}{\Theta \,;\, \epsilon \;\Rightarrow\; \epsilon' \,;\, \Upsilon, \vartheta \succ \upsilon}$$

ACC.2: $\succ$L:
$$\frac{\Theta; \;\Rightarrow \vartheta \,;\, \Upsilon \qquad \Theta; \upsilon \;\Rightarrow\,;\, \Upsilon}{\Theta \,;\, \vartheta \succ \upsilon \;\Rightarrow\,;\, \Upsilon}$$

ACC.3: $\curlywedge$R:
$$\frac{\Theta \,;\, \Rightarrow\; \vartheta \,;\, \Upsilon, \vartheta \curlywedge \upsilon \qquad \Theta \,;\, \epsilon \;\Rightarrow\; \epsilon' \,;\, \Upsilon, \upsilon}{\Theta \,;\, \epsilon \;\Rightarrow\; \epsilon' \,;\, \Upsilon, \vartheta \curlywedge \upsilon}$$

ACC.4: $\curlywedge$L:
$$\frac{\Theta, \vartheta \,;\, \Rightarrow\,;\, \Upsilon}{\Theta \,;\, \vartheta \curlywedge \upsilon \;\Rightarrow\,;\, \Upsilon}$$

ACC.5: $\curlywedge$L:
$$\frac{\Theta \,;\, \upsilon \;\Rightarrow\,;\, \Upsilon}{\Theta \,;\, \vartheta \curlywedge \upsilon \;\Rightarrow\,;\, \Upsilon}$$

ACC.6: $\curlyvee$R:
$$\frac{\Theta \,;\, \Rightarrow\; \vartheta \,;\, \Upsilon, \upsilon, \vartheta \curlyvee \upsilon}{\Theta \,; \epsilon \Rightarrow \epsilon';\; \Upsilon, \vartheta \curlyvee \upsilon}$$

ACC.7: $\curlyvee$R:
$$\frac{\Theta \,;\, \epsilon \;\Rightarrow\; \epsilon' \,;\, \Upsilon, \upsilon, \vartheta \curlyvee \upsilon}{\Theta \,;\, \epsilon \;\Rightarrow\; \epsilon' \,;\, \Upsilon, \vartheta \curlyvee \upsilon}$$

ACC.8: $\curlyvee$L:
$$\frac{\Theta, \vartheta \,;\, \Rightarrow\,;\, \Upsilon \qquad \Theta \,;\, \upsilon \;\Rightarrow\,;\, \Upsilon}{\Theta \,;\, \vartheta \curlyvee \upsilon \;\Rightarrow\,;\, \Upsilon}$$

logical rules, connectives of type $\upsilon \times \vartheta \to \upsilon$

CAC.1: $\succ$R:
$$\frac{\Theta \,;\, \Rightarrow\; \vartheta \,;\, \Upsilon, \upsilon \succ \vartheta}{\Theta \,; \epsilon \Rightarrow \epsilon';\; \Upsilon, \upsilon \succ \vartheta}$$

CAC.2: $\succ$L:
$$\frac{\Theta \,;\, \upsilon \;\Rightarrow\,;\, \Upsilon, \upsilon \succ \vartheta}{\Theta \,; \epsilon \Rightarrow \epsilon';\; \Upsilon, \upsilon \succ \vartheta}$$

CAC.3: $\succ$L:
$$\frac{\Theta; \;\Rightarrow\,;\, \Upsilon, \upsilon \qquad \vartheta, \Theta \,;\, \Rightarrow\,;\, \Upsilon}{\Theta \,;\, \upsilon \succ \vartheta \;\Rightarrow\,;\, \Upsilon}$$

CAC.4: $\curlywedge$R:
$$\frac{\Theta \,;\, \epsilon \;\Rightarrow\; \epsilon' \,;\, \Upsilon, \upsilon \qquad \Theta \,;\, \Rightarrow\; \vartheta \,;\, \Upsilon, \upsilon \curlywedge \vartheta}{\Theta \,;\, \epsilon \;\Rightarrow\; \epsilon' \,;\, \Upsilon, \upsilon \curlywedge \vartheta}$$

CAC.5: $\curlywedge$L:
$$\frac{\Theta \,;\, \upsilon \;\Rightarrow\,;\, \Upsilon}{\Theta \,;\, \upsilon \curlywedge \vartheta \;\Rightarrow\,;\, \Upsilon}$$

CAC.6: $\curlywedge$L:
$$\frac{\Theta, \vartheta \,;\, \Rightarrow\,;\, \Upsilon}{\Theta \,;\, \upsilon \curlywedge \vartheta \;\Rightarrow\,;\, \Upsilon}$$

CAC.7: $\curlyvee$R:
$$\frac{\Theta \,;\, \Rightarrow\; \vartheta \,;\, \Upsilon, \upsilon, \upsilon \curlyvee \vartheta}{\Theta \,; \epsilon \Rightarrow \epsilon';\; \Upsilon, \upsilon \curlyvee \vartheta}$$

CAC.8: $\curlyvee$R:
$$\frac{\Theta \,;\, \epsilon \;\Rightarrow\; \epsilon' \,;\, \Upsilon, \upsilon, \upsilon \curlyvee \vartheta}{\Theta \,;\, \epsilon \;\Rightarrow\; \epsilon' \,;\, \Upsilon, \upsilon \curlyvee \vartheta}$$

CAC.9: $\curlyvee$L:
$$\frac{\Theta \,;\, \upsilon \;\Rightarrow\,;\, \Upsilon \qquad \Theta, \vartheta \,;\, \Rightarrow\,;\, \Upsilon}{\Theta \,;\, \upsilon \curlyvee \vartheta \;\Rightarrow\,;\, \Upsilon}$$

logical rules, connectives of type $\vartheta \times \vartheta \to \upsilon$

AAC.1: $\succ$R:
$$\frac{\Theta, \vartheta_1 \,;\, \Rightarrow\; \vartheta_2 \,;\, \Upsilon, \vartheta_1 \succ \vartheta_2}{\Theta \,; \epsilon \Rightarrow \epsilon';\; \Upsilon, \vartheta_1 \succ \vartheta_2}$$

AAC.2: $\succ$L:
$$\frac{\Theta, \vartheta_1 \,;\, \epsilon \;\Rightarrow\; \epsilon' \,;\, \Upsilon, \vartheta_1 \succ \vartheta_2}{\Theta \,;\, \epsilon \;\Rightarrow\; \epsilon' \,;\, \Upsilon, \vartheta_1 \succ \vartheta_2}$$

AAC.3: $\succ$L:
$$\frac{\Theta; \;\Rightarrow \vartheta_1 \,;\, \Upsilon \qquad \vartheta_2, \Theta \,;\, \Rightarrow\,;\, \Upsilon}{\Theta \,;\, \vartheta_1 \succ \vartheta_2 \;\Rightarrow\,;\, \Upsilon}$$

AAC.4: $\curlywedge$R:
$$\frac{\Theta \,;\, \Rightarrow\; \vartheta_1 \,;\, \Upsilon, \vartheta_1 \curlywedge \vartheta_2 \qquad \Theta \,;\, \Rightarrow\; \vartheta_2;\; \Upsilon, \vartheta_1 \curlywedge \vartheta_2}{\Theta \,; \epsilon \Rightarrow \epsilon';\; \Upsilon, \vartheta_1 \curlywedge \vartheta_2}$$

AAC.5,6: $\curlywedge$L:
$$\frac{\Theta, \vartheta_i \,;\, \Rightarrow\,;\, \Upsilon}{\Theta \,;\, \vartheta_0 \curlywedge \vartheta_1 \;\Rightarrow\,;\, \Upsilon}$$

AAC.7,8: $\curlyvee$R:
$$\frac{\Theta \,;\, \Rightarrow\; \vartheta_i \,; \vartheta_0 \curlyvee \vartheta_1, \Upsilon}{\Theta \,; \epsilon \Rightarrow \epsilon';\; \vartheta_0 \curlyvee \vartheta_1, \Upsilon}$$

AAC.9: $\curlyvee$L:
$$\frac{\Theta, \vartheta_0 \,;\, \Rightarrow\,;\, \Upsilon \qquad \Theta, \vartheta_1 \,;\, \Rightarrow\,;\, \Upsilon}{\Theta \,;\, \vartheta_0 \curlyvee \vartheta_1 \;\Rightarrow\,;\, \Upsilon}$$

Table 6: **ILP**, mixed conjectural rules

Strong subtraction

A.10: $\backslash\!\!\backslash R$:

$$\frac{\Theta;\ \Rightarrow \vartheta_1\ ;\ \Upsilon \qquad \vartheta_2, \Theta; \Rightarrow\ ;\ \Upsilon}{\Theta\ ;\ \Rightarrow\ \vartheta_1 \backslash\!\!\backslash \vartheta_2;\ \Upsilon}$$

A.11: $\backslash\!\!\backslash L$:

$$\frac{\vartheta_1 \backslash\!\!\backslash \vartheta_2, \vartheta_1, \Theta;\ \Rightarrow\ \vartheta_2;\ \Upsilon}{\vartheta_1 \backslash\!\!\backslash \vartheta_2, \Theta\ ;\ \epsilon\ \Rightarrow\ \epsilon'\ ;\ \Upsilon}$$

A.12: $\backslash\!\!\backslash L$:

$$\frac{\vartheta_1 \backslash\!\!\backslash \vartheta_2, \vartheta_1, \Theta;\ \epsilon \Rightarrow\ \epsilon';\ \Upsilon}{\vartheta_1 \backslash\!\!\backslash \vartheta_2, \Theta\ ;\ \epsilon\ \Rightarrow\ \epsilon'\ ;\ \Upsilon}$$

ACA.9: $\backslash\!\!\backslash R$:

$$\frac{\Theta\ ;\ \Rightarrow\ \vartheta\ ;\ \Upsilon \qquad \Theta\ ;\ \upsilon\ \Rightarrow\ ;\ \Upsilon}{\Theta\ ;\ \Rightarrow\ \vartheta \backslash\!\!\backslash \upsilon;\ \Upsilon}$$

ACA.10: $\backslash\!\!\backslash L$:

$$\frac{\vartheta, \Theta; \epsilon \Rightarrow\ \epsilon';\ \upsilon}{\vartheta \backslash\!\!\backslash \upsilon, \Theta\ ;\ \epsilon\ \Rightarrow\ \epsilon';\ \Upsilon}$$

CAA.10: $\backslash\!\!\backslash R$:

$$\frac{\Theta\ ;\ \Rightarrow\ ;\ \Upsilon, \upsilon \qquad \vartheta, \Theta\ ;\ \Rightarrow\ ;\ \Upsilon}{\Theta\ ; \Rightarrow \upsilon \backslash\!\!\backslash \vartheta;\ \Upsilon}$$

CAA.11: $\backslash\!\!\backslash L$:

$$\frac{\upsilon \backslash\!\!\backslash \vartheta, \Theta\ ;\ \upsilon\ \Rightarrow\ ;\ \Upsilon}{\upsilon \backslash\!\!\backslash \vartheta, \Theta\ ;\ \epsilon\ \Rightarrow\ \epsilon'\ ;\ \Upsilon}$$

CAA.12: $\backslash\!\!\backslash L$:

$$\frac{\upsilon \backslash\!\!\backslash \vartheta, \Theta\ ;\ \Rightarrow \vartheta;\ \Upsilon}{\upsilon \backslash\!\!\backslash \vartheta, \Theta\ ;\ \epsilon\ \Rightarrow\ \epsilon'\ ;\ \Upsilon}$$

CCA.9: $\backslash\!\!\backslash R$:

$$\frac{\Theta\ ;\ \Rightarrow\ ;\ \Upsilon, \upsilon_1 \qquad \Theta\ ; \upsilon_2 \Rightarrow\ ;\ \Upsilon}{\Theta\ ;\ \Rightarrow\ \upsilon_1 \backslash\!\!\backslash \upsilon_2;\ \Upsilon}$$

CCA.10: $\backslash\!\!\backslash L$:

$$\frac{\upsilon_1 \backslash\!\!\backslash \upsilon_2, \Theta\ ;\ \upsilon_1\ \Rightarrow\ ;\ \Upsilon, \upsilon_2}{\upsilon_1 \backslash\!\!\backslash \upsilon_2, \Theta\ ;\ \epsilon \Rightarrow \epsilon';\ \Upsilon}$$

CCA.11: $\backslash\!\!\backslash L$:

$$\frac{\upsilon_1 \backslash\!\!\backslash \upsilon_2, \Theta\ ;\ \epsilon\ \Rightarrow\ \epsilon'\ ;\ \Upsilon, \upsilon_2}{\upsilon_1 \backslash\!\!\backslash \upsilon_2, \Theta\ ;\ \epsilon \Rightarrow \epsilon';\ \Upsilon}$$

Weak subtraction

C.11: $\smallsetminus R$:

$$\frac{\Theta\ ;\ \upsilon_2\ \Rightarrow\ ;\ \Upsilon, \upsilon_1 \smallsetminus \upsilon_2 \qquad \Theta\ ;\ \epsilon\ \Rightarrow\ \epsilon'\ ;\ \upsilon_1, \Upsilon}{\Theta\ ;\ \epsilon\ \Rightarrow\ \epsilon'\ ;\ \Upsilon, \upsilon_1 \smallsetminus \upsilon_2}$$

C.12: $\smallsetminus L$:

$$\frac{\Theta\ ; \upsilon_1 \Rightarrow\ ;\ \Upsilon, \upsilon_2}{\Theta\ ;\ \upsilon_1 \smallsetminus \upsilon_2\ \Rightarrow\ ;\ \Upsilon}$$

ACC.9: $\smallsetminus R$:

$$\frac{\Theta;\ \Rightarrow\ \vartheta\ ;\ \Upsilon, \vartheta \smallsetminus \upsilon \qquad \Theta; \upsilon \Rightarrow\ ;\ \Upsilon, \vartheta \smallsetminus \upsilon}{\Theta\ ;\ \epsilon\ \Rightarrow\ \epsilon'\ ;\ \Upsilon, \vartheta \smallsetminus \upsilon}$$

ACC.10: $\smallsetminus L$:

$$\frac{\Theta, \vartheta;\ \Rightarrow;\ \Upsilon, \upsilon}{\Theta\ ;\ \vartheta \smallsetminus \upsilon\ \Rightarrow\ ;\ \Upsilon}$$

CAC.10: $\smallsetminus R$:

$$\frac{\Theta\ ; \epsilon \Rightarrow \epsilon';\ \Upsilon, \upsilon \qquad \Theta\ , \vartheta; \epsilon \Rightarrow \epsilon';\ \Upsilon,}{\Theta\ ; \epsilon \Rightarrow \epsilon';\ \Upsilon, \upsilon \smallsetminus \vartheta}$$

CAC.11: $\smallsetminus L$:

$$\frac{\Theta; \upsilon \Rightarrow\ ;\ \Upsilon}{\Theta\ ;\ \upsilon \smallsetminus \vartheta\ \Rightarrow\ ;\ \Upsilon}$$

CAC.12: $\smallsetminus L$:

$$\frac{\Theta;\ \Rightarrow \vartheta;\ \Upsilon}{\Theta\ ;\ \upsilon \smallsetminus \vartheta\ \Rightarrow\ ;\ \Upsilon}$$

AAC.10: $\smallsetminus R$:

$$\frac{\Theta;\ \Rightarrow\ \vartheta_1\ ;\ \Upsilon, \vartheta_1 \smallsetminus \vartheta_2 \qquad \vartheta_2, \Theta\ ; \epsilon \Rightarrow \epsilon';\ \Upsilon}{\Theta\ ; \epsilon \Rightarrow \epsilon';\ \Upsilon, \vartheta_1 \smallsetminus \vartheta_2}$$

AAC.11: $\smallsetminus L$:

$$\frac{\Theta, \vartheta_1;\ \Rightarrow \vartheta_2;\ \Upsilon}{\Theta\ ;\ \vartheta_1 \smallsetminus \vartheta_2\ \Rightarrow\ ;\ \Upsilon}$$

Table 7: **ILP**, subtractions